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SPECTRUM
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QUARK

Relativistic 2-body Bottomonium decays.

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Setting

- A **relativistic wave equation**
- for **two fermions**
- with **arbitrary masses**
- with the correct one particle **Dirac equation limit** when the mass of one component tends to infinity (not as Bethe-Salpeter Eq.!)
- with the correct two-body **Schrödinger Equation** limit in the non-relativistic regime.
- including a **vector Coulomb**-like **interaction** and a **Breit** term responsible for the hyperfine splitting
- possibly including also a confining **scalar interaction** linearly growing at infinity (phenomenological term from QCD)
- Later on the **vertex with a photon field** will be needed for the covariant treatment of radiative decays
- and the **vertex for leptonic decays** will be studied.

Benefits

- The obvious inclusion of the **complete relativistic kinematics** without any need of corrections.
- **Spin-orbit**: **2 spin-orbit couplings**, one for **each** fermion, directly implied by the Dirac equations for the two particles
- **Spin-spin**: a plain perturbation treatment of the spin-spin interaction ' without the puzzle of **δ-function smearing** in the Fermi coupling approximation.
- Correct introduction of the confining **scalar potential** - coupled to the mass - and of the **vector potential** (*e.g.* electrostatic) - coupled to the energy -.
- Eigenfunctions (16 component spinors) and their use for perturbative calculations as the radiative decays.

Kinematical stuff

- **Particle** variables $\quad m_i\,,\quad x^\mu_{(i)}\,,\quad p^\mu_{(i)}\,,\qquad i=1,2$ $\rightsquigarrow$
- **Global** and **relative** variables

$$P^\mu = p^\mu_{(1)} + p^\mu_{(2)}\,, \qquad X^\mu = \frac{1}{2}\left(x^\mu_{(1)} + x^\mu_{(2)}\right),$$

$$q^\mu = \frac{1}{2}\left(p^\mu_{(1)} - p^\mu_{(2)}\right), \qquad r^\mu = x^\mu_{(1)} - x^\mu_{(2)}\,,$$ $\rightsquigarrow$

- **Canonical** variables

$$P^\mu$$

$$Z^\mu = X^\mu + \frac{\varepsilon_{abc}\,P_a\,\eta^\mu_b\,L_c}{\sqrt{P^2}\,[P_0+\sqrt{P^2}]} + \frac{\varepsilon^\mu_a}{\sqrt{P^2}}\left(q_a\,\mathring{r} - r_a\,\mathring{q}\right) + \frac{P^\mu}{P^2}\,\mathring{q}\,\mathring{r}$$

$$\mathring{q} = \varepsilon^\mu_0\,q_\mu\,, \qquad \mathring{r} = \varepsilon^\mu_0\,r_\mu\,, \qquad q_a = \varepsilon^\mu_a\,q_\mu\,, \qquad r_a = \varepsilon^\mu_a\,r_\mu$$

- $\boldsymbol{r}$, $\boldsymbol{q}$ Newton-Wigner 3-vectors: $\boldsymbol{r}^2$, $\boldsymbol{q}^2$, $\mathring{r}$, $\mathring{q}$ **Lorentz invariant.**

Coupled Dirac Operators

- The **Dirac operators** and the γ matrices of the two particles are

$$\boldsymbol{D_1} = \left(\frac{1}{2}P_\mu + q_\mu\right)\gamma^\mu_{(1)} - m_1\,, \quad \boldsymbol{D_2} = \left(\frac{1}{2}P_\mu - q_\mu\right)\gamma^\mu_{(2)} - m_2$$
$$\gamma^\mu_{(1)} = \gamma^\mu \otimes \mathbf{I}_4\,, \qquad \gamma^\mu_{(2)} = \mathbf{I}_4 \otimes \gamma^\mu\,,$$

- In terms of the **canonical variables**

$$(1/2)\,\lambda\,\mathring{\gamma}_{(1)} + \mathring{q}\,\mathring{\gamma}_{(1)} - q_a\gamma_{(1)\,a} = m_1\,,$$
$$(1/2)\,\lambda\,\mathring{\gamma}_{(2)} - \mathring{q}\,\mathring{\gamma}_{(2)} + q_a\gamma_{(2)\,a} = m_2$$

with

$$\mathring{\gamma}_{(i)} \equiv \mathring{\gamma}_{(i)}(P) = \varepsilon^\mu_0(P)\gamma_{(i)\mu}\,, \qquad \gamma_{(i)\,a} \equiv \gamma_{(i)\,a}(P) = \varepsilon^\mu_a(P)\gamma_{(i)\,\mu}$$

