Abstract
We investigate the magnetohydrodynamics in the presence of an external magnetic field following the power-law decay in proper time and having spatial inhomogeneity characterized by a Gaussian distribution in one of the transverse coordinates under the Bjorken expansion. The leading-order solution is obtained in the weak-field approximation, where both energy density and fluid velocity are modified. It is found that the spatial gradient of the magnetic field results in transverse flow, where the flow direction depends on the decay exponents of the magnetic field. We suggest that such a magnetic-field-induced effect might influence anisotropic flow in heavy ion collisions.

Magnetohydrodynamics (MHD)

The ideal and magnetized fluid in the MDH:

\[ T^{\mu \nu} = (\epsilon + p + B^2)v^\mu v^\nu + (p + B^2)u^\mu u^\nu - B^\mu B^\nu, \]

\[ \eta_{\mu \nu} = \text{diag}(\epsilon, +, +, +). \]

Note \( T^{\mu \nu} = 0 \) given that \( \epsilon = 3 \rho \).

Conservation equations:

\[ u_\mu \nabla_\mu T^{\mu \nu} = -u_\mu \nabla_\mu (\epsilon + \frac{1}{2}B^2) = -u_\mu \nabla_\mu (\epsilon + \frac{1}{2}B^2) - u_\mu \nabla_\mu (B^\mu B^\nu) = 0, \]

\[ \Delta_{\mu \nu} \nabla_\mu T^{\mu \nu} = (\epsilon + p + B^2)(\nabla \cdot u) + \Delta_{\mu \nu} \nabla_\mu (B^\mu B^\nu) = 0. \]

We discard the induced E fields (assuming \( \epsilon \rightarrow \infty \)) and consider zero magnetic susceptibility.

Setup
We consider an “external” B field under the Bjorken expansion. We consider the weak-field approximation:

\[ u_\mu = (1, \lambda u_\lambda(\tau, x), 0, 0), \]

\[ \epsilon = \epsilon_0(\tau) + \lambda^2 \epsilon_1(\tau, x), \]

\[ B = B_0 B_\perp(\tau, x). \]

The expansion parameter \( \epsilon_0(\tau) \) rescales all spacetime coords. by \( \tau_0 \).

Time-dependent solutions
For \( B_\perp = B_0 \tau_0 \), \( u_\perp = 0 \) and one can directly solve for \( \epsilon(\tau) \) without making the approximation. S. Pu, V. Roy, L. Rezzolla, D. Rischke, Phys. Rev. D93 (2016) no.7, 074022

Conservation equations:

\[ \partial_\tau \epsilon + \frac{4}{3} \frac{\epsilon_0}{\tau} + B_0 \partial_\tau B_\perp + \frac{B_0^2}{\tau} = 0 \]

Power-law decay: \( B_\perp(\tau) = B_0 \tau^{-n/2} \) with \( n < 0 \).

Solutions:

\[ \epsilon(\tau) = \epsilon_0 \frac{3B_0^2(2+n)\tau^{-n}}{8 + 6n} \quad \text{for } n \neq -4/3 \]

\[ \epsilon(\tau) = \epsilon_0 \frac{B_0^2}{32} \log\frac{\tau}{\tau_0} \quad \text{for } n = -4/3 \]

When \( n = -2, \) \( \delta \epsilon(\tau) = 0 \) due to the “Frozen flux theorem”.

\[ \epsilon = \frac{1}{8} |B \cdot \nabla| u^\mu u^\nu \nabla \cdot B |. \]

The medium “does not feel” the presence of the B field.

Spatial dependence
In the presence of spatial inhomogeneity \( B_\perp(\tau, x) \sim x^{n/2} \), we seek for the perturbative solution up to \( O(\tau^2) \). S. Pu, V. Roy, L. Rezzolla, D. Rischke, Phys. Rev. D93 (2016) no.7, 074022

Conservation equations:

\[ \partial_\tau \epsilon_1 + \frac{4}{3} \frac{\epsilon_0}{\tau} + B_0 \partial_\tau B_\perp + \frac{B_0^2}{\tau} = 0, \]

\[ \partial_\tau \epsilon_2 - \frac{4}{3} \frac{\epsilon_0}{\tau} + 3 \frac{\epsilon_0}{\tau} B_\perp + 3 B_0 \partial_\tau B_\perp = 0. \]

A special case when \( n = -1 \): \( u_\perp(\tau, x) = 0 \) (an exact solution)

Perturbative solutions

Trick: we then approximate \( B_\perp(x)^2 \) by a Fourier series.

\[ B_\perp(x)^2 = |B_\perp(\tau, x)|^2 \approx \sum_n B_n^2 \cos(k_n x) \]

An ansatz:

\[ u_\perp(\tau, x) = \sum_n \beta_{n,0}(\tau) \cos(k_n x) + \beta_{n,1}(\tau) \sin(k_n x) \]

\[ (3\tau^2 - \tau \partial_\tau + k_n^2 + 1) \beta_{n,0}(\tau) + \frac{3B_n^2}{4\tau} k_n^2 (n + 1)^n = 0 \]

Solve for \( \beta_{n,1}(\tau) \):

Perturbative solutions are complicated but analytic.

We fix the integrational constants by asymptotic solutions in late times.

\[ u_\perp(\tau, x) = \sum_n B_n^2 \cos(k_n x) \]

\[ \epsilon_n(\tau) = -\frac{3B_n^2}{2} \frac{(n+1)^{n-3/2}}{n} \]

Properties of solutions

A Gaussian distribution:

\[ B_\perp(x)^2 \sim e^{-2x^2/\sigma^2} \]

The Fourier series:

\[ B_n^2 \sim \frac{1}{2\pi} \int e^{-x^2/\sigma^2} \cos(k_n x) dx \]

Transverse flow moving inward \( n < -1 \):

\[ B_n^2 \sim e^{-x^2/\sigma^2} \]

Transverse flow moving outward \( n > -1 \):

\[ B_n^2 \sim e^{-x^2/\sigma^2} \]

Conservation of the magnetic flux:

\[ n \neq -1(\tau \rightarrow 0) \rightarrow \text{A static magnetic field } B_\perp(\tau, x) \text{ with longitudinal expansion.} \]

\[ \text{Magnetic flux increases with respect to time.} \]

\[ \text{To reduce the flux, the medium expands transversely.} \]

\[ n < -1 \rightarrow \text{A time-decreasing magnetic field } B_\perp(\tau, x) \text{ in a static medium} \]

\[ \text{Magnetic flux decreases with respect to time.} \]

\[ \text{To increase the flux, the medium is compressed transversely.} \]

\[ n = -1 \rightarrow \text{The decreasing B field compensates the expansion of the medium.} \]