The QCD equation of state at finite density from analytical continuation

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The Taylor coefficients of $\frac{P}{T^4} = c_0 + c_2 \left( \frac{\mu_B}{T} \right)^2 + c_4 \left( \frac{\mu_B}{T} \right)^4 + c_6 \left( \frac{\mu_B}{T} \right)^6$

Equation of state

\[ \frac{P}{T^4} = \frac{S}{N} = 51 \]

\[ \frac{S}{N} = 420 \]

\[ \frac{\epsilon - 3P}{T^4} \]

\[ \frac{S}{N} = 51 \]

\[ \frac{S}{N} = 420 \]
The \((T, \mu_B)\)-phase diagram of QCD

Our observables:
Last Year: \(T_c\)

\(R. \ Bellwied \ et \ al., \ Phys. \ Lett. \ B751, \ 559 \ (2015), \ arXiv:1507.07510\)

This year: The Equation of State along trajectories of constant \(\frac{S}{N_B}\) and its Taylor coefficients determined by the method of analytical continuation

\(J. \ Günther \ et \ al., \ arXiv:1607.02493\)
Analytic continuation

$\frac{d(p/T^4)}{d\mu}$

$T_c(\mu)$

Roberge-Weiss

real chemical potentials

lattice simulations

1.2

2

1.6

2

2.0

2

2.4

2

continuation

$\frac{d(p/T^4)}{d\mu}$

$K$

$T_c(\mu)$

continuation

lattice simulations

real chemical potentials

$\mu^2/T^2$
Overview over the Analysis

1. Do the simulations at $\langle n_s \rangle \approx 0$
2. Extrapolate to $\langle n_s \rangle = 0$ and $\langle n_Q \rangle = 0.4 \langle n_B \rangle$
3. Make a fit in the $T$ direction
4. Determine everything you need for the observables
5. Make a fit in the $\mu_B$ direction
6. Make a fit in the $\frac{1}{N_t^2}$ direction
7. Determine the observables
Simulation details

- Action: tree-level Symanzik improved gauge action, with four times stout smeared staggered fermions
- 2+1+1 flavour, on LCP with pion and kaon mass
- Simulation at \( \langle n_S \rangle = 0 \) (as for heavy ion collisions, in contrast to simulations with \( \mu_s = 0 \) or \( \mu_S = 0 \) where \( \mu_S = \frac{1}{3}\mu_B - \mu_s \))
- Lattice sizes: \( 40^3 \times 10 \), \( 48^3 \times 12 \) and \( 64^3 \times 16 \)
- \( \frac{\mu_B}{T} = i^{\frac{j\pi}{8}} \) with \( j = 0, 3, 4, 5, 6, 6.5 \) and 7
- Two methods of scale setting: \( f_\pi \) and \( w_0 \), \( Lm_\pi > 4 \)
Tuning to $\langle n_S \rangle = 0$

Aim: For a given $\mu_B$ determine $\mu_S$ so that $\langle n_S \rangle = 0$. This means solving the differential equation

$$\langle n_S \rangle = 0 \iff \frac{\partial \log Z}{\partial \mu_S} = 0$$

Notation:

$$\chi_{udsc} = -\frac{1}{T^4} \frac{\partial^4}{\partial (\mu_u/T) \partial (\mu_d/T) \partial (\mu_s/T) \partial (\mu_c/T)} \frac{T}{V} \log Z$$

Assuming we know the value for $\mu_S(\mu_B)$ so that $\langle n_s \rangle = 0$ for $\mu_S(\mu_B^0)$ and $\mu_S(\mu_B^0 - \Delta \mu_B)$ with all the derivatives. Then (Runge-Kutta):

$$\mu_S(\mu_B^0 + \Delta \mu_B) = \mu_S(\mu_B^0 - \Delta \mu_B) + 2\Delta \mu_B \frac{d\mu_S}{d\mu_B}(\mu_B^0).$$

In the simulations with $\mu_B^0$ and $\mu_B^0 - \Delta \mu_B$, $\mu_S$ might not precisely tuned. There we want to extrapolate to a better value. We assume that correct value of $\mu_S$ is $\tilde{\mu}_S = \mu_S' + \Delta \mu_S'$. Then:

$$\langle n_S \rangle = \frac{\partial \log Z}{\partial \tilde{\mu}_S} = \frac{\partial \log Z}{\partial \mu_S'} + \frac{\partial^2 \log Z}{\partial \mu_S'^2} \Delta \mu_S' = 0$$
Tuning to $\langle n_S \rangle = 0$

This yields

$$\Delta \mu'_S = -\frac{\chi_S}{\chi_{SS}}$$

Similar for the derivative we get:

$$\frac{d\tilde{\mu}_S}{d\mu_B} = -\tilde{\chi}_{SB} \frac{1}{\tilde{Z}_{SS}} \bigg|_{\langle n_S \rangle = 0} = -\frac{\chi_{SB}}{\chi_{SS}} - \frac{\chi_{SSB} \chi_{SS} - \chi_{SSS} \chi_{SB}}{(\chi_{SS})^2} \Delta \mu'_S + O(\Delta \mu'_S^2)$$

