Probing QCD perturbation theory at high energies with continuum extrapolated lattice data

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Wanted: QCD observables, O, which ...

- are gauge invariant & non-perturbatively defined through the (Euclidean) QCD path integral:

$$\langle O \rangle = \mathcal{Z}^{-1} \int D[A,\psi,\overline{\psi}] O[A,\psi,\overline{\psi}] \exp\{-S\}$$

- depend on a single scale $\mu = 1/L$, with L^4 the space-time volume. Other dimensionful parameters (momenta, distances,..) are scaled with L or set to zero (quark masses);
- can be expanded perturbatively in $\alpha_s(\mu) = \bar{g}^2(L)/(4\pi)$:

$$\langle O \rangle = c_0 + c_1 \alpha_s(\mu) + c_2 \alpha_s^2(\mu) + \ldots$$

$\Rightarrow$ give rise to *non-perturbatively defined couplings*:

$$\alpha_O(\mu) \stackrel{\text{def}}{=} \frac{\langle O \rangle - c_0}{c_1} = \alpha_s(\mu) + c_1' \alpha_s^2(\mu) + c_2' \alpha_s^3(\mu) + \ldots$$

Example: a family of SF couplings

- Dirichlet b.c.'s in Euclidean time, abelian boundary values C_k, C'_k:

$$A_k(x)|_{x_0=0} = C_k(\eta,\nu), \qquad A_k(x)|_{x_0=L} = C'_k(\eta,\nu)$$

$\Rightarrow$ induce family of abelian, spatially constant background fields B_μ with parameters η, ν ($\rightarrow$ 2 abelian generators of SU(3)):

$$B_k(x) = C_k(\eta,\nu) + \frac{x_0}{L}\left(C'_k(\eta,\nu) - C_k(\eta,\nu)\right), \qquad B_0 = 0.$$

- Induced background field is unique up to gauge equivalence
- Effective action

$$\mathrm{e}^{-\Gamma[B]} = \int D[A,\psi,\overline{\psi}]\mathrm{e}^{-S[A,\psi,\overline{\psi}]}, \qquad \Gamma[B] = \tfrac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + \mathrm{O}(g_0^2)$$

- Define family of SF couplings, parameter ν:

$$\frac{1}{\bar{g}_\nu^2(L)} \overset{\text{def}}{=} \left.\frac{\partial_\eta \Gamma[B]}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0} = \left.\frac{\langle \partial_\eta S\rangle}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0} = \frac{1}{\bar{g}^2(L)} - \nu\bar{v}(L)$$

$\Rightarrow$ response of the system to a change of a colour electric background field. [Narayanan et al. '92]

Testing perturbation theory: use the Λ-parameter I

- Non-perturbatively defined coupling $\bar{g}^2(L)$ implies non-perturbative definition of β-function:

$$\beta(\bar{g}) \stackrel{\text{def}}{=} -L\frac{\partial \bar{g}(L)}{\partial L}, \qquad \beta(g) = -b_0 g^3 - b_1 g^5 + \ldots$$

with universal coefficients b_0, b_1 (i.e. b_k, $k \geq 2$ scheme dependent)

$$b_0 = (11 - \tfrac{2}{3}N_\mathrm{f})/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}N_\mathrm{f})/(4\pi)^4 \,.$$

- *Exact* solution of Callan-Symanzik equation $[L\partial/\partial L - \beta(\bar{g})\partial/\partial \bar{g}]\, L\Lambda = 0$

$$\begin{aligned} L\Lambda &= \varphi\left(\bar{g}(L)\right) \\ \varphi\left(\bar{g}\right) &= \left[b_0\bar{g}^2\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2}} \exp\left\{-\int_0^{\bar{g}} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\} \end{aligned}$$

- Scheme dependence of Λ almost trivial:

$$g_\mathrm{X}^2(\mu) = g_\mathrm{Y}^2(\mu) + c_\mathrm{XY} g_\mathrm{Y}^4(\mu) + \ldots \quad \Rightarrow \quad \frac{\Lambda_\mathrm{X}}{\Lambda_\mathrm{Y}} = \mathrm{e}^{c_\mathrm{XY}/2b_0}$$

$\Rightarrow$ use $\Lambda = \Lambda_{\mathsf{SF},\nu=0}$ as reference.

