Goldstone-Type Pseudoscalar Mesons: Instantaneous Bethe–Salpeter Models

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Goldstonic quark–antiquark bound states

Within quantum chromodynamics, the pions or, as a matter of fact, all light pseudoscalar mesons must be interpretable as both quark–antiquark bound states and almost massless (pseudo) Goldstone bosons of the spontaneously —and to a minor degree even explicitly—broken chiral symmetries of QCD.

Relativistic quantum field theory describes bound states by Bethe–Salpeter amplitudes $\Phi(p)$ controlled by some homogeneous Bethe–Salpeter equation defined, for two bound particles of individual and relative momenta $p_{1,2}$ and p, by their propagators $S(p_{1,2})$ and an integral kernel $K(p,q)$ encompassing their interactions, suppressing dependences on the total momentum p_1+p_2:

$$\Phi(p) = \frac{\mathrm{i}}{(2\pi)^4}\, S_1(p_1) \int \mathrm{d}^4 q\, K(p,q)\, \Phi(q)\, S_2(-p_2) \ .$$

Suitably adapted inversion techniques [1] allow us to retrieve the underlying interactions analytically in form of a (configuration-space) central potential $V(r)$, $r \equiv |\boldsymbol{x}|$, from presumed solutions to the Bethe–Salpeter equation [2].

By that, we are put in a position to construct exact analytic Bethe–Salpeter solutions for all massless pseudoscalar mesons [3] in the sense of establishing rigorous analytic relationships between interactions and resulting solutions: all analytic findings [4] may then be confronted with numerical outcomes [5].

Crucial simplifying-assumptions sequence

1. Assuming, for any involved quark, both instantaneous interactions and free propagation with a mass dubbed constituent, simplifies the Bethe–Salpeter equation to a bound-state equation for the Salpeter amplitude
$$\phi(\boldsymbol{p}) \propto \int \mathrm{d}p_0\, \Phi(p) \ .$$
For a spin-$\frac{1}{2}$ fermion and a spin-$\frac{1}{2}$ antifermion of equal masses m bound to a spin-singlet state (which is the case for, e.g., pseudoscalar mesons), this wave function involves only two independent components, $\varphi_{1,2}(\boldsymbol{p})$:
$$\phi(\boldsymbol{p}) = \left[\varphi_1(\boldsymbol{p})\,\frac{\gamma_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)}{E(p)} + \varphi_2(\boldsymbol{p})\right]\gamma_5 \ ,$$
$$E(p) \equiv \sqrt{\boldsymbol{p}^2+m^2} \ , \qquad p \equiv |\boldsymbol{p}| \ .$$

2. Upon assuming the quark interactions in the kernel to respect spherical and Fierz symmetries, the bound-state equation for $\phi(\boldsymbol{p})$ becomes a set of two coupled radial eigenvalue equations for the bound-state mass $\widehat{M}$:
$$2\,E(p)\,\varphi_2(p) + 2\int_0^\infty \frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = \widehat{M}\,\varphi_1(p) \ ,$$
$$2\,E(p)\,\varphi_1(p) = \widehat{M}\,\varphi_2(p) \ ,$$
$$V(p,q) \equiv \frac{8\pi}{p\,q}\int_0^\infty \mathrm{d}r\,\sin(p\,r)\sin(q\,r)\,V(r) \ , \qquad q \equiv |\boldsymbol{q}| \ .$$

3. In the truly massless Goldstone case $\widehat{M}=0$, the system decouples, one component vanishes [$\varphi_1(p)\equiv 0$], and the surviving component satisfies
$$E(p)\,\varphi_2(p) + \int_0^\infty \frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = 0 \ .$$
Denoting by $T(r)$ the Fourier transform of the kinetic term $E(p)\,\varphi_2(p)$, $V(r)$ can be found from the latter's configuration-space representation:
$$V(r) = -\frac{T(r)}{\varphi_2(r)} \ .$$