The free eigenvalues

- Solve in the **mass** $\lambda_0 = \sqrt{P^2}$ and in the **relative energy** $\mathring{q}$

$$\lambda_0 = q_a \left(\mathring{\gamma}_{(1)} \, \gamma_{(1)a} - \mathring{\gamma}_{(2)} \, \gamma_{(2)a} \right) + \mathring{\gamma}_{(1)} m_1 + \mathring{\gamma}_{(2)} m_2$$

$$\mathring{q} = \frac{1}{2} \, q_a \left(\mathring{\gamma}_{(1)} \, \gamma_{(1)a} + \mathring{\gamma}_{(2)} \, \gamma_{(2)a} \right) + \mathring{\gamma}_{(1)} m_1 - \mathring{\gamma}_{(2)} m_2$$

- **Free eigenvalues:** the 4 combinations :

$$\lambda_0 = \pm \left(q_a \, q_a + m_1^2 \right)^{1/2} \pm \left(q_a \, q_a + m_2^2 \right)^{1/2}$$

each one with **multiplicity 4**.

The interacting $q\bar{q}$ problem

- Mass parameters $M = m_1 + m_2\,, \qquad \mu = m_1 - m_2$
- **Meson interaction** in terms of the **Cornell potential:**
  - ∗ a linear **scalar** interaction σr is added to the total mass.
  - ∗ a Coulomb-like **vector** interaction α/r is added to the energy.
- The **q$\bar{\text{q}}$ wave equation** reads:

$$\Big[\left(\gamma^0_{(1)}\gamma_{(1)a} - \gamma^0_{(2)}\gamma_{(2)a}\right) q_a + \frac{1}{2}\left(\gamma^0_{(1)} + \gamma^0_{(2)}\right)\left(M + \sigma r\right) + \frac{1}{2}\left(\gamma^0_{(1)} - \gamma^0_{(2)}\right)\mu - \left(\lambda + \frac{b}{r}\right) + \varepsilon V_B(r)\Big]\,\Psi(\boldsymbol{r}) = 0.$$

- $V_B(r)$ is the **Breit term** to be treated **at the first perturbation order** It reads:

$$V_B(r) = \frac{b}{2r}\,\gamma^0_{(1)}\gamma_{(1)a}\gamma^0_{(2)}\gamma_{(2)b}\left(\delta_{ab} + \frac{r_a r_b}{r^2}\right)$$

(For hydrogenic atoms $b = \kappa_1\kappa_2\alpha$ where κ_i = total magnetic moments of the components and α = fine structure constant.)

The interacting problem: equation and states

- The **unperturbed interacting problem** is

$$\left[\begin{pmatrix} M(r)\,\mathbf{I}_{4,4} & \mathcal{H}_0 \\ \mathcal{H}_0 & -\mu\,\mathbf{I}_{4,4} \end{pmatrix} - \left(\lambda + \frac{b}{r}\right)\right]\Psi = 0\,, \quad \mathbf{I}_{4,4} = \mathrm{diag}(\mathbf{I}_4, -\mathbf{I}_4)$$

- The $\boldsymbol{\Psi}_{\pm}$ states have 16 components, collected in 4 groups labeled by the free eigenvalues $+M\,,\ -M\,,\ -\mu\,,\ +\mu\,.$ In each group the components have the in singlet-triplet order, *i.e.*

$$\boldsymbol{\Psi}_{\pm} = {}^t\!\left(\Psi_{\pm}^{(M)},\, \Psi_{\pm}^{(-M)},\, \Psi_{\pm}^{(-\mu)},\, \Psi_{\pm}^{(\mu)}\right).$$

with $$\Psi_{\pm}^{(\Lambda)} = {}^t\!\left(\psi_{\pm\,0}^{(\Lambda)}\,,\, \psi_{\pm\,1_+}^{(\Lambda)}\,,\, \psi_{\pm\,1_0}^{(\Lambda)}\,,\, \psi_{\pm\,1_-}^{(\Lambda)}\right),$$

- Odd and even spinors are related by

$$\boldsymbol{\Psi}_{-} = \begin{pmatrix} 0 & \mathbf{I}_8 \\ \mathbf{I}_8 & 0 \end{pmatrix} \boldsymbol{\Psi}_{+}$$

