A precise determination of the $N_\mathrm{f} = 3$ QCD Λ-parameter from lattice QCD

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DESY

in collaboration with:

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ALPHA Collaboration

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Introduction

Where do we stand?

High-energy: (S. Sint's talk) (MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '16a)

- The running of $\overline{g}^2_{\rm SF}(L)$ allowed the precise determination

$$L_0\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3} = 0.0791(21) \quad \text{with} \quad \overline{g}^2_{\rm SF}(L_0) = 2.012$$

- To be useful L_0 must be related to some physical quantity

$$f_\pi,\, f_K,\, m_p,\, \ldots$$

- Reaching low-energies is difficult following the same strategy
  - $\mathrm{var}(\overline{g}^2_{\rm SF})\,/\,\overline{g}^4_{\rm SF} \propto \overline{g}^4_{\rm SF} \Rightarrow$ expensive for large couplings!
  - $\mathrm{var}(\overline{g}^2_{\rm SF})$ is large in general

Question: Can we do better than simply brute force?

Introduction

The gradient flow coupling

Solution: We introduce a new coupling definition:[1] (Lüscher '10; Fritzsch, Ramos '13)

1. Finite volume with Schrödinger functional (SF) bc.'s
2. $m_{u,d,s}(L) = 0 \rightarrow$ mass-independent scheme
3. $\bar{g}^2_{\mathrm{GF}}(L) \propto t^2 \langle \mathrm{tr}\{G_{\mu\nu}(t,x)G_{\mu\nu}(t,x)\}\rangle\big|_{\sqrt{8t}=0.3\times L}$

Gradient flow (GF): (Lüscher '10)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x), \quad B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$

- Gauge invariant fields are finite (Lüscher, Weisz '11)
- Simple to evaluate in simulations
- $\mathrm{var}(\bar{g}^2_{\mathrm{GF}})$ is small, and $\mathrm{var}(\bar{g}^2_{\mathrm{GF}}) \,/\, \bar{g}^4_{\mathrm{GF}} \propto \mathrm{const.}$

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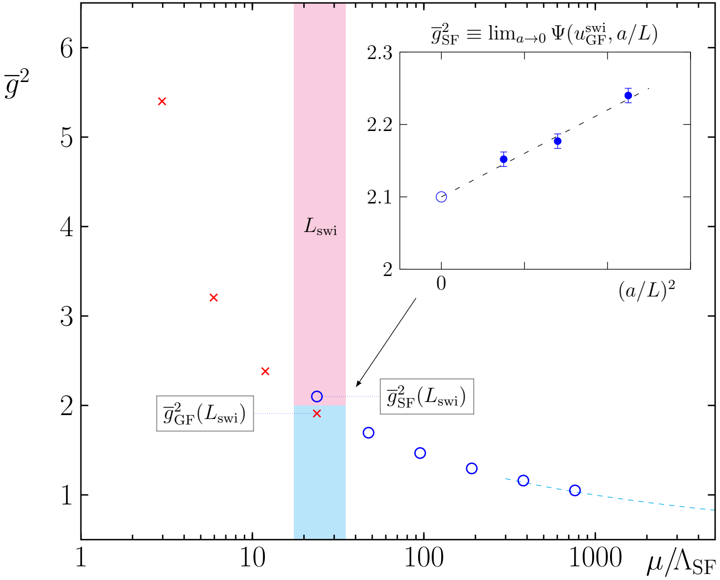
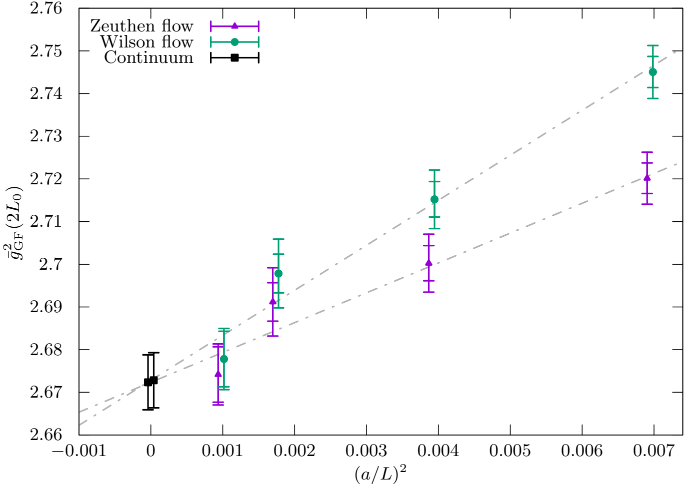
[1] In fact, we need something a little more complicated (MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '16b)

The strategy in a picture

(Fritzsch, Lattice '14)



Step I: matching the couplings



$$\bar{g}^2_{\mathrm{SF}}(L_0) = 2.012 \quad \Rightarrow \quad \bar{g}^2_{\mathrm{GF}}(2L_0) = 2.6723(64)$$

Step II: running to low-energy

How do we do it?

(MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '16b)

Step scaling function:

(Lüscher, Weisz, Wolff '91)

$$\sigma(u) = \lim_{a/L\to 0} \Sigma(u, a/L), \qquad \Sigma(u, a/L) = \bar{g}^2_{\mathrm{GF}}(2L)\Big|^{\bar{g}^2_{\mathrm{GF}}(L)=u}_{m(L)=0}$$

β-function:

$$\log 2 = -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{\mathrm{d}x}{\beta(x)}, \qquad \beta(\bar{g}_{\mathrm{GF}}) = -L\frac{\partial \bar{g}_{\mathrm{GF}}(L)}{\partial L}$$

Ratios of scales:

$$\frac{L_2}{L_1} = \exp\left\{-\int_{\bar{g}_{\mathrm{GF}}(L_1)}^{\bar{g}_{\mathrm{GF}}(L_2)} \frac{\mathrm{d}x}{\beta(x)}\right\}$$

Step II: running to low-energy

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Ratios of scales:

$$\frac{L_{\mathrm{had}}}{L_0} = 2\times\exp\left\{-\int_{\bar{g}_{\mathrm{GF}}(2L_0)}^{\bar{g}_{\mathrm{GF}}(L_{\mathrm{had}})} \frac{\mathrm{d}x}{\beta(x)}\right\} = 21.86(42)$$

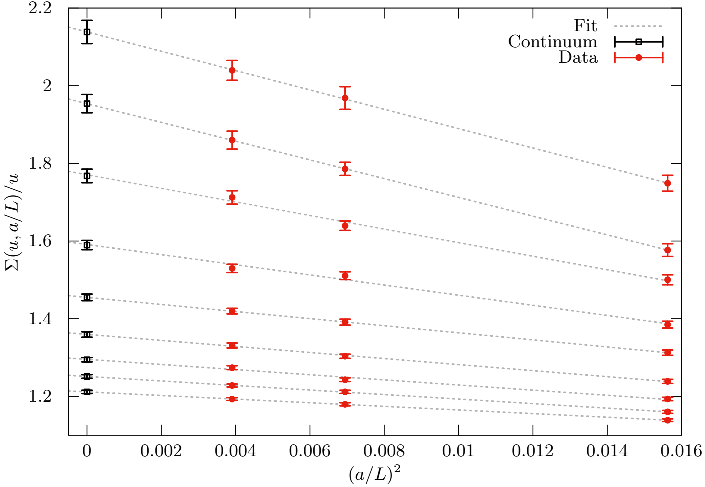
$$\bar{g}^2_{\mathrm{GF}}(L_{\mathrm{had}}) = 11.31 \quad \Rightarrow \quad L_{\mathrm{had}} \approx 200\,\mathrm{MeV}$$

NOTE: L_{had} can safely be connected to hadronic physics (**Step III**)!

Step II: running to low-energy

Taking the continuum limit

(MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '16b)

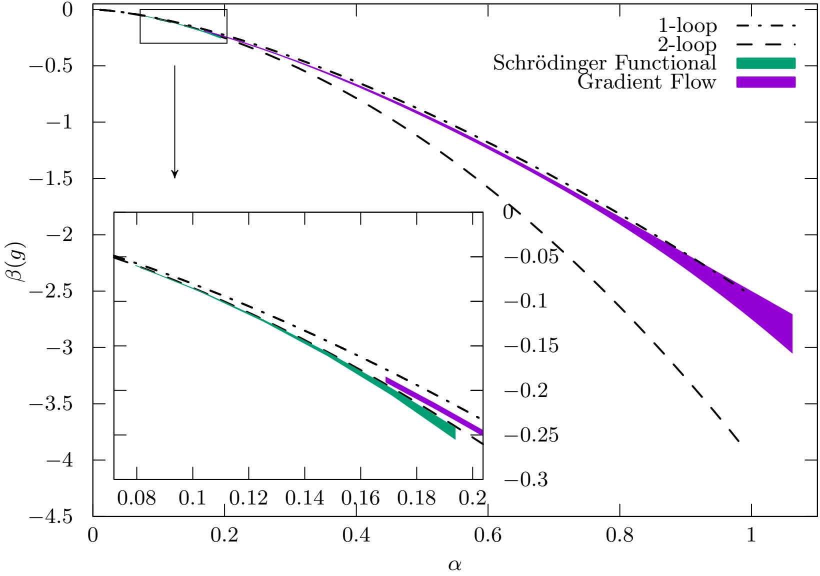


- sizable discretization effects → **careful** extrapolations are needed!
- continuum results are nonetheless very **precise**!

