# Generalized dispersion relations for unphysical particles with complex masses


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XIIth Quark Confinement and the Hadron Spectrum
Thessaloniki, 29 August - 3 September 2016

# Abstract
## More details in arXiv:1606.03769

A generalized dispersion relation is discussed for confined degrees of freedom that are not present in the physical spectra but can give rise to observable bound states. The propagator of the unphysical particles can have complex poles and cannot be reconstructed from the knowledge of the imaginary part.
Under reasonable assumptions the missing piece of information is shown to be in the rational function that contains the poles and must be added to the integral representation.

For pure Yang-Mills theory, the rational part and the spectral term can be sorted out by the explicit analytical expressions of the one-loop massive expansion, where a massive gluon propagator is inserted in the loops. The spectral function turns out to be very small and, from first principles, the simple rational part provides an approximate propagator that is equivalent to the tree-level result of simple phenomenological models like the refined Gribov-Zwanziger model.

# 1. Introduction

In many interacting theories and, notably, in non-Abelian gauge theories, some of the quantum fields describe *confined* particles that are not present in real spectra. They can be regarded as *unphysical* internal degrees of freedom. We are interested in the *physical* class of unphysical particles that give rise to observable bound states[1, 2, 3], like gluons and quarks.

If the particle does not appear in the spectra, the general Källen-Lehmann representation does not hold and the whole existence of a spectral representation can be questioned. The usual dispersion relation between real and imaginary part of the propagator does not hold for a generic unphysical particle.

Under general physical assumptions, an extension of the standard dispersion relation can be proven, with a rational part that replaces the discrete spectral term of the physical particles. The generalized dispersion relation might be useful for the study of physical bound states. Several model propagators, like that emerging from the refined Gribov-Zwanziger[4] and replica[5] models, are immediately shown to be equivalent to the most general propagator at tree level.

# 2. General Assumptions (from physical motivations)

- Assume that a generally analytic function $G(z)$ does exist, real and regular on the negative real axis $p^2 = -p_E^2 < 0$ where $\mathrm{Im}\, G(-p_E^2) = 0$ and that $G(-p_E^2)$ reproduces the numerical data points for the propagator in the Euclidean space.

- Assume that the most general propagator $G(z)$ of an unphysical particle might have any number of simple poles in the complex plane but no other branch cuts except for $p^2 > \theta \geq 0$ on the real axis.

- Assume that the function $G(z)$ goes to zero fast enough in the limit $z \to \infty$, say at least like $1/z$, as satisfied by all the numerical data in the Euclidean space.

# . . . and the physical motivations

Bound states of unphysical particles arise in two-point correlators of composite operators that can be built[1, 2, 3] by Feynman graphs in terms of the elementary propagator $G(z)$.
The physical two-point correlators that describe observable bound states must satisfy the general Källen-Lehmann dispersion relations and must be analytic out of the real axis, with a single branch cut along the positive real axis. That would pose a limit to the singularities of $G(z)$: we can hardly believe that entire branch cuts can totally disappear in the correlator of the composite operators, without leaving any unphysical feature.
On the other hand, it has been shown[1] that even when G(z) has complex poles at $z = m_\pm^2 = \alpha \pm i\beta$, the poles might combine in the composite correlator giving a physical branch cut along the real axis and a real multi-particle threshold at

$$\theta' = (m_+ + m_-)^2 = 2\left[\alpha + \sqrt{\alpha^2 + \beta^2}\right] \tag{1}$$

as if they were real mass poles. Thus we cannot exclude the presence of simple poles everywhere in the complex plane.

# 3. Immediate consequences

- Being real on the negative real axis, the function $G(z)$ must satisfy

$$G(z)^\star = G(z^\star) \tag{2}$$

so that, in the limit $\epsilon \to 0$, for any $p^2 > 0$

$$\begin{aligned} \mathrm{Re}\, G(p^2 + i\epsilon) &= \mathrm{Re}\, G(p^2 - i\epsilon) \\ \mathrm{Im}\, G(p^2 + i\epsilon) &= -\,\mathrm{Im}\, G(p^2 - i\epsilon) \end{aligned} \tag{3}$$

and the imaginary part must be discontinuous across the real axis unless it is exactly zero. If $\mathrm{Im}\, G(p^2 \pm i\epsilon) = 0$ below a generic threshold $p^2 < \theta$, then $G(z)$ must have a branch cut along the positive real axis for $p^2 > \theta \geq 0$.