The Hamiltonian submatrix *

The $\mathcal{H}_0$ bloc is given by

$$\mathcal{H}_0 = \begin{pmatrix} 0 & X_+ & X_0 & X_- & 0 & X_+ & X_0 & X_- \\ -X_- & X_0 & X_- & 0 & -X_- & -X_0 & -X_- & 0 \\ X_0 & -X_+ & 0 & X_- & X_0 & X_+ & 0 & -X_- \\ -X_+ & 0 & -X_+ & -X_0 & -X_+ & 0 & X_+ & X_0 \\ 0 & X_+ & X_0 & X_- & 0 & X_+ & X_0 & X_- \\ -X_- & -X_0 & -X_- & 0 & -X_- & X_0 & X_- & 0 \\ X_0 & X_+ & 0 & -X_- & X_0 & -X_+ & 0 & X_- \\ -X_+ & 0 & X_+ & X_0 & -X_+ & 0 & -X_+ & -X_0 \end{pmatrix}$$

where

$$X_\pm = -(\pm q_x + i\, q_y)/\sqrt{2}\,, \qquad X_0 = q_z\,, \qquad q_a \to -i\partial/\partial r_a$$

The even state components (i-viii) *

- The four $\Psi_{+}^{(M)}$ components

$$\psi_{+\,0}^{(M)} = Y_m^j(\theta,\,\phi)\ a_0(r)$$

$$\psi_{+\,1_+}^{(M)} = -\frac{\sqrt{j-m+1}\sqrt{j+m}}{\sqrt{2j}\sqrt{j+1}}\,Y_{m-1}^j(\theta,\,\phi)\ b_0(r)$$

$$\psi_{+\,1_0}^{(M)} = \frac{m}{\sqrt{j}\,\sqrt{1+j}}\,Y_m^j(\theta,\,\phi)\ b_0(r)$$

$$\psi_{+\,1_-}^{(M)} = \frac{\sqrt{j-m}\,\sqrt{j+m+1}}{\sqrt{2j}\sqrt{j+1}}\,Y_{m+1}^j(\theta,\,\phi)\ b_0(r)$$

- The four $\Psi_{+}^{(-M)}$ components

$$\psi_{+\,0}^{(-M)} = Y_m^j(\theta,\,\phi)\ a_1(r)$$

$$\psi_{+\,1_+}^{(-M)} = -\frac{\sqrt{j-m+1}\sqrt{j+m}}{\sqrt{2j}\sqrt{j+1}}\,Y_{m-1}^j(\theta,\phi)\ b_1(r)$$

$$\psi_{+\,1_0}^{(-M)} = \frac{m}{\sqrt{j}\,\sqrt{1+j}}\,Y_m^j(\theta,\,\phi)\ b_1(r)$$

$$\psi_{+\,1_-}^{(-M)} = \frac{\sqrt{j-m}\sqrt{j+m+1}}{\sqrt{2j}\sqrt{j+1}}\,Y_{m+1}^j(\theta,\phi)\ b_1(r)$$

The even state components (ix-xvi) *

- The four $\Psi_{+}^{(-\mu)}$ components

$$\psi_{+\,0}^{(-\mu)} = 0$$

$$\psi_{+\,1_+}^{(-\mu)} = \frac{\sqrt{j+m-1}\,\sqrt{j+m}}{\sqrt{2j}\,\sqrt{2j-1}}\,Y_{m-1}^{j-1}(\theta,\,\phi)\,i\,c_0(r) + \frac{\sqrt{j-m+1}\,\sqrt{j-m+2}}{\sqrt{2j+2}\,\sqrt{2j+3}}\,Y_{m-1}^{j+1}(\theta,\,\phi)\,i\,d_0(r)$$

$$\psi_{+\,1_0}^{(-\mu)} = \frac{\sqrt{j-m}\,\sqrt{j+m}}{\sqrt{j}\,\sqrt{2j-1}}\,Y_{m}^{j-1}(\theta,\,\phi)\,i\,c_0(r) - \frac{\sqrt{j-m+1}\,\sqrt{j+m+1}}{\sqrt{1+j}\,\sqrt{2j+3}}\,Y_{m}^{j+1}(\theta,\,\phi)\,i\,d_0(r)$$

$$\psi_{+\,1_-}^{(-\mu)} = \frac{\sqrt{j-m-1}\,\sqrt{j-m}}{\sqrt{2j}\,\sqrt{2j-1}}\,Y_{m+1}^{j-1}(\theta,\,\phi)\,i\,c_0(r) + \frac{\sqrt{j+m+1}\,\sqrt{j+m+2}}{\sqrt{2j+2}\,\sqrt{2j+3}}\,Y_{m+1}^{j+1}(\theta,\,\phi)\,i\,d_0(r)$$