Step II: jogging to low-energy

The β-function(s)

$\alpha \equiv g^2/(4\pi)$



Step III: matching to hadronic physics

Setting the scale

The story so far:

$$L_{\rm had}\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3} = L_0\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3} \times \frac{L_{\rm had}}{L_0} = 1.729(57) \;\Leftarrow\; L_{\rm had} = ???\,{\rm fm}$$

A (relative) scale: (Lüscher '10)

$$t_0^2\,\langle\,{\rm tr}\{G_{\mu\nu}(t_0,x)G_{\mu\nu}(t_0,x)\}\rangle = 0.3$$

- ✓ simply and accurately measurable in simulations
- ✓ gluonic quantity $\to$ very mild m_π dependence
- ✗ not directly measurable in experiments

Strategy: (Bruno, Korzec, Schaefer '16)

$$\left.\begin{array}{c}\text{CLS effort + PDG}\\ m_\pi, m_K, f_\pi, f_K\end{array}\right\} \Rightarrow \lim_{\substack{m_{\pi,K}\to m_{\pi,K}^{\rm phys}\\ a\to 0}} \sqrt{t_0}\cdot f_{\pi K}, \quad f_{\pi K} = \tfrac{2}{3}(f_K + \tfrac{1}{2}f_\pi)$$

Result: $\sqrt{8\,t_0} = 0.413(4)\,{\rm fm}$ $\to$ very **precise** relative scale!

Step III: matching to hadronic physics

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Result: $\sqrt{8\,t_0^{\rm sym}} = 0.413(5)\,{\rm fm}$, with $m_\pi^{\rm sym} = m_K^{\rm sym} \approx 400\,{\rm MeV}$

Step III: matching to hadronic physics

The CLS $N_\mathrm{f} = 2 + 1$ ensembles (Bruno *et. al.* '14)

id	$a\,[\mathrm{fm}]$	L/a	T/a	m_π[MeV]	m_K[MeV]	$m_\pi L$
H101	≈ 0.086	32	96	420	420	5.8
H102		32	96	350	440	4.9
H105		32	96	280	460	3.9
C101		48	96	220	470	4.7
H400	≈ 0.076	32	96	420	420	5.2
H401		32	96	550	550	7.3
H402		32	96	450	450	5.7
H200	≈ 0.064	32	96	420	420	4.3
N202		48	128	420	420	6.5
N203		48	128	340	440	5.4
N200		48	128	280	460	4.4
D200		64	128	200	480	4.2
N300	≈ 0.050	48	128	420	420	5.1
J303		64	192	260	470	4.1

- state of the art lattice simulations: O(a) improved Wilson-fermions, open bc.'s, twisted-mass reweighting, …

Step III: matching to hadronic physics

The Λ-parameter

Relative scale:

$6/g_0^2$	t_0^{sym}/a^2	a [fm]
3.40	2.860(10)	≈ 0.086
3.46	3.659(15)	≈ 0.076
3.55	5.164(17)	≈ 0.064
3.70	8.595(29)	≈ 0.050
3.85*	13.8(3)	≈ 0.040

* Very **preliminary**: used simply for illustration

Coupling:

$$\bar{g}^2_{\mathrm{GF}}(L_{\mathrm{had}}) = 11.31 \Rightarrow (L_{\mathrm{had}}/a)(g_0)$$

Note: (Lüscher, Sint, Sommer, Weisz '96; Sint, Sommer '96)

- $m(L) = 0$
- $g_0^2 \to \tilde{g}_0^2 = g_0^2(1 + b_g(g_0)am_q)$
  - In PTh, $b_g^{N_\mathrm{f}=3}(g_0) \approx 0.04\, g_0^2$
  - $(am_q)^{\mathrm{CLS}} \lesssim 5 \times 10^{-3}$

