The QCD equation of state and fluctuations of conserved charges at non-vanishing temperature and density

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BNL-Bi-CCNU Collaboration:
Motivation: The QCD phase diagram

- The QCD phase diagram is extensively studied by heavy ion collisions (HIC)
- The QCD equation of state (EoS) is essential input to hydrodynamic modeling of HIC
- Hadronic abundances and fluctuations are measured at freeze-out

Lattice QCD can provide both, the EoS and fluctuations of conserved charges at small $\mu_B/T$
Motivation: The QCD phase diagram

Quark mass dependance of the phase diagram:

Not just one critical point!

Is physics on the freeze-out line sensitive to QCD critical behavior?

(one possible scenario, still under debate)
\[ \kappa \sigma^2 = \chi_4 / \chi_2 \]

The energy dependence tends to be more pronounced with wider \( p_T \) acceptance, relative to published results. The values are smaller for wider \( p_T \) acceptance.

Intriguing non-monotonic behavior in the cumulant ratio of net-proton number fluctuations.

Can this data be understood in terms of equilibrium thermodynamics?
Beam Energy Scan at RHIC

\[ \kappa \sigma^2 = \chi_4 / \chi_2 \]

(a) net-p \( \kappa \sigma^2 \)

\[
\begin{align*}
p_T \text{ Range (GeV/c)} & \\
0.4 < p_T < 0.8 & \text{(STAR: PRL112)} \\
0.4 < p_T < 1.2 & \\
0.4 < p_T < 1.4 & \\
0.4 < p_T < 1.6 & \\
0.4 < p_T < 2.0 &
\end{align*}
\]

\( \Rightarrow \) intriguing non-monotonic behavior in the cumulant ratio of net-proton number fluctuations

Can this data be understood in terms of equilibrium thermodynamics?

How far do we get with a low order Taylor expansion?

\( \sqrt{s_{NN}} \text{ GeV} \)

X. Luo, CPOD’14

STAR Preliminary

\( \chi^2 = 4/2 \)
Big obstacle on the lattice: the sign problem

- The QCD partition function

\[
Z(V, T, \mu) = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\{-S_F(U, \psi, \bar{\psi}) - \beta S_G(U)\}
\]

\[
= \int \mathcal{D}U \det[M](U, \mu) \exp\{-\beta S_G(U)\}
\]

complex for \( \mu > 0 \) can not be interpreted as probability

we find: \([\det M(\mu)]^* = \det M(-\mu^*)\)

\[\rightarrow \text{determinant is real only for } \mu = 0 \text{ or } \mu = i\mu I\]

Here we follow the Taylor expansion approach, all quantities are expanded in \( \mu/T \), at fixed \( T \).
1) Introduction and Motivation

2) Taylor expansion of the equation of state
   • definitions, state-of-the-art, convergence estimate
   • constraints: strangeness neutrality, constant baryon number to electric charge ratio

2) Cumulant ratios at nonzero baryon number density
   • expressing $\mu_B/T$ by $M_B/\sigma_B^2$
   • RHIC data vs. QCD equilibrium thermodynamics

3) Conclusions and Summary
Taylor expansion of the pressure
**Conserved charge fluctuations**

Expansion of the pressure:

$$\frac{p}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{1}{i!j!k!} \chi_{BQS}^{ijk,0} \left( \frac{\mu_B}{T} \right)^i \left( \frac{\mu_Q}{T} \right)^j \left( \frac{\mu_S}{T} \right)^k$$

**Lattice**

$$\chi_n^X = \left. \frac{\partial^n [p/T^4]}{\partial (\mu_X/T)^n} \right|_{\mu_X = 0}$$

Generalized susceptibilities

$$\Rightarrow \text{ only at } \mu_X = 0!$$

**Experiment**

\begin{align*}
VT^3 \chi_2^X &= \langle (\delta N_X)^2 \rangle \\
VT^3 \chi_4^X &= \langle (\delta N_X)^4 \rangle - 3 \langle (\delta N_X)^2 \rangle^2 \\
VT^3 \chi_6^X &= \langle (\delta N_X)^4 \rangle - 15 \langle (\delta N_X)^4 \rangle \langle (\delta N_X)^2 \rangle + 30 \langle (\delta N_X)^2 \rangle^3
\end{align*}