- The four $\Psi_{+}^{(\mu)}$ components

$$\psi_{+\,0}^{(\mu)} = 0$$

$$\psi_{+\,1_+}^{(\mu)} = \frac{\sqrt{j+m-1}\,\sqrt{j+m}}{\sqrt{2j}\,\sqrt{2j-1}}\,Y_{m-1}^{j-1}(\theta,\,\phi)\,i\,c_1(r) + \frac{\sqrt{j-m+1}\,\sqrt{j-m+2}}{\sqrt{2j+2}\,\sqrt{2j+3}}\,Y_{m-1}^{j+1}(\theta,\,\phi)\,i\,d_1(r)$$

$$\psi_{+\,1_0}^{(\mu)} = \frac{\sqrt{j-m}\,\sqrt{j+m}}{\sqrt{j}\,\sqrt{2j-1}}\,Y_{m}^{j-1}(\theta,\,\phi)\,i\,c_1(r) - \frac{\sqrt{j-m+1}\,\sqrt{j+m+1}}{\sqrt{j+1}\,\sqrt{2j+3}}\,Y_{m}^{j+1}(\theta,\,\phi)\,i\,d_1(r)$$

$$\psi_{+\,1_-}^{(\mu)} = \frac{\sqrt{j-m-1}\,\sqrt{j-m}}{\sqrt{2j}\,\sqrt{2j-1}}\,Y_{m+1}^{j-1}(\theta,\,\phi)\,i\,c_1(r) + \frac{\sqrt{j+m+1}\,\sqrt{j+m+2}}{\sqrt{2j+2}\,\sqrt{2j+3}}\,Y_{m+1}^{j+1}(\theta,\,\phi)\,i\,d_1(r)$$

The originary radial system *

Substitute $\boldsymbol{\Psi}_{\pm}$ in wave-equation and let the coefficient of each spherical harmonics vanish. Get **34** equations, **8** of which are independent:

$$\sqrt{j}\left(\frac{d}{dr}+\frac{j+1}{r}\right)a_+(r)-\sqrt{j+1}\left(\frac{d}{dr}+\frac{j+1}{r}\right)b_-(r)+\sqrt{2j+1}\,c_0(r)\,(\mu+h(r))=0$$

$$\sqrt{j}\left(\frac{d}{dr}+\frac{j+1}{r}\right)a_+(r)+\sqrt{j+1}\left(\frac{d}{dr}+\frac{j+1}{r}\right)b_-(r)-\sqrt{2j+1}\,c_1(r)\,(\mu-h(r))=0$$

$$\sqrt{j+1}\left(\frac{d}{dr}-\frac{j}{r}\right)a_+(r)+\sqrt{j}\left(\frac{d}{dr}-\frac{j}{r}\right)b_-(r)-\sqrt{2j+1}\,d_0(r)\,(\mu+h(r))=0$$

$$\sqrt{j+1}\left(\frac{d}{dr}-\frac{j}{r}\right)a_+(r)-\sqrt{j}\left(\frac{d}{dr}-\frac{j}{r}\right)b_-(r)+\sqrt{2j+1}\,d_1(r)\,(\mu-h(r))=0$$

$$\sqrt{j}\left(\frac{d}{dr}-\frac{j-1}{r}\right)c_+(r)-\sqrt{j+1}\left(\frac{d}{dr}+\frac{j+2}{r}\right)d_+(r)+\sqrt{2j+1}\,a_0(r)\,(M(r)-h(r))=0$$

$$\sqrt{j}\left(\frac{d}{dr}-\frac{j-1}{r}\right)c_+(r)-\sqrt{j+1}\left(\frac{d}{dr}+\frac{j+2}{r}\right)d_+(r)-\sqrt{2j+1}\,a_1(r)\,(M(r)+h(r))=0$$

$$\sqrt{j+1}\left(\frac{d}{dr}-\frac{j-1}{r}\right)c_-(r)+\sqrt{j}\left(\frac{d}{dr}+\frac{j+2}{r}\right)d_-(r)-\sqrt{2j+1}\,b_0(r)\,(M(r)-h(r))=0$$

$$\sqrt{j+1}\left(\frac{d}{dr}-\frac{j-1}{r}\right)c_-(r)+\sqrt{j}\left(\frac{d}{dr}+\frac{j+2}{r}\right)d_-(r)-\sqrt{2j+1}\,b_1(r)\,(M(r)+h(r))=0$$

where

$$s_\pm(r)=s_0(r)\pm s_1(r)\,,\qquad s=a,\,b,\,c,\,d$$

The reduced 4th order radial system *

- The system gives 4 **algebraic** and 4 **differential** equations.
- The reduced radial equation for each parity is the 4×4 system

$$\frac{d}{dr}\begin{bmatrix} y_1(r) \\ y_2(r) \\ y_3(r) \\ y_4(r) \end{bmatrix} + \begin{bmatrix} 0 & A_{\varepsilon=0}(r) & -B_{\varepsilon=0}(r) & 0 \\ A_\varepsilon(r) & 1/r & 0 & B_\varepsilon(r) \\ C_\varepsilon(r) & 0 & 2/r & A_\varepsilon(r) \\ 0 & D_\varepsilon(r) & A_{\varepsilon=0}(r) & 1/r \end{bmatrix}\begin{bmatrix} y_1(r) \\ y_2(r) \\ y_3(r) \\ y_4(r) \end{bmatrix} = 0$$