- Because of Eq.(2), out of the real axis the poles can only occur in pairs of complex conjugated points, while single poles might occur on the real axis below the threshold $\theta$.

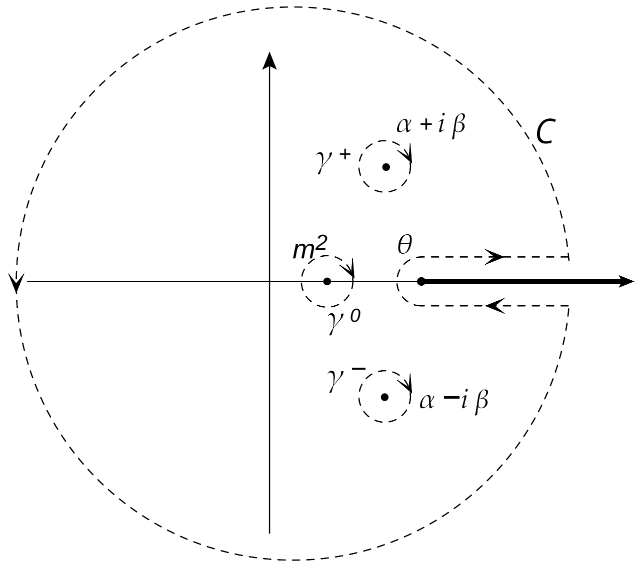
- A generalized dispersion relation can be proven by Cauchy formula, with a rational part that generalizes the standard discrete term.

# 4. Formal Proof

By Cauchy formula

$$G(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{G(\omega)}{\omega - z} \, \mathrm{d}\omega \tag{4}$$

where $\Gamma = C \cup \gamma_i$. The circles $\gamma_i$ are around the simple poles at the complex conjugated points $z_i = m_{\pm}^2 = \alpha_i \pm i\beta_i$ and on the positive real axis at $z_i = m^2 > 0$ (just three of them are displayed). Since $G(\omega)/(z - \omega) \sim 1/\omega^2$ in the limit $\omega \to \infty$, we can neglect the contribution of the external circle $C$ and by the residue theorem




$$\boxed{G(z) = \sum_i \frac{R_i}{z - z_i} + \int_{\theta}^{+\infty} \frac{\rho_c(\omega)}{z - \omega} \, \mathrm{d}\omega} \tag{5}$$

where $R_i$ is the residue of $G(z)$ in $z_i$ and $\rho_c(\omega)$ is the ususal continuos part of the spectral function $\rho_c(\omega) = -\frac{1}{\pi} \, \mathrm{Im}\, G(\omega + i\epsilon)$, with $\omega > \theta$.

# 5. Two very special cases

- Single pole on the real axis: the standard dispersion relation is recovered. Incorporating the discrete term in a total spectral function $\rho(\omega) = -\frac{1}{\pi}\,\mathrm{Im}\, G(\omega + i\epsilon) = R\,\delta(\omega - m^2) + \rho_c(\omega)$ the general Eq.(5) reads

$$\mathrm{Re}\, G(p^2) = \int_0^{+\infty} \frac{\rho(\omega)}{p^2 - \omega}\, \mathrm{d}\omega \tag{6}$$

- Single pair of complex conjugated poles at $z = m_\pm^2 = \alpha \pm i\beta$ and no real pole: the rational part in Eq.(5) reads

$$G^R(z) = \frac{R}{z - \alpha - i\beta} + \frac{R^\star}{z - \alpha + i\beta} \tag{7}$$

  and is analytic on the real axis where $\mathrm{Im}\, G^R = 0$. On the real axis the imaginary part of $G(z)$ does not know anything about the rational part $G^R$ and the poles. The real part cannot be reconstructed from the imaginary part and $G^R$ must be added to the integral as in Eq.(5).

In the last case, along the real axis we can write

$$\boxed{\mathrm{Re}\, G(p^2) = G^R(p^2) + \delta G(p^2)} \tag{8}$$

where the subtracted function $\delta G = G - G^R$ has no poles and is entirely determined by its imaginary part through the standard dispersion relation

$$\begin{aligned} \mathrm{Re}\, \delta G(p^2) &= PV \int_0^{+\infty} \frac{\rho(\omega)}{p^2 - \omega} \mathrm{d}\omega \\ \rho(\omega) &= -\frac{1}{\pi} \mathrm{Im}\, \delta G(\omega + i\epsilon) = \rho_c(\omega). \end{aligned} \tag{9}$$

We may argue that, if the multiparticle continuous contribution $\delta G$ is small, the whole propagator could be approximated by the rational part $G(p^2) \approx G^R(p^2)$ that would capture the main physical properties. That would be equivalent to the usual approximation $G(p^2) \approx R/(p^2 - m^2)$ for a physical particle, when only the discrete part of the spectral function is retained in Eq.(6).