With similar arguments we get to extrapolate to $\langle n_Q \rangle = 0.4 \langle n_B \rangle$
Fit in the $T$ direction

\[ \frac{d\hat{P}}{d\hat{\mu}_B} = a + bT + \frac{c}{T} + d\text{ arctan}(e(T - f)) \]

\[ A_1(T) = a + bT + c/T + d \text{ arctan}(e(T - f)) \]

\[ A_2(T) = a + bT + c/T + d/(1 + e(T - f)^g)^{1/g}, \]

\[ A_3(T) = a + bT + cT^2 + d\text{ arctan}(e(T - f)) \]

\[ A_4(T) = a + bT + cT^2 + d/(1 + e(T - f)^g)^{1/g}. \]
Fit in the $\mu_B$ direction

$$B_1(\hat{\mu}) = a + b\hat{\mu}^2 + c\hat{\mu}^4$$

$$B_2(\hat{\mu}) = \frac{a + b\hat{\mu}^2}{1 + c\hat{\mu}^2}$$

$$B_3(\hat{\mu}) = a + b\hat{\mu}^2 + c\sin(\hat{\mu})/\hat{\mu}$$
Extrapolation from different fit functions

Analytical continuation on $N_t = 12$ raw data

\[
\frac{T}{\mu_B} \frac{d(p/T^4)}{d(\mu_B/T)} = \frac{a + b\hat{\mu}^2 + c\hat{\mu}^4}{1 + c\hat{\mu}^2}
\]

\[
T = 170\text{MeV}
\]

\[
T = 145\text{MeV}
\]

\[
(\mu_B/T)^2 = -\hat{\mu}^2
\]
Entropy

- $S = \left. \frac{\partial P}{\partial T} \right|_{\mu_i}$
- We have $\frac{d\hat{P}}{d\hat{\mu}_B}$
- And we can only do a total derivative in $T$

$$
\frac{d\hat{P}}{dT} = \left. \frac{\partial P}{\partial T} \right|_{\mu_i} + \frac{d\mu_B}{dT} \hat{\mu}_B + \frac{d\mu_Q}{dT} \hat{\mu}_Q
$$

How we correct this:

$$
\frac{d\mu_B}{dT} = \frac{d(\hat{\mu}_B T)}{dT} = \hat{\mu}_B
$$

$$
\frac{d\mu_Q}{dT} = \frac{d(\hat{\mu}_Q T)}{dT} = \hat{\mu}_Q + T \frac{d\hat{\mu}_Q}{dT}
$$

$$
\hat{S} = 4\hat{P} + T \frac{d\hat{P}}{dT} - \hat{\mu}_B \hat{\mu}_B - \hat{\mu}_Q \hat{\mu}_Q - T \frac{d\hat{\mu}_Q}{dT}
$$

$$
= 4\hat{P} + T \frac{d\hat{P}}{dT} - \hat{\mu}_B (\hat{\mu}_B + 0.4\hat{\mu}_Q) - T \frac{d\hat{\mu}_Q}{dT}
$$

New terms we have to determine
Error estimation

- **Statistical error:**
  - Bootstrap method

- **Systematic error:**
  - Using different way of analysis, combining them in a histogram:
    - 4 fit functions for the $T$ direction
    - 3 fit functions in the $\mu_B$ direction
    - Doing continuum extrapolation and $\mu_B$-fit in one or two steps
    - 2 methods of scale setting: $f_\pi$ and $w_0$
    - 2 temperatures from where we use the extrapolated data
  
  This adds up to 96 ways of analysis
The Taylor coefficients of \( \frac{P}{T^4} \) are given by

\[ \frac{P}{T^4} = c_0 + c_2 \left( \frac{\mu_B}{T} \right)^2 + c_4 \left( \frac{\mu_B}{T} \right)^4 + c_6 \left( \frac{\mu_B}{T} \right)^6 \]

Influence of different orders

\[ T = 145 \text{MeV} \]

\[ n_B/T^3 \]

\[ \mathcal{O}(\mu_B) \]
\[ \mathcal{O}(\mu_B^3) \]
\[ \mathcal{O}(\mu_B^5) \]
Influence of different orders

\[ T = 170 \text{MeV} \]

\[ n_B / T^3 \]

\[ \mathcal{O}(\mu_B) \]
\[ \mathcal{O}(\mu_B^3) \]
\[ \mathcal{O}(\mu_B^5) \]
Trajectories

$\frac{S}{N_B} = 420 \ (200 \ GeV)$
$\frac{S}{N_B} = 144 \ (62.4 \ GeV)$
$\frac{S}{N_B} = 94 \ (39 \ GeV)$
$\frac{S}{N_B} = 70 \ (27 \ GeV)$
$\frac{S}{N_B} = 51 \ (19.6 \ GeV)$
$\frac{S}{N_B} = 30 \ (14.5 \ GeV)$
Equation of state

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