Constraints on Bethe–Salpeter amplitude

Information on $\varphi_2(p)$ can be extracted from the full quark propagator $S(p)$, determined by its mass function $M(p^2)$ and a renormalization factor $Z(p^2)$:

$$S(p)=\frac{\mathrm{i}\,Z(p^2)}{\not p-M(p^2)+\mathrm{i}\,\varepsilon}\ ,\qquad \not p\equiv p^\mu\,\gamma_\mu\ ,\qquad \varepsilon\downarrow 0\ .$$

Studies of $S(p)$ within the Dyson–Schwinger framework, preferably done in Euclidean space indicated by underlined variables, entail crucial insights. In the chiral limit, a Ward–Takahashi identity relates [6] this quark propagator to the flavour-nonsinglet pseudoscalar-meson Bethe–Salpeter amplitude [3]:

$$\Phi(\underline{k})\approx\frac{M(\underline{k}^2)}{\underline{k}^2+M^2(\underline{k}^2)}\,\gamma_5+\text{subleading contributions}\ .$$

In order to devise an analytic scenario, we exploit two pieces of information:

1. Phenomenologically sound Dyson–Schwinger models [7] get for $M(\underline{k}^2)$, in the chiral limit, at large $\underline{k}^2$ a decrease basically proportional to $1/\underline{k}^2$.
2. From axiomatic QFT, we may infer [8] that the presence in $M(\underline{k}^2)$ of an inflexion point at spacelike momenta $\underline{k}^2>0$ entails quark confinement.

Of course, such requirements on $M(\underline{k}^2)$ are reflected by $\Phi(\underline{k})$. A compatible ansatz for $\Phi(\underline{k})$, involving a mass parameter μ and a mixing parameter η, is

$$\Phi(\underline{k})=\left[\frac{1}{(\underline{k}^2+\mu^2)^2}+\frac{\eta\,\underline{k}^2}{(\underline{k}^2+\mu^2)^3}\right]\underline{\gamma}_5\ ,\qquad \mu>0\ ,\qquad \eta\in\mathbb{R}\ .$$

An integration w.r.t. the Euclidean momentum's time component results in

$$\varphi_2(p)\propto\frac{1}{(p^2+\mu^2)^{3/2}}+\eta\,\frac{p^2+\mu^2/4}{(p^2+\mu^2)^{5/2}}\ ,\qquad p\equiv|\boldsymbol{p}|\ ,$$

in configuration space expressible in terms of modified Bessel functions K_n:

$$\varphi_2(r)\propto 4\,(1+\eta)\,K_0(\mu\,r)-\eta\,\mu\,r\,K_1(\mu\,r)\ .$$

If $\eta<-1$ or $\eta>0$, $\varphi_2(r)$ has one zero, which induces a singularity in $V(r)$.

For special values of m/μ, $V(r)$ can be given by an analytic expression [3,4]. Henceforth, any quantity is understood in units of the adequate power of μ.

Confining potentials: Analytic results [3,4]

As consequence of our particular ansatz for $\varphi_2(r)$, for $\eta \neq -1$ all $V(r)$ must develop, at spatial origin, a logarithmically softened Coulombic singularity:

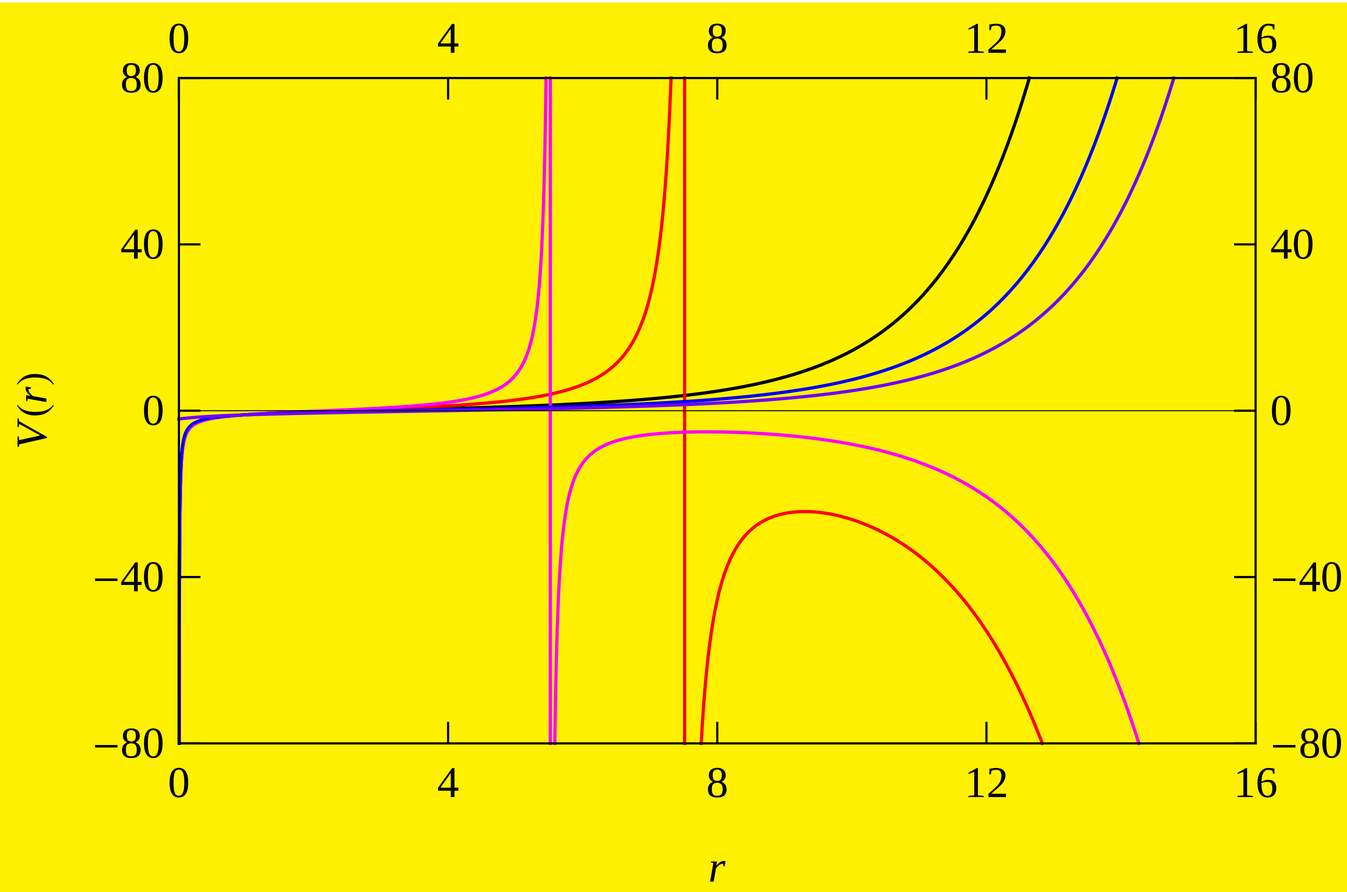
$$V(r) \xrightarrow[r\to 0]{} \frac{\text{const}}{r \ln r} \xrightarrow[r\to 0]{} -\infty \qquad (\text{const} > 0) \qquad \text{for } \eta \neq -1\ .$$

Analytically manageable scenario of massless quarks ($m = 0$)

For our $\varphi_2(r)$, $V(r)$ involves modified Bessel (I_n) and Struve ($\mathbf{L}_n$) functions and rises—confiningly—to infinity either at the zero of $\varphi_2(r)$ or for $r \to \infty$:

$$V(r) = \frac{N(r)}{D(r)}\ , \qquad \begin{aligned} N(r) &\equiv \pi\,[4 + \eta\,(4 + r^2)]\,[\mathbf{L}_0(r) - I_0(r)] \\ &\quad + \pi\,(4 + 5\,\eta)\,r\,[\mathbf{L}_1(r) - I_1(r)] + 4\,(2 + 3\,\eta)\,r\ , \\ D(r) &\equiv 2\,r\,[4\,(1 + \eta)\,K_0(r) - \eta\,r\,K_1(r)]\ . \end{aligned}$$