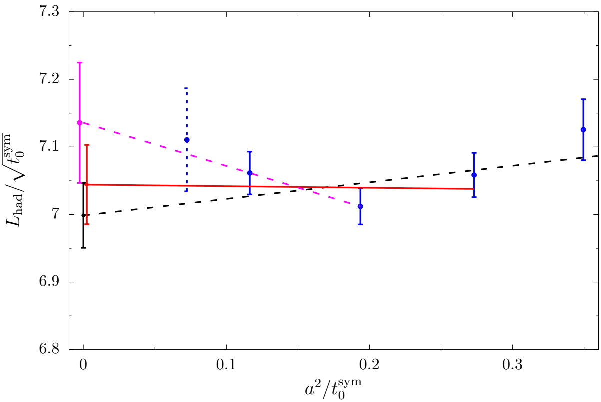
Λ-parameter:

$$\Lambda^{N_\mathrm{f}=3}_{\overline{\mathrm{MS}}} = L_{\mathrm{had}}\Lambda^{N_\mathrm{f}=3}_{\overline{\mathrm{MS}}} \times \frac{\sqrt{t_0^{\mathrm{sym}}}}{L_{\mathrm{had}}} \times \frac{1}{\sqrt{t_0^{\mathrm{sym}}}} = ???$$

Step III: matching to hadronic physics

The Λ-parameter

Extrapolation:



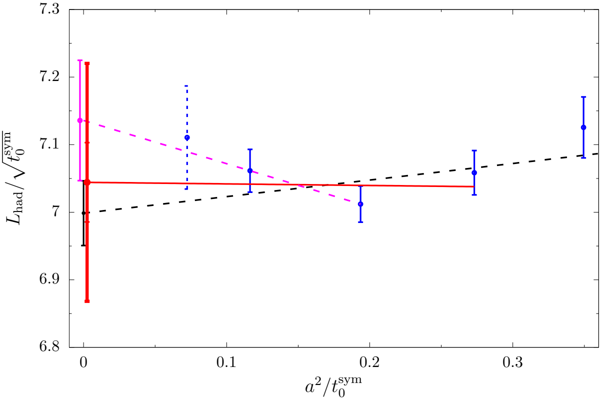
Λ-parameter:

$$\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3} = L_{\rm had}\Lambda_{\overline{\rm MS}}^{N_{\rm f}=3} \times \frac{\sqrt{t_0^{\rm sym}}}{L_{\rm had}} \times \frac{1}{\sqrt{t_0^{\rm sym}}} = ???$$

Step III: matching to hadronic physics

The Λ-parameter

Extrapolation:



Λ-parameter:

$$\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = L_{\mathrm{had}}\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} \times \frac{\sqrt{t_0^{\mathrm{sym}}}}{L_{\mathrm{had}}} \times \frac{1}{\sqrt{t_0^{\mathrm{sym}}}} = 332(14)\,\mathrm{MeV} \;\sim 4\%$$

Conclusions & Outlook

Conclusions:

- Several ideas applied: finite-size scaling, gradient flow, open bc.'s, ...
- All together these allow us to have errors under control
  $\rightarrow$ not only **precision**, but also **accuracy**!
- The effect of the charm and bottom quarks may be included using **perturbation theory**, given $\overline{m}^c_{\overline{\rm MS}}(\overline{m}^c_{\overline{\rm MS}})$ and $\overline{m}^b_{\overline{\rm MS}}(\overline{m}^b_{\overline{\rm MS}})$ (PDG),

$$\Lambda^{N_{\rm f}=3}_{\overline{\rm MS}} \rightarrow \Lambda^{N_{\rm f}=4}_{\overline{\rm MS}} \rightarrow \Lambda^{N_{\rm f}=5}_{\overline{\rm MS}} = 207(11)\,\text{MeV}$$

cf. (Bruno *et. al.* Lattice '15)

- Having $\Lambda^{N_{\rm f}=5}_{\overline{\rm MS}}$ we quote,

$$\alpha_{\overline{\rm MS}}(m_Z) = 0.1179(10)(2) \quad \text{(this work)},$$

$$\alpha^{\text{FLAG '16}}_{\overline{\rm MS}}(m_Z) = 0.1182(12), \quad \alpha^{\text{PDG '16}}_{\overline{\rm MS}}(m_Z)|_{\text{w/o LQCD}} = 0.1175(17)$$

Outlook:

- Include the charm quark non-perturbatively ... challenging!