Cumulants of net-charge fluctuations

$$\delta N_X \equiv N_X - \langle N_X \rangle$$

$$\Rightarrow \text{ only at freeze-out } (\mu_f(\sqrt{s}), T_f(\sqrt{s}))!$$
Conserved charge fluctuations

Expansion of the pressure:

\[
\frac{p}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{1}{i!j!k!} \chi_{ijk,0}^{BQS} \left( \frac{\mu_B}{T} \right)^i \left( \frac{\mu_Q}{T} \right)^j \left( \frac{\mu_S}{T} \right)^k
\]

\[X = B, Q, S: \text{conserved charges}\]

consider cumulant ratios to eliminate the freeze-out volume

**Lattice**

\[
\frac{\chi_1^X(\mu_B, T)}{\chi_2^X(\mu_B, T)} = \frac{M_X}{\sigma_X^2}
\]

\[
\frac{\chi_3^X(\mu_B, T)}{\chi_2^X(\mu_B, T)} = S_X \sigma_X
\]

\[
\frac{\chi_4^X(\mu_B, T)}{\chi_2^X(\mu_B, T)} = \kappa_X \sigma_X^2
\]

**Experiment**

\[M := \text{mean}\]

\[\sigma^2 := \text{variance}\]

\[S := \text{skewness}\]

\[\kappa := \text{kurtosis}\]
State-of-the-art equation of state for (2+1)-flavor QCD

Pressure $p$, energy density $\varepsilon$ and entropy density $s$, at $\mu_B = \mu_Q = \mu_S = 0$:

Bazavov et al. [HotQCD], Phys. Rev. D90 (2014) 094503.

-Improves over earlier HotQCD calculation Bazavov et al. [HotQCD], Phys. Rev. D80 (2009) 014504.


-Until the crossover region the QCD EoS agrees quite well with the HRG EoS, however, QCD results are systematically above HRG

$\Rightarrow$ evidence for additional hadronic states?
The equation of state at $\mu_B > 0$

**Lattice setup:** HISQ-action, (2+1)-flavor, quark mass ratios $m_l/m_s = 1/27$, and $m_l/m_s = 1/20$.

**Methodology:** stochastic noise method

\[
\frac{\partial \ln Z}{\partial \mu} = \frac{1}{Z} \int \mathcal{D}U \; \text{Tr} \left[ M^{-1} M' \right] e^{\text{Tr} \ln M} e^{-\beta S_G} \\
= \left\langle \text{Tr} \left[ M^{-1} M' \right] \right\rangle \\
 \frac{\partial^2 \ln Z}{\partial \mu^2} = \left\langle \text{Tr} \left[ M^{-1} M'' \right] \right\rangle - \left\langle \text{Tr} \left[ M^{-1} M' M^{-1} \right] \right\rangle + \left\langle \text{Tr} \left[ M^{-1} M' \right]^2 \right\rangle \\
\vdots \\
\text{Tr} [Q] \approx \frac{1}{N} \sum_{i=1}^{N} \eta_i^\dagger Q \eta_i \\
\text{with } \eta_i \text{ being a vector with uncorrelated random entries, normalized such that } \\
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \eta_{i,x} \eta_{i,y} = \delta_{x,y}
\]
The equation of state at $\mu_B > 0$

Pressure correction due to non-vanishing baryon chemical potential:

$$\frac{\Delta p(T, \mu_B)}{T^4} = \frac{p(T, \mu_B) - p(T, 0)}{T^4} = \frac{1}{2} \chi_2^B \left( \frac{\mu_B}{T} \right)^2 \left( 1 + \frac{1}{12} \chi_4^B \left( \frac{\mu_B}{T} \right)^2 + \cdots \right)$$