$V(r)$ of the Fierz-symmetric kernel $K(p,q)$ for $m = 0$ and mixture $\eta = 0$ [3] (black), $\eta = 1$ (red), $\eta = 2$ (magenta), $\eta = -0.5$ (blue), or $\eta = -1$ (violet):

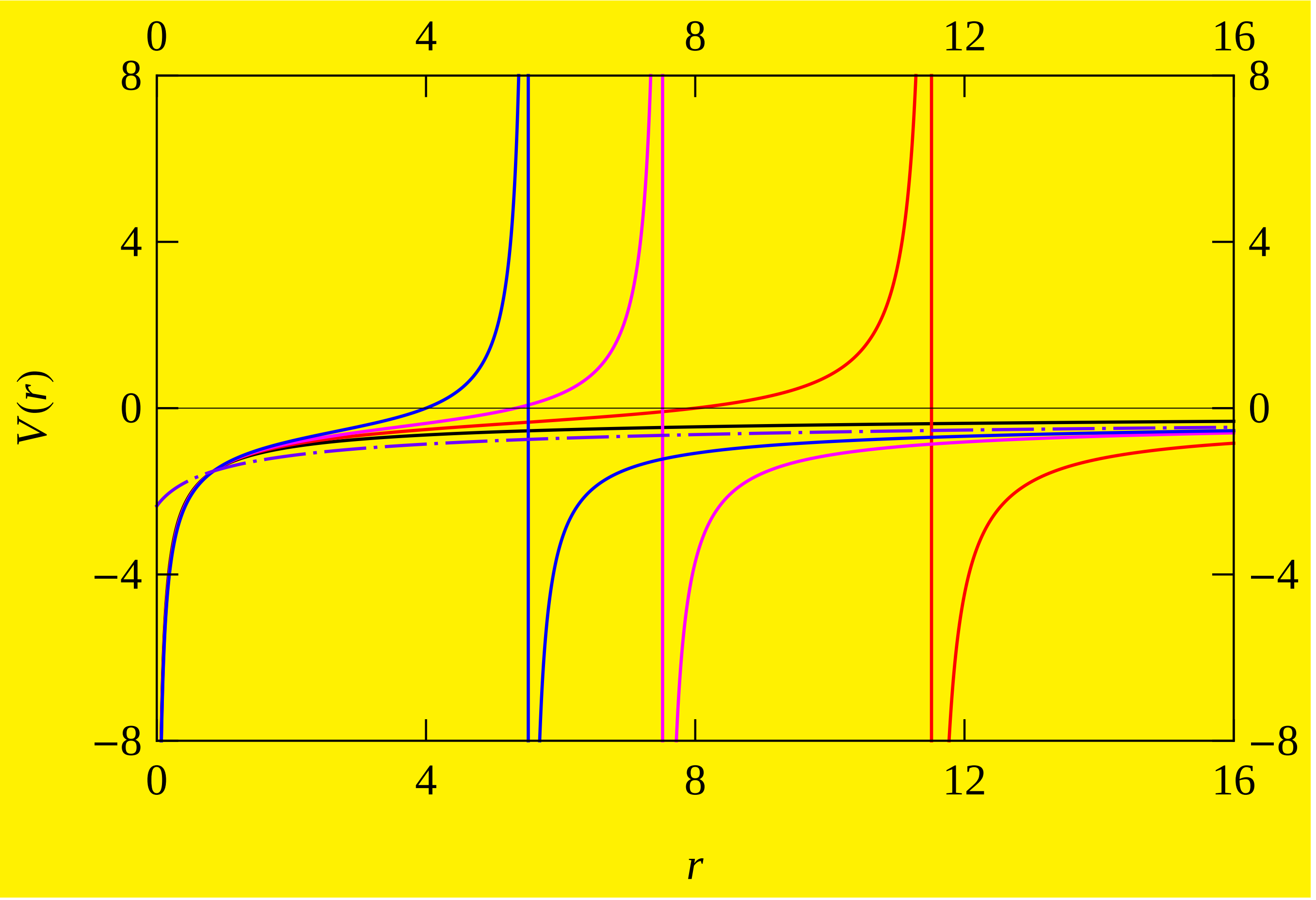


Analytically expressible case: quarks of common mass $m = \mu$

Here, $T(r)$ exhibits a mixture of Yukawa and exponential behaviour. Thus,