- The **Schrödinger** and the **1-particle Dirac** limits are satisfied.
- Solution by double shooting and Padé approximants.
- **Match** the four component functions at a crossing point x_c

$$K_1\, y_j^{(0,1)}(x_c) + K_2\, y_j^{(0,2)}(x_c) = K_3\, y_j^{(\infty,1)}(x_c) + K_4\, y_j^{(\infty,2)}(x_c)$$

- **Spectral condition** = vanishing of a 4x4 determinant

HFS for some Hydrogenic atoms

- Hyperfine splitting for **1S** and **2S** states (e in MHz, μ in meV).

Atom	$\Delta_{HFS}(1s)$		$\Delta_{HFS}(2s)$	
(p, e)	1420.595	1420.405	177.580	177.557
(μ^+, e)	4464.481	4463.302	558.078	558.
($^3He^+$, e)	-8665.637	-8665.650	-1083.347	-1083.355
(p, μ)	182.621	182.638	22.828	22.815
($^3He^+$, μ)	-1372.194	-1334.730	-171.544	-166.645

- Hyperfine splitting for **2P** states (e in MHz, μ in meV).

Atom	$\Delta_{HFS}(2p^{1/2})$		$\Delta_{HFS}(2p^{3/2})$	
(p, e)	59.196	59.221	23.678	24.0
(μ^+, e)	186.252	187.0	74.629	74.0
($^3He^+$, e)	-361.100	-	-144.385	-
(p, μ)	76.82	78.20	31.15	32.48
($^3He^+$, μ)	-57.028	58.713	-22.700	-24.291

Procedure

- We have **first** calculated the spectra of heavy quarkonia $(b\bar{b})$, $(c\bar{c})$, $(b\bar{c})$.
  From minimization of the square differences with respect to the experimental data the **same string tension** is obtained.
- We have fitted m_b and m_c.
- We have **fitted** two parameters $\alpha_{b\bar{b}}$ and $\alpha_{c\bar{c}}$ **independently** for the whole spectra of $(b\bar{b})$ and $(c\bar{c})$.
- We have done the **same** for $(s\bar{s})$, $(b\bar{s})$, $(c\bar{s})$. For the last two we have **used** the **already found masses**.
- We have finally studied the **light mesons** $(u\bar{d})$
- We comment later on the string tension and on α parameters.

$b\bar{b}$ Mesons (1)

State	Exp	Num
$(1^1s_0)\ 0^+(0^{-+})\ \eta_b$	9390.90±2.8*	9390.39
$(1^3s_1)\ 0^-(1^{--})\ \Upsilon$	9460.30±.25	9466.10
$(1^3p_0)\ 0^+(0^{++})\ \chi_{b0}$	9859.44±.73	9857.41
$(1^3p_1)\ 0^+(1^{++})\ \chi_{b1}$	9892.78±.57	9886.70
$(1^1p_1)\ 0^-(1^{+-})\ h_b$	9898.60±1.4	9895.35
$(1^3p_2)\ 0^+(2^{++})\ \chi_{b2}$	9912.21±.57	9908.14
$(2^3s_1)\ 0^-(1^{--})\ \Upsilon$	10023.26±.0003	10009.04
$(1^3d_2)\ 0^-(2^{--})\ \Upsilon_2$	10163.70±1.4	10152.69
$(2^3p_0)\ 0^+(0^{++})\ \chi_{b0}$	10232.50±.0009	10232.36
$(2^3p_1)\ 0^+(1^{++})\ \chi_{b1}$	10255.46±.0005	10256.58
$(2^3p_2)\ 0^+(2^{++})\ \chi_{b2}$	10268.65±.0007	10274.26

$b\bar{b}$ Mesons (2)

State	Exp	Num
$(3^3s_1)\ 0^-(1^{--})\ \Upsilon$	**10355.20±.0005**	**10364.52**
$(3^3p_0)\ 0^+(0^{++})\ \chi_{b0}$		**10534.86**
$(3^3p_1)\ 0^+(1^{++})\ \chi_{b1}$	$<$**10530±.014**$>_J$	**10556.59**
$(3^3p_2)\ 0^+(2^{++})\ \chi_{b2}$		**10572.44**
$(4^3s_1)\ 0^-(1^{--})\ \Upsilon$	**10579.40±.0012**	**10655.34**
$(5^3s_1)\ 0^-(1^{--})\ \Upsilon$	**10876±11**	**10910.35**

Table: The **b$\bar{\text{b}}$** levels in MeV. First column: term symbol, $I^G(JPC)$ numbers , particle name. σ=**1.111** GeV/fm, α=**0.3272**, m_b=**4725.5** MeV. Experimental data from [PdG].