**LO:** Variance of baryon number fluctuation

**NLO:** Kurtosis x variance
The equation of state at \( \mu_B > 0 \)

pressure correction due to non-vanishing baryon chemical potential:

\[
\frac{\Delta p(T, \mu_B)}{T^4} = \frac{1}{2} \chi_2^B \left( \frac{\mu_B}{T} \right)^2 \left( 1 + \frac{1}{12} \frac{\chi_4^B}{\chi_2} \left( \frac{\mu_B}{T} \right)^2 + \frac{1}{360} \frac{\chi_6^B}{\chi_2} \left( \frac{\mu_B}{T} \right)^4 + \cdots \right)
\]

NNLO

pressure is well controlled for \( \mu_B/T \leq 2 \) or equivalently \( \sqrt{\frac{s}{NN}} \geq 12 \) GeV
The equation of state at $\mu_B > 0$

correction to energy density:

- energy coefficients are readily obtained from the pressure coefficients
  \[
  \frac{\Delta \epsilon}{T^4} = \sum_{i=1}^{\infty} \left( 3\chi_{2i}^B - \tilde{\chi}_{2i}^B \right) \left( \frac{\mu_B}{T} \right)^{2i}
  \]
  with \[ \tilde{\chi}_{2i}^B = T \frac{d\chi_{2i}^B(T)}{dT} \]

- convergence deferred due to additional temperature derivative
Introducing $\mu_S > 0$ and $\mu_Q > 0$

Apply: initial conditions as in HIC

- strangeness neutrality: $\langle N_S \rangle = 0$
- isospin asymmetry: $\langle N_Q \rangle = r \langle N_B \rangle$

expand in powers of $\mu_B, \mu_Q, \mu_S$
solve for $\mu_Q, \mu_S$

$\mu_Q(T, \mu_B) = q_1(T)\hat{\mu}_B + q_3(T)\hat{\mu}_B^3$
$\mu_S(T, \mu_B) = s_1(T)\hat{\mu}_B + s_3(T)\hat{\mu}_B^3$

$\hat{\mu}_B = \mu_B/T$

LO  NLO

define strangeness neutral coefficients $p_n$

$\frac{\Delta p}{T^4} = \frac{1}{2} \chi_2 B \hat{\mu}_B^2 + \frac{1}{2} \chi_2 Q \hat{\mu}_Q^2 + \frac{1}{2} \chi_2 S \hat{\mu}_S^2 + \chi_{11}^{BQ} \hat{\mu}_B \hat{\mu}_Q + \chi_{11}^{BS} \hat{\mu}_B \hat{\mu}_S + \chi_{11}^{QS} \hat{\mu}_Q \hat{\mu}_S + \cdots$

$= \frac{1}{2} \left( \chi_2 B + \chi_2 Q q_1^2 + \chi_2 S s_1^2 + 2\chi_{11}^{BQ} q_1 + 2\chi_{11}^{BS} s_1 + 2\chi_{11}^{QS} q_1 s_1 \right) \hat{\mu}_B^2 + \cdots$

$r \approx 0.4$ for Au-Au and Pb-Pb
Introducing $\mu_S > 0$ and $\mu_Q > 0$

Strangeness neutral pressure coefficients:

$$\frac{\Delta p}{T^4} = \frac{1}{2} p_2 \left( \frac{\mu_B}{T} \right)^2 + \frac{1}{24} p_4 \left( \frac{\mu_B}{T} \right)^4 + \cdots$$

- considerable strangeness dependence
- mild isospin dependence
Introducing $\mu_S > 0$ and $\mu_Q > 0$

Strangeness neutral pressure coefficients:

$$\frac{\Delta p}{T^4} = \frac{1}{2} p_2 \left( \frac{\mu_B}{T} \right)^2 + \frac{1}{24} p_4 \left( \frac{\mu_B}{T} \right)^4 + \cdots$$

$p_2(r, T)$

$p_4(r, T)$

$m_\nu/m_1 = 20$ (open)
$m_\nu/m_1 = 27$ (filled)

HRG

$\mu_O = \mu_S = 0$, continuum
$N_S = 0$, $N_Q/N_B = 0.4$, $N_\tau = 16$
12
8
6

$m_\nu/m_1 = 20$ (open)
$m_\nu/m_1 = 27$ (filled)

HRG

$N_S = 0$, $N_Q/N_B = 0.4$, cont. est.
$N_S = 0$, $N_Q/N_B = 0.4$, $N_\tau = 6$

$m_\nu/m_1 = 20$ (open)
$m_\nu/m_1 = 27$ (filled)