Testing perturbation theory: use the Λ-parameter II

- Introduce a reference scale $1/L_0$ through:

$$\bar{g}^2(L_0) = 2.012 \quad \Rightarrow \quad \frac{1}{\bar{g}_\nu^2(L_0)} = \frac{1}{2.012} - \nu \times \bar{v}(L_0) \quad \text{(s. below)}$$

- Consider

$$L_0\Lambda = \underbrace{L_0/L}_{\text{known}} \quad \times \quad \underbrace{\Lambda/\Lambda_\nu}_{\exp(-\nu\times 1.25516)} \quad \times \quad \varphi\left(\bar{g}_\nu(L)\right)$$

- Non-perturbative results for $1/L_0 \leq \mu \leq 1/L$ (s. below)
- Perturbation theory for $\mu > 1/L$ by replacing $\beta_\nu(g) \rightarrow \beta_{\nu,\text{3-loop}}(g)$ in:

$$\varphi\left(\bar{g}_\nu(L)\right) \quad \propto \quad \exp\left\{-\int_0^{\bar{g}_\nu(L)} dg \left[\frac{1}{\beta_\nu(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

$$\beta_{\nu,\text{3-loop}}(g) \quad = \quad -b_0 g^3 - b_1 g^5 - b_{2,\nu}\, g^7,$$

$$b_{2,\nu} \quad = \quad (-0.06(3) - \nu \times 1.26)/(4\pi)^3$$ [Bode, Weisz, Wolff '99]

- N.B.: $L_0\Lambda$ must be independent of L and $\nu \Rightarrow$ excellent test of PT!

Non-perturbative running in steps and determination of L_0/L

Vary scale by factor 2, define step-scaling function [Lüscher, Weisz, Wolff '91]:

$$\sigma(u) = \bar{g}^2(2L)\Big|_{u=\bar{g}^2(L)}\,,$$

- Connection to β-function:

$$\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{dg}{\beta(g)} = -\ln 2$$

- $\sigma(u)$ can be constructed as the continuum limit of lattice approximants (s. below)
- Once $\sigma(u)$ is available for a range of values $u \in [u_{\text{min}}, u_0]$

$\Rightarrow$ iteratively step up the energy scale:

$$u_0 = \bar{g}^2(L_0), \quad u_n = \sigma(u_{n+1}) = \bar{g}^2(L_n) = \bar{g}^2(2^{-n}L_0), \quad n = 0, 1, \ldots$$

$\Rightarrow$ scale ratios are $L_0/L_n = 2^n$, where n is the number of steps.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (defines value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

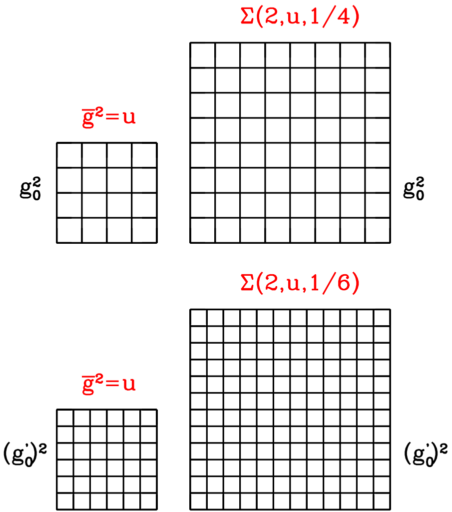
$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- ...

$$\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$$

.

- change u and repeat...