One photon emission

- **Photon wavefunction** with polarization ϵ_σ

$$A(k,\sigma) = \frac{\sqrt{4\pi}}{\sqrt{2\omega}}\,\epsilon_\sigma\, e^{-ik\cdot x}$$

- **Interaction Hamiltonian** in the two particle coordinates, z=A.N.

$$H_{int} = -q\left(z\alpha_{(1)}\cdot A(x_1) - \alpha_{(2)}\cdot A(x_2)\right)$$

- **Reference frame** with vanishing initial global momentum, $P_i = 0$
- **2-fermion wavefunction** in canonical global-relative coordinates

$$\Psi_\ell = V^{-1/2}\, e^{-iP_\ell Z}\,\psi_\ell(r), \qquad\qquad \ell = i,\, f$$

- **S-matrix element**, with $\Delta = (m_1^2 - m_2^2)/2P^2$,

$$S_{fi} = -iq\,\frac{(2\pi)^4}{\sqrt{2\omega V}}\,\delta^4(P_f + k - P_i)\int d^3r\,\frac{\sqrt{4\pi}}{\sqrt{V^2}}\,\psi_f^*(r)\,\epsilon_\sigma^*$$
$$\left[\widetilde{\alpha}_{(1)}\, e^{-i(\frac{1}{2}-\Delta)k\cdot r} - \widetilde{\alpha}_{(2)}\, e^{i(\frac{1}{2}+\Delta)k\cdot r}\right]\psi_i(r)\,.$$

The S-matrix element

- **Conservations**: $P_i^0 = P_f^0 + \omega$, $\boldsymbol{P}_i = \boldsymbol{P}_f + \boldsymbol{k}$ (**recoil included !**)
- If $2\pi/\omega >>$ atomic scale length,

$$e^{\pm i(1/2 \pm \Delta)\boldsymbol{k}\cdot\boldsymbol{r}} \simeq 1 \pm (1/2 \pm \Delta)\,\boldsymbol{k}\cdot\boldsymbol{r}$$

  This corresponds to the "*dipole approximation*", but has different origin.
- **By angular properties**

$$\int d^3\boldsymbol{r}\;\; \psi_f^*(\boldsymbol{r})\,\widetilde{\alpha}_{(j)}\,\psi_i(\boldsymbol{r}) = 0, \quad j = 1, 2.$$

- The **final form of the S-matrix element** in this approximation is

$$S_{fi} = (4\pi\omega/V^3)^{1/2}\,(2\pi)^4\,\delta^4(P_f + k - P_i)\;(\epsilon_\sigma^* \cdot \boldsymbol{d}_{fi})\,,$$

$$\boldsymbol{d}_{fi} = -i\,q \int d^3\boldsymbol{r}\;(\boldsymbol{n}\cdot\boldsymbol{r})\;\psi_f^*(\boldsymbol{r})\left[\widetilde{\alpha}_{(1)}\left(\frac{1}{2} - \Delta\right) + \widetilde{\alpha}_{(2)}\left(\frac{1}{2} + \Delta\right)\right]\psi_i(\boldsymbol{r})\,.$$

The transition rate

- The **differential transition** rate is

$$dw = \frac{\omega}{2\pi}\,\delta^4(P_f + k - P_i)\sum_\sigma |\epsilon^*_\sigma \cdot \boldsymbol{d}_{fi}|^2 \; d^3\boldsymbol{k} \; d^3\boldsymbol{P}_f$$

- Integrate over P with the **invariant measure.** Use the identity

$$\int (d^3\boldsymbol{P}/2P^0) = \int (d^4P)\;\theta(P^0)\,\delta(P^2 - \lambda^2)$$

- Get

$$\frac{dw}{d\omega\; d\Omega_n} = \frac{\omega^3}{2\pi\lambda_i}\,(\lambda_i - \omega)\,\delta\left(\omega - \frac{\lambda_f^2 - \lambda_i^2}{2\lambda_i}\right)\sum_\sigma |\epsilon^*_\sigma \cdot \boldsymbol{d}_{fi}|^2$$