Free quark gas

$m_\nu/m_1 = 20$ (open)
$m_\nu/m_1 = 27$ (filled)

HRG

$N_S = 0$, $N_Q/N_B = 0.4$, cont. est.
$N_S = 0$, $N_Q/N_B = 0.4$, $N_\tau = 6$

Free quark gas
Introducing $\mu_S > 0$ and $\mu_Q > 0$

total pressure corrections (comparison):

$\mu_Q = \mu_S = 0$

$N_S = 0, \ N_Q/N_B = 0.4$

• note scale difference
Cumulant ratios at $\mu_B > 0$
expanding ratios of baryon number fluctuations:

\[
\frac{M_B}{\sigma_B^2} = \frac{\mu_B}{T} \frac{1 + \frac{1}{6} \frac{\chi_4^B}{\chi_2^B} \left( \frac{\mu_B}{T} \right)^2}{1 + \frac{1}{2} \frac{\chi_4^B}{\chi_2^B} \left( \frac{\mu_B}{T} \right)^2}
\]

\[
S_B \sigma_B = \frac{\mu_B}{T} \frac{\chi_4^B}{\chi_2^B} \frac{1 + \frac{1}{6} \frac{\chi_6^B}{\chi_4^B} \left( \frac{\mu_B}{T} \right)^2}{1 + \frac{1}{2} \frac{\chi_4^B}{\chi_2^B} \left( \frac{\mu_B}{T} \right)^2}
\]

\[
\kappa_B \sigma_B^2 = \frac{\chi_4^B}{\chi_2^B} \frac{1 + \frac{1}{2} \frac{\chi_6^B}{\chi_4^B} \left( \frac{\mu_B}{T} \right)^2}{1 + \frac{1}{2} \frac{\chi_4^B}{\chi_2^B} \left( \frac{\mu_B}{T} \right)^2}
\]

• in QCD all the ratios \( \chi_{n+2}^B / \chi_n^B \) are a functions of temperature, in HRG they are unity

current simplifications:

• on the left: \( \mu_Q = \mu_S = 0 \)

• approximate freeze-out line by a constant

\[
T^f(\mu_B) = T^f(0) \left( 1 - \kappa_f \left( \frac{\mu_B}{T} \right)^2 \right)
\]

\( \kappa_f \approx 0 \)
Conserved charge fluctuations and freeze-out

Aim for a comparison with RHIC data:

how to translate $\sqrt{s_{NN}}$ into $\mu_B$ without making further approximations?

→ trick: express all ratios as function of $M_B/\sigma_B^2 = \chi_1^B/\chi_2^B = R_{12}^B$

• caution: RHIC measures net-proton and not net-baryon number fluctuations

• may use HRG motivated conversion factor:

\[
R_{12}^P = \tanh(\hat{\mu}_B + \hat{\mu}_Q) = \hat{\mu}_B + \mathcal{O}(\hat{\mu}_B^3)
\]

no $\hat{\mu}_S$-dependence, neglect $\hat{\mu}_Q$-dependence

HRG:

QCD:

\[
R_{12}^B = R_{12}^{B,1} \hat{\mu}_B + \mathcal{O}(\hat{\mu}_B^3)
\]

(strangeness neutral, $r=0.4$)

⇒ to leading order: $R_{12}^B/R_{12}^P = R_{12}^{B,1}$

A. Bazavov et al., PRD 93 (2016) 014512
Skewness at $\mu_B > 0$

$$R_{31}^B \equiv \frac{S_B \sigma_B^3}{M_B} = R_{31}^{B,0} + \frac{R_{31}^{B,2}}{R_{12}^{B,1}} \left( \frac{M_B}{\sigma_B^2} \right)^2 + \cdots$$

(strangeness neutral, $r=0.4$)

F. Karsch et al., arXiv:1512.06987
Skewness at $\mu_B > 0$

\[ R_{31}^B \equiv \frac{S_B \sigma_B^3}{M_B} = R_{31}^{B,0} + \frac{R_{31}^{B,2}}{(R_{12}^{B,1})^2} \left( \frac{M_B}{\sigma_B^2} \right)^2 + \cdots \]