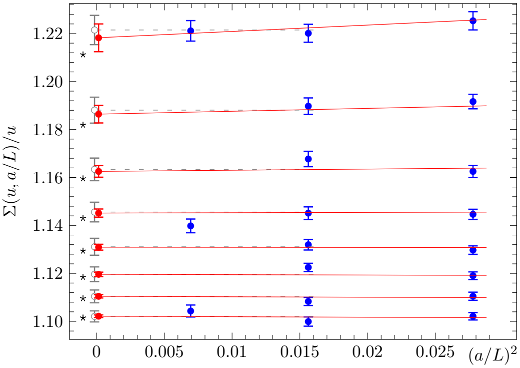


Continuum extrapolation of $\Sigma(u, a/L)$

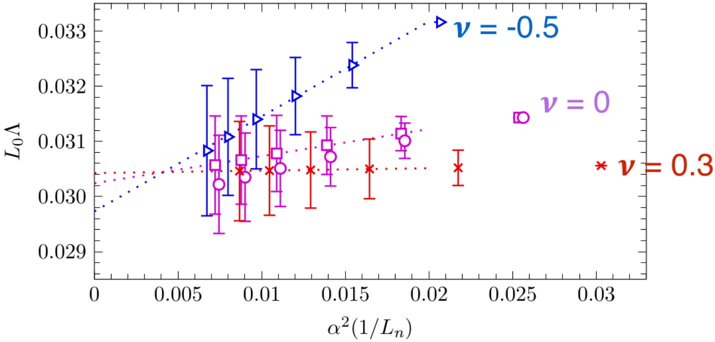
Example for global fit ansatz:

$$\Sigma(u, a/L) = u + s_0 u^2 + s_1 u^3 + c_1 u^4 + c_2 u^5 + \rho_1 u^4 \frac{a^2}{L^2} + \rho_2 u^5 \frac{a^2}{L^2}$$

- s_0, s_1 fixed to perturbative values: $s_0 = 2b_0 \ln 2, \quad s_1 = s_0^2 + 2b_1 \ln 2$
- 4 parameters: c_1, c_2, ρ_1, ρ_2; 19 data points, χ^2/d.o.f. ≈ 1



Result for $L_0\Lambda$

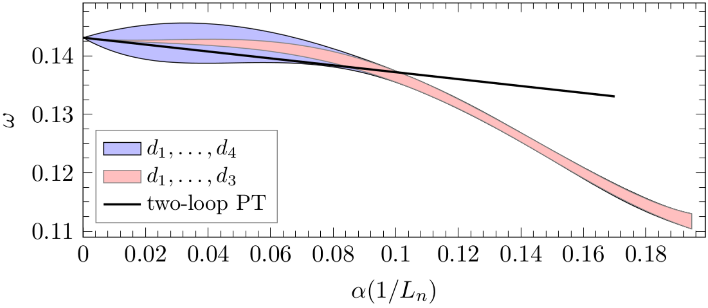


- All results agree at $\alpha = 0.1$, we quote

$$L_0\Lambda = 0.0303(8) \qquad \text{error} < 3\% \;!$$

- For $\nu = 0.3$ this result could be inferred from larger values of α, but not for $\nu = -0.5$!

Continuum results $\bar{v} = \omega(u) = v_1 + u \times v_2 + \ldots$



- Continuum extrapolation analogous to $\sigma(u)$
- $L_0\Lambda$ calculation for $\nu \neq 0$ requires $\bar{v}(L_0) = \omega(2.012) = 0.1199(10)$ $(u = 2.012 \Leftrightarrow \alpha = 0.16)$
- Observe large deviation from perturbation theory at $\alpha = 0.19$:

$$\left(\omega(\bar{g}^2) - v_1 - v_2\bar{g}^2\right)/v_1 = -3.7(2)\alpha^2$$

Conclusions

SF$_\nu$ schemes have very advantageous properties for perturbative expansion:

1. Euclidean, finite space-time volume $\Rightarrow$ no renormalon issues;
2. perturbative coefficients of β-function well-behaved for any choice $\nu = \mathrm{O}(1)$;
3. large gap to secondary minimum of the action: $\Delta S = 10\pi^2/3 \Rightarrow$ negligible perturbative effect on observables for $\alpha < 0.2$.

Stringent test of PT by studying remnant L, ν-dependence of $L_0\Lambda$ for $1/L_0 \approx 4\,\mathrm{GeV} < 1/L < 128\,\mathrm{GeV}$

- 3% accuracy for Λ can be safely quoted provided $\alpha = 0.1$ is reached:

$$L_0\Lambda = 0.0303(8) \quad \Rightarrow \quad L_0\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = 0.0791(21)$$

- For $\nu = -0.5$ uncertainties much larger if data limited to $\alpha > 0.15$

$\Rightarrow$ perturbative uncertainties may be to blame for inconsistent α_s determinations.

- For $\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3}$ in physical units and an estimate of $\alpha_s(m_Z)$
  cf. next talk by Mattia Dalla Brida!
- For running quark mass results cf. talk by Patrick Fritzsch on August 29th.