- The **integtrated transition** rate is

$$w = \frac{4}{3}\,\frac{\omega^3}{\hbar c^3}\,\Lambda^2_{fi}\,|\,\mu_{fi}\,|^2 \qquad |\,\mu_{fi}\,|^2 = \left|\frac{(\epsilon_1 \pm \epsilon_2)}{\sqrt{2}} \cdot \boldsymbol{d}_{fi}\right|^2$$

with

$$\hbar\omega = \frac{\lambda_i + \lambda_f}{2\lambda_i}\,(\lambda_i - \lambda_f)\,, \qquad \Lambda^2_{fi} = \frac{\lambda_i^2 + \lambda_f^2}{2\lambda_i^2}$$

Atomic hyperfine transitions

Using for the matrix element the Breit corrected wavefunctions

$$\Psi_f = \Psi_{13s1}(r, \theta, \phi) \qquad \text{and} \qquad \Psi_i = \Psi_{11s0}(r, \theta, \phi)$$

we have calculated the decay rate for some cases.
For the first two levels of the **electronic Hydrogen**:

$$n = 1 \quad \mathcal{P}_{HFS_1} = \mathbf{2.866\ 10^{-15}\ s^{-1}} \qquad \text{approximately known}$$

$$n = 2 \quad \mathcal{P}_{HFS_2} = \mathbf{1.871\ 10^{-15}\ s^{-1}} \qquad \text{not found in literature}$$

For the first levels of the **muonic Hydrogen**:

$$n = 1 \quad \mathcal{P}_{HFS_1} = \mathbf{2.496\ 10^{-6}\ s^{-1}} \qquad \text{not found in literature}$$

Meson radiative decays. Remarks

Comparison and difficulties w.r.t atomic transitions.

- We calculate branching ratios and widths of the measured radiative decays of $\Upsilon(3s)$, $\chi_{b2}(2p)$, $\chi_{b1}(2p)$, $\chi_{b0}(2p)$, $\Upsilon(2s)$
- Good but less accurate results *w.r.t.* atomic transitions. Indeed:

  (a) The **Cornell potential** is an effective potential more adapted to the description of a stationary situation contrary to the fundamental **atomic electrodynamical interaction**.

  (b) **Atoms**: same fine structure coupling constant α_{em} for Coulomb potential, Breit spin-spin interaction and decay process $\looparrowright$ proper calculation of first order corrections to wave functions.

  (c) **Mesons**: a remnant of Breit term correction already present at lowest order by means of the fitted Cornell parameters entering the wave functions $\looparrowright$ no proper (even) first order perturbative corrections.
- Difficulties with **QCD e.w.** and **strong radiative effects**.

The Breit corrections for some levels

State	$\Delta_B(b\bar{b})$	$\Delta_B(c\bar{c})$	$\Delta_B(s\bar{s})$	$\Delta_B(u\bar{d})$
$(1^1s_0)\ 0^+(0^{-+})$	92.31	155.22	296.81	600.12
$(1^3s_1)\ 0^-(1^{--})$	18.09	38.80	94.37	106.21
$(1^3p_0)\ 0^+(0^{++})$	44.30	117.41	297.14	
$(1^3p_1)\ 0^+(1^{++})$	19.98	52.14	127.83	
$(1^1p_1)\ 0^-(1^{+-})$	15.95	43.24	110.77	
$(1^3p_2)\ 0^+(2^{++})$	7.51	21.10	55.93	63.72
$(2^3s_1)\ 0^-(1^{--})$	24.31	60.02	134.22	
$(1^3d_1)\ 0^-(1^{--})$	17.49	49.32	123.85	

Table: The Breit correction Δ_B in MeV for some levels of $b\bar{b}$, $c\bar{c}$, $s\bar{s}$, $u\bar{d}$.

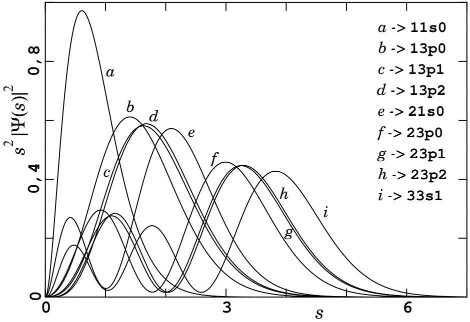
Wave equation & densities

- The 2-fermion equation with vector-scalar **Cornell** potential and with the **Breit** term

$$\left[\left(\gamma^0_{(1)}\gamma_{(1)a}-\gamma^0_{(2)}\gamma_{(2)a}\right)q_a+\frac{1}{2}\left(\gamma^0_{(1)}+\gamma^0_{(2)}\right)\left(2m_b+\sigma r\right)-\left(E+\frac{b}{r}\right)+V_B(r)\right]\Psi(\mathbf{r})=0$$

$$V_B(r)=\frac{b}{2r}\,\gamma^0_{(1)}\gamma_{(1)a}\gamma^0_{(2)}\gamma_{(2)b}\left(\delta_{ab}+\frac{r_a r_b}{r^2}\right)$$