(strangeness neutral, $r=0.4$)

F. Karsch et al., arXiv:1512.06987

- intercept consistent with QCD result
Skewness at $\mu_B > 0$

$$
R_{31}^B \equiv \frac{S_B \sigma_B^3}{M_B} = R_{31}^{B,0} + \frac{R_{31}^{B,2}}{R_{12}^{B,1}} \left( \frac{M_B}{\sigma_B^2} \right)^2 + \cdots
$$

(strangeness neutral, $r=0.4$)

F. Karsch et al., arXiv:1512.06987

- intercept consistent with QCD result
- curvature consistent with QCD result (still large statistical error)
Kurtosis at $\mu_B > 0$

$$R_{42}^B \equiv \kappa_B \sigma_B^2 = R_{42}^{B,0} + \frac{R_{42}^{B,2}}{R_{12}^{B,1}} \left( \frac{M_B}{\sigma_B^2} \right)^2 + \cdots$$

(strangeness neutral, $r=0.4$)

F. Karsch et al., arXiv:1512.06987

• find similar results for the kurtosis, i.e. intercept and curvature are in agreement with QCD results

• especially we find $\chi_{42}^{B,0} \simeq \chi_{31}^{B,0}$ and $\chi_{42}^{B,2} \simeq 3\chi_{31}^{B,2}$ (exact for $\mu_Q = \mu_S = 0$)

  which is also supported by the RHIC data

• in general: need to understand systematics
  - **non-equilibrium effects** (S. Mukherjee et al., arXiv:1506.00645)
  - **proton vs. baryon number fluctuations** (M. Kitazawa et al., arXiv: 1205.3292, arXiv:1303.3338)
Conclusions and Summary
Conclusions and Summary

• Cumulants of conserved charge fluctuations can be obtained on the lattice and are measured in heavy ion collision. They can be used to infer freeze-out parameter.

• Results on bulk thermodynamics based on Taylor expansion of the QCD partition function are currently well controlled for $\mu_B/T \leq 2$, i.e. for $\sqrt{s} \gtrsim 20$ GeV.

• in the range $20$ GeV $\leq \sqrt{s_{NN}} \leq 200$ GeV the pattern seen in the beam energy dependance of up to 4th order cumulants of net-proton (baryon) number and electric charge fluctuations can be understood in terms of QCD equilibrium thermodynamics.

• QCD equilibrium thermodynamics sets a baseline for the discussion of the systematic effects which have to be taken into account for a more quantitative comparison.
Backup
(analytic) one-link smearing:

- Improved flavor (taste) symmetry breaking

3-link term:

- Improved SB limit
The constraints $\langle N_S \rangle = 0$ and $\langle N_Q \rangle = r \langle N_B \rangle$ are fulfilled by choosing the strangeness and electric chemical potentials as

$$\mu_S(T, \mu_B) = s_1(T)\mu_B + s_3(T)\mu_B^3$$

$$\mu_Q(T, \mu_B) = q_1(T)\mu_B + q_3(T)\mu_B^3$$
Kurtosis at $\mu_B > 0$

$$R_{31}^B \equiv \frac{S_B \sigma_B^3}{M_B} = R_{31}^{B,0} + \frac{R_{31}^{B,2}}{(R_{12}^B)^2} \left( \frac{M_B}{\sigma_B^2} \right)^2 + \cdots$$

(strangeness neutral, $r=0.4$)

$$R_{42}^B \equiv \kappa_B \sigma_B^2 = R_{42}^{B,0} + \frac{R_{42}^{B,2}}{(R_{12}^B)^2} \left( \frac{M_B}{\sigma_B^2} \right)^2 + \cdots$$

In Fig. 1(right) we show the ratio of the NLO expansion coefficients, $\kappa_B$.

In Fig. 2 we show the ratio of net-proton number fluctuations measured in the BES at RHIC. To do so, we also note that we may eliminate the (4,8) flavor QCD on lattices by solving the Taylor series expansion for another ratio, e.g. $\chi^2$ with respect to the data on $R_{12}^B$. Eliminating this figure shows the di