# 6. The Gluon propagator at tree-level

Complex poles are found in the gluon propagator $\Delta(p)$ from first principles by a massive expansion[6, 7] (Talk on Thursday 2 September - Parallel A, 18:20) and were predicted by the refined version of the Gribov-Zwanziger[4] model. Denoting by $\Delta^R(p)$ the rational part, if the multiparticle term $\delta\Delta = \Delta - \Delta^R$ is small, in the Euclidean space, Eq.(8) reads

$$\Delta(-p_E^2) \approx \Delta^R(-p_E^2) = -N\left[\frac{p_E^2 + (\alpha + t\beta)}{p_E^4 + 2\alpha p_E^2 + (\alpha^2 + \beta^2)}\right]. \tag{10}$$

where $N$ is a renormalization constant and $t = (\mathrm{Im}\, R)/(\mathrm{Re}\, R) = \tan[\arg(R)]$.
That would just be a re-parametrization of the refined Gribov-Zwanziger propagator[4]
Thus, the propagator arising from that model can be seen as the most general tree-level propagator with a pair of complex conjugated poles and its phenomenological success seems to be related to the relatively small weight of the continuos term in the general dispersion relation of Eq.(5).

Moreover, the dispersion relations Eqs.(8),(9) can be checked by the explicit analytical one-loop result arising for $\Delta(p)$ from the massive expansion[6, 7], predicting complex masses $m_\pm^2 = (\alpha, \pm\beta) = (0.163, \pm 0.602)$ GeV$^2$ with $t = \pm 3.17$. As expected, the continuous multiparticle term $\delta\Delta$ is very small.