$$V(r) = -\frac{\pi\,[8+\eta\,(8-3\,r)]\exp(-r)}{4\,r\,[4\,(1+\eta)\,K_0(r)-\eta\,r\,K_1(r)]} \xrightarrow[r\to\infty]{} -\frac{\text{const}}{\sqrt{r}} \xrightarrow[r\to\infty]{} 0\ .$$

$V(r)$ of the Fierz-symmetric kernel $K(p,q)$ for $m = 1$ and mixture $\eta = 0$ [3] (black), $\eta = 0.5$ (red), $\eta = 1$ (magenta), $\eta = 2$ (blue), and $\eta = -1$ (violet):



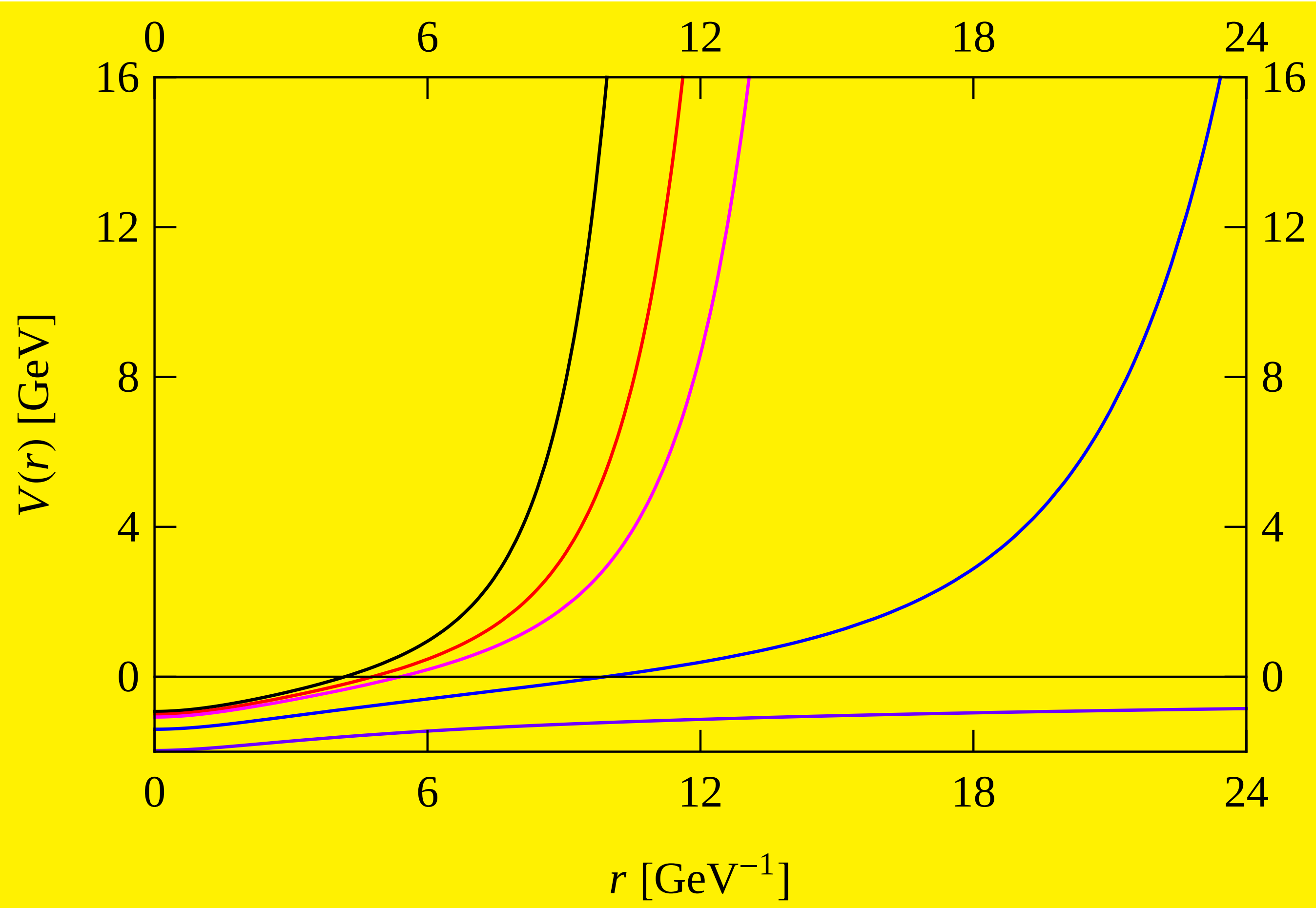
Test of reliability: Numerical derivation [5]