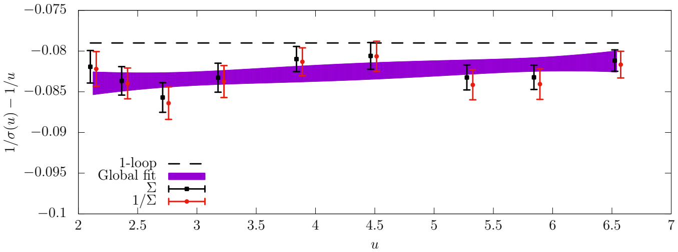

Thanks

BACKUP

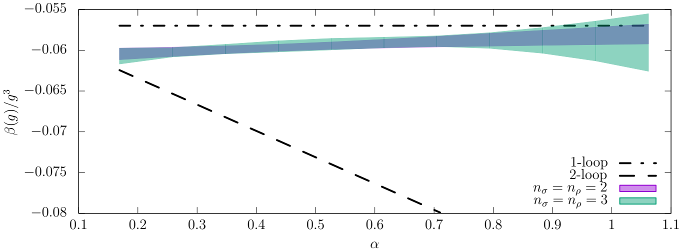
Step II: jogging to low-energy

Continuum results

Step-scaling function:



β-function:



Setting the scale

(Bruno, Korzec, Schaefer '16)

Ingredients:

$$t_0^2\langle\,\mathrm{tr}\{G_{\mu\nu}G_{\mu\nu}\}\rangle = 0.3, \quad \phi_2 = 8t_0 m_\pi^2, \quad \phi_4 = 8t_0(m_K^2 + \tfrac{1}{2}m_\pi^2)$$

Input: CLS ensembles

```
while  t̃_0 ≠ t_0^phys
do
   choose t̃_0   ⇒  (φ̃_2, φ̃_4)  using  m_π, m_K (PDG)
   shift to φ̃_4 ⇒  lim_{φ_2→φ̃_2, a→0} √t_0 · f_πK|_φ̃_4,   f_πK = 2/3 (f_K + 1/2 f_π)
   extract t_0^phys using f_π, f_K (PDG)
done
```

while $\tilde{t}_0 \neq t_0^{\mathrm{phys}}$

do

choose $\tilde{t}_0 \;\Rightarrow\; (\tilde{\phi}_2,\, \tilde{\phi}_4)$ using m_π, m_K (PDG)

shift to $\tilde{\phi}_4 \;\Rightarrow\; \lim_{\substack{\phi_2\to\tilde{\phi}_2 \\ a\to 0}} \sqrt{t_0}\cdot f_{\pi K}|_{\tilde{\phi}_4}, \quad f_{\pi K} = \frac{2}{3}(f_K + \frac{1}{2}f_\pi)$

extract t_0^{phys} using f_π, f_K (PDG)

done

Results:

- $\phi_4^{\mathrm{phys}} = 1.11(2)$
- $\sqrt{8\,t_0^{\mathrm{phys}}} = 0.413(4)\,\mathrm{fm}$

Decoupling from $N_\mathrm{q} \to N_\mathrm{l}$ flavours

(Bruno *et. al.* Lattice '15)

Λ-parameter:

$$\Lambda^{N_\mathrm{f}}/\mu = \exp\left\{I_g^{N_\mathrm{f}}(\overline{g}(\mu))\right\} \propto \exp\left\{-\int^{\overline{g}(\mu)} \frac{\mathrm{d}x}{\beta_{N_\mathrm{f}}(x)}\right\}$$

Invariant mass M:

$$M/\overline{m}(\mu) = \exp\left\{I_m^{N_\mathrm{f}}(\overline{g}(\mu))\right\} \propto \exp\left\{-\int^{\overline{g}(\mu)} \mathrm{d}x\, \frac{\tau_{N_\mathrm{f}}(x)}{\beta_{N_\mathrm{f}}(x)}\right\}$$

Coupling: ($\overline{m}(m_*) = m_*$)

$$\overline{g}^2_{N_\mathrm{l}}(m_*) = \overline{g}^2_{N_\mathrm{q}}(m_*)\, C(\overline{g}_{N_\mathrm{q}}(m_*)), \quad C(g) = 1 + c_2 g^4 + c_3 g^6 + \ldots$$

Decoupling: ($g_* \equiv \overline{g}_{N_\mathrm{q}}(m_*)$)

$$\Lambda^{N_\mathrm{l}}/\Lambda^{N_\mathrm{q}} = \exp\left\{I_g^{N_\mathrm{l}}(g_*\sqrt{C(g_*)}) - I_g^{N_\mathrm{q}}(g_*)\right\} \equiv P_{\mathrm{l,q}}(M/\Lambda^{N_\mathrm{q}})$$

$$\Lambda^{N_\mathrm{q}}/M = \exp\left\{I_g^{N_\mathrm{q}}(g_*) - I_m^{N_\mathrm{q}}(g_*)\right\}$$