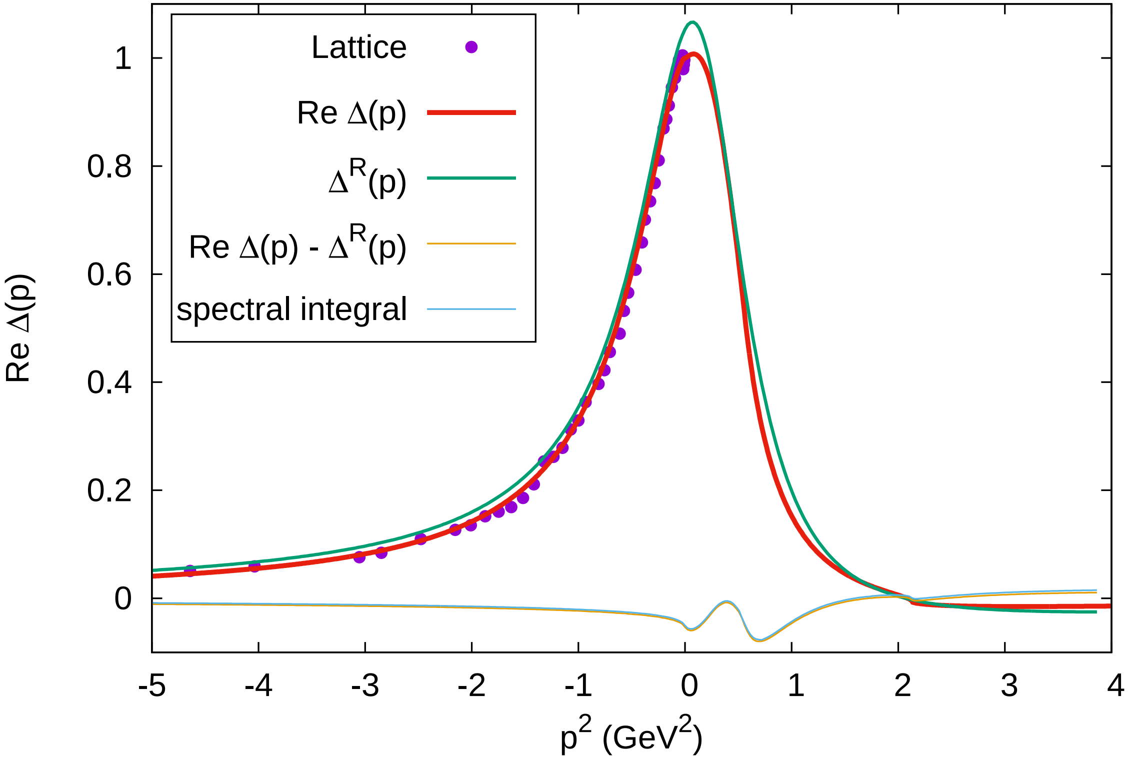
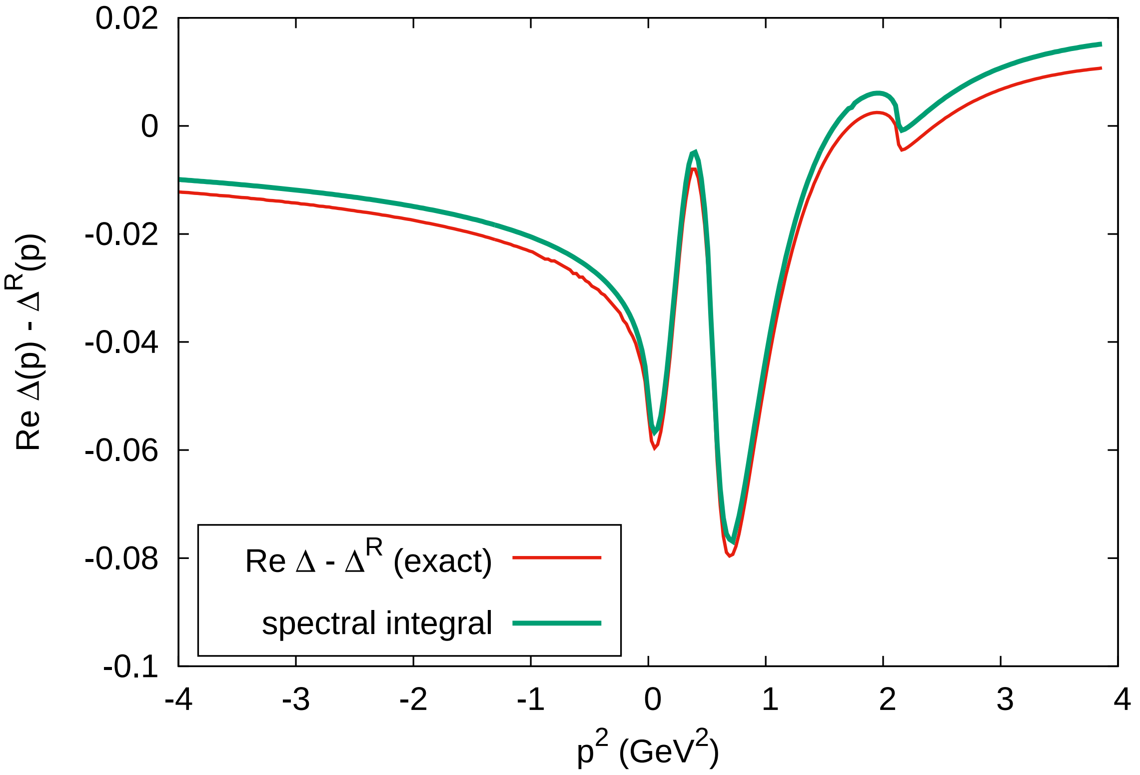



Figure 1. The real part of the one-loop gluon propagator $\Delta(p)$ (red line) is evaluated for $SU(3)$ pure Yang-Mills theory by the optimized massive expansion of Ref.[6] and analytically continued to Minkowski space as discussed in Ref.[7]. The points are the lattice data of Ref.[8]. The green line is the rational part $\Delta^R$ by Eq.(10).

At the bottom, the subtracted function $\delta\Delta = \mathrm{Re}\,\Delta - \Delta^R$ is reported twice, as arising from the exact difference of the analytical expressions and from the numerical integral of the spectral function in Eq.(9). The two curves are not distinguishable at the scale of the figure.

Figure 2. The small term $\delta\Delta = \mathrm{Re}\,\Delta - \Delta^R$ is shown in more detail. The red line is the exact difference between the analytical expressions. The green line is a numerical reconstruction by the spectral integral in Eq.(9). The slight difference arises because of the numerical accuracy and of the finite cut-off in the integral.




Even if $\delta\Delta$ has a rich behaviour that arises from the multiparticle continuous spectral function, its absolute value is very small and the total propagator seems to be very well described by the simple rational part.

Higher-order corrections are probably still smaller, enforcing the idea that the one-loop propagators of the massive expansion[6, 7] provide a very good approximation in the whole complex plane.

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