We check our results using the pointwise form of the chiral-limit quark mass function $M(\underline{k}^2)$, provided graphically in Ref. [7]: We parametrize $M(\underline{k}^2)$ by

$$M(\underline{k}^2) = 0.708\ \text{GeV}\exp\left(-\frac{\underline{k}^2}{0.655\ \text{GeV}^2}\right) + \frac{0.0706\ \text{GeV}}{\left[1+\left(\frac{\underline{k}^2}{0.487\ \text{GeV}^2}\right)^{1.48}\right]^{0.752}}\ .$$

N.B.: $1.48 \times 0.752 = 1.1$, pretty close to unity. Feeding this parametrization into our inversion procedure, we get potentials which are finite at $r = 0$ and rise, with r, to infinity for sufficiently small m but stay negative for large m.

$V(r)$ numerically fixed from $M(\underline{k}^2)$ for $m = 0$ (black), $m = 0.35$ GeV (red) $m = 0.50$ GeV (magenta), $m = 1.0$ GeV (blue), $m = 1.69$ GeV (violet) [5]:



Minimal-Damage Instantaneous Equation

Desisting from our analytic intentions, we may use an approximation [9] to the Bethe–Salpeter formalism undeniably closer [10] to its ancestor, in order to exploit the full information encoded in the quark propagator. Herein, the spin-singlet equal-mass quark–antiquark bound-state Salpeter amplitude is

$$\phi(\boldsymbol{p}) = \left[\varphi_1(\boldsymbol{p})\,\frac{\gamma_0\,[\boldsymbol{\gamma}\cdot\boldsymbol{p} + M(\boldsymbol{p}^2)]}{E(\boldsymbol{p})} + \varphi_2(\boldsymbol{p})\right]\gamma_5\ ,$$
$$E(\boldsymbol{p}) \equiv \sqrt{\boldsymbol{p}^2 + M^2(\boldsymbol{p}^2)}\ .$$

For Fierz-invariant interactions, our coupled bound-state equations become

$$2\,E(p)\,\varphi_2(p) + 2\,Z^2(p^2)\int_0^\infty \frac{dq\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = \widehat{M}\,\varphi_1(p)\ ,$$

$$2\,E(p)\,\varphi_1(p) = \widehat{M}\,\varphi_2(p)\ ,\qquad E(p) \equiv \sqrt{p^2 + M^2(p^2)}\ ,\qquad p \equiv |\boldsymbol{p}|\ .$$

Hence, for $\widehat{M} = 0$ only a single nontrivial relation governs $\phi(\boldsymbol{p}) = \varphi_2(\boldsymbol{p})\,\gamma_5$:

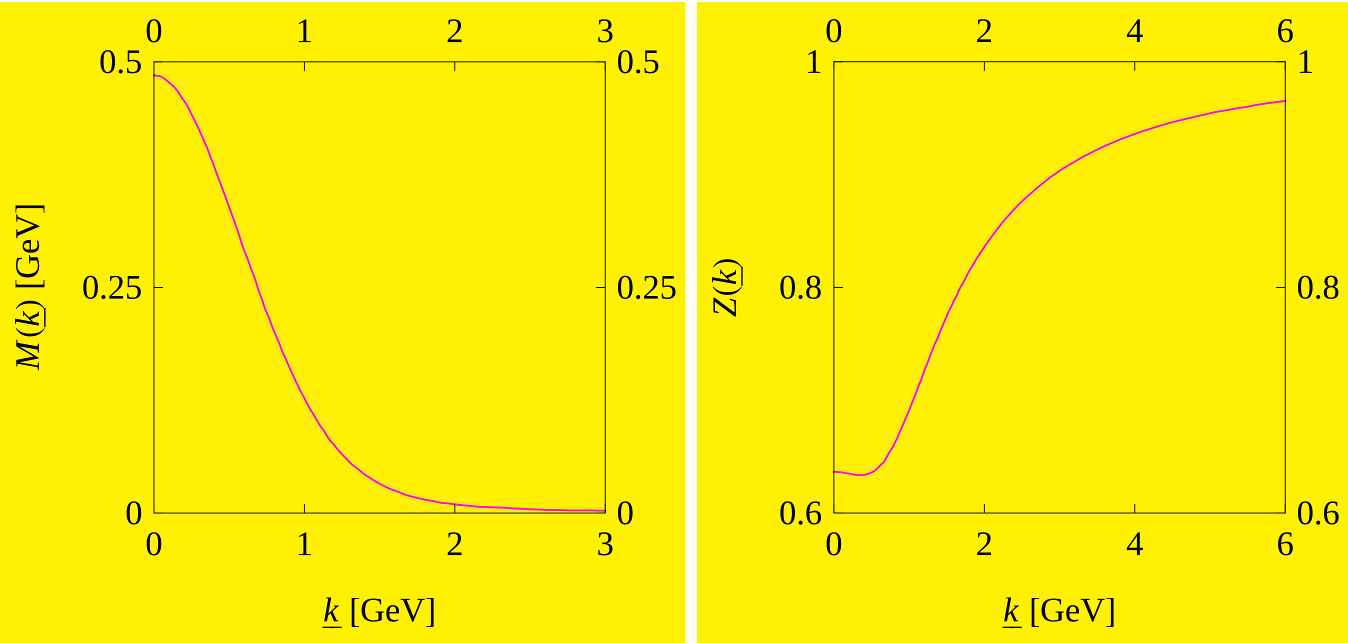
$$E(p)\,\varphi_2(p) + Z^2(p^2)\int_0^\infty \frac{dq\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = 0\ .$$

No longer omitting the impact of $Z(\underline{k}^2)$, the chiral-limit relation [3] between pseudoscalar-meson Bethe–Salpeter amplitude and quark propagator reads

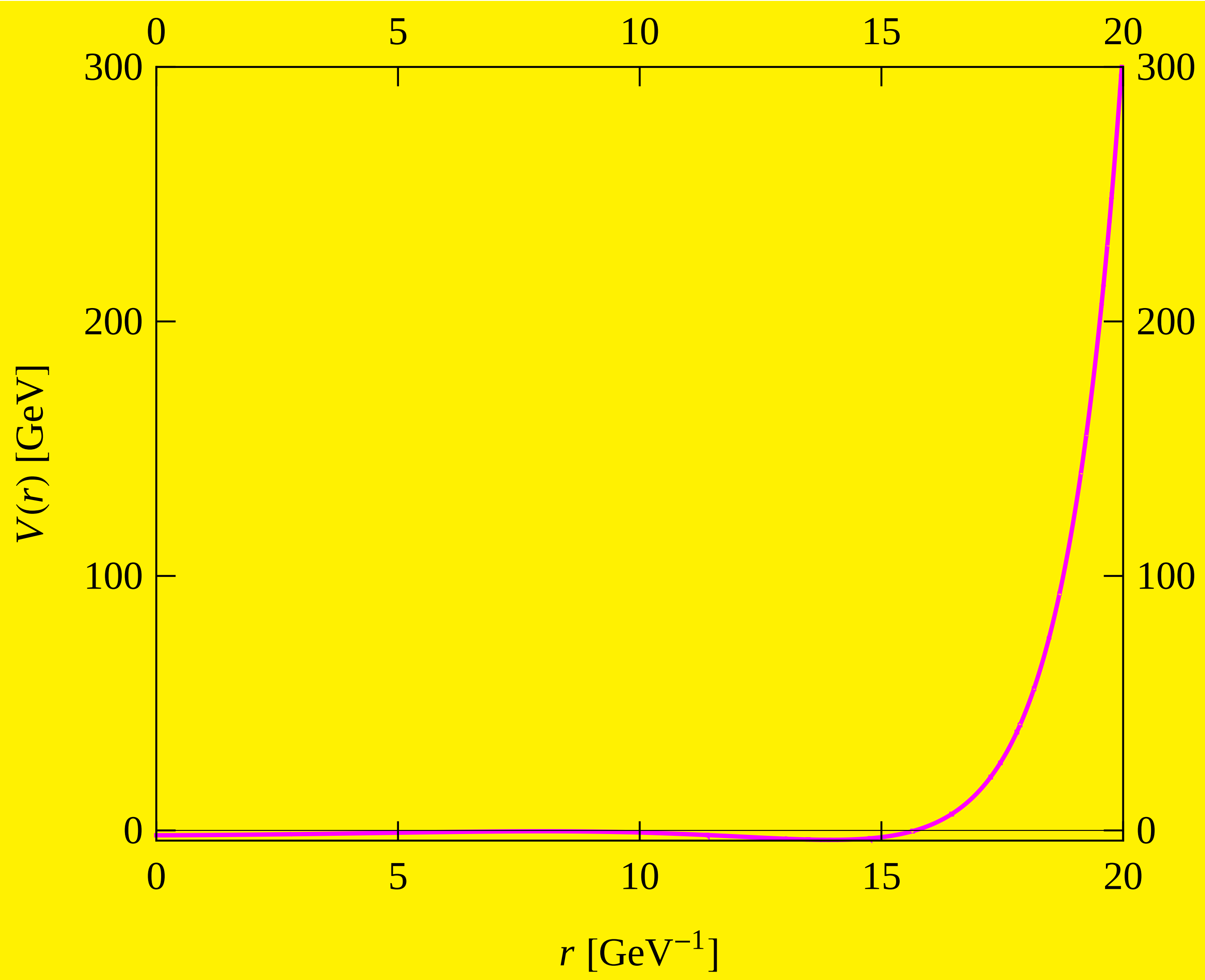
$$\Phi(\underline{k}) \propto \frac{Z(\underline{k}^2)\,M(\underline{k}^2)}{\underline{k}^2 + M^2(\underline{k}^2)}\,\underline{\gamma}_5 + \text{subleading contributions}\ .$$

By this relation, we derive $\Phi(\underline{k})$ and thereafter $\varphi_2(p)$ from a solution [11] for the quark propagator $S(\underline{k})$ for which $Z(\underline{k}^2)$ too is available in the literature.

Quark propagator functions $M(\underline{k})$ (left) and $Z(\underline{k})$ (right) vs. $\underline{k} \equiv \sqrt{\underline{k}^2}$ [11]:



The resulting interquark potential $V(r)$ must be extracted numerically [12], from the ratio of the Fourier transform of $E(p)\,\varphi_2(p)/Z^2(p^2)$ and $\varphi_2(r)$. So, of all insights clearly most essential is the square-well shape of the potential.



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