- **Completely covariant** probability normalized **densities** for the states



Decay rate expressions

- Photon wavefunction $\boldsymbol{A}(\boldsymbol{k},\boldsymbol{\sigma}) = \frac{\sqrt{4\pi}}{\sqrt{2\omega\, V}}\ \boldsymbol{\epsilon}_\sigma\ e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\ .$
- Transition matrix element

$$\boldsymbol{M}_{fi} = \int d^3\boldsymbol{r}\ \psi_f^*(\boldsymbol{r}) \left(\widetilde{\boldsymbol{\alpha}}_{(1)}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}/2} - \widetilde{\boldsymbol{\alpha}}_{(2)}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}/2} \right) \psi_i(\boldsymbol{r})$$

- Differential transition rate

$$dw = \frac{e_b^2}{2\pi\omega}\, \delta^4(P_f + k - P_i) \sum \frac{|\boldsymbol{\epsilon}_\sigma^* \cdot \boldsymbol{M}_{fi}|^2}{2j_i + 1} d^3\,\boldsymbol{k}\ d^3\,\boldsymbol{P}_f$$

- Differential transition rate per unit (frequency,angle) from level λ_i to λ_f

$$\frac{dw}{d\omega\, d\Omega_n} = \frac{e_b^2\,\omega}{2\pi\lambda_i}\,(\lambda_i - \omega)\,\delta\!\left(\omega - \frac{\lambda_i^2 - \lambda_f^2}{2\lambda_i}\right) \sum \frac{|\boldsymbol{\epsilon}_\sigma^* \cdot \boldsymbol{M}_{fi}|^2}{2j_i + 1}$$

- Total transition rate

$$w = \frac{4}{3}\,\frac{e_b^2}{\hbar c}\,\omega_{fi}\,\Lambda_{fi}^2 \sum \frac{|\boldsymbol{\epsilon}_\sigma^* \cdot \boldsymbol{M}_{fi}|^2}{2j_i + 1}$$

where

$$\omega_{fi} = \frac{c}{\hbar}\,\frac{\lambda_i + \lambda_f}{2\lambda_i}\,(\lambda_i - \lambda_f)\,, \qquad \Lambda_{fi}^2 = \frac{\lambda_i^2 + \lambda_f^2}{2\lambda_i^2}\,,$$

are the frequency of the emitted photon that completely includes the recoil and the relativistic kinematic correction factor.

Branching Ratios

Branching Ratios	Theor	Exp
$\Upsilon(3s) \to \gamma\chi_{b1}(2p)/\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	.812	.96±.21
$\Upsilon(3s) \to \gamma\chi_{b0}(2p)/\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	.433	.45±.10
$\Upsilon(3s) \to \gamma\eta_b(2s)/\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	.002	$<$.005
$\Upsilon(3s) \to \gamma\chi_{b2}(1p)/\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	.042	.075±.019
$\Upsilon(3s) \to \gamma\chi_{b1}(1p)/\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	.010	.007±.005
$\Upsilon(3s) \to \gamma\chi_{b0}(1p)/\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	.010	.021±.006
$\Upsilon(3s) \to \gamma\eta_b(1s)/\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	.003	.004±.001
$\Upsilon(2s) \to \gamma\chi_{b1}(1p)/\Upsilon(2s) \to \gamma\chi_{b2}(1p)$	.812	.96±.10
$\Upsilon(2s) \to \gamma\chi_{b0}(1p)/\Upsilon(2s) \to \gamma\chi_{b2}(1p)$	.410	.53±.08
$\Upsilon(2s) \to \gamma\eta_b(1s)/\Upsilon(2s) \to \gamma\chi_{b2}(1p)$	.006	.006±.002
$\chi_{b2}(2p) \to \gamma\Upsilon(1s)/\chi_{b2}(2p) \to \gamma\Upsilon(2s)$	.55	.66±.23
$\chi_{b1}(2p) \to \gamma\Upsilon(1s)/\chi_{b1}(2p) \to \gamma\Upsilon(2s)$	.46	.46±.08
$\chi_{b0}(2p) \to \gamma\Upsilon(1s)/\chi_{b0}(2p) \to \gamma\Upsilon(2s)$	.13	(.20±.20)$^{(*)}$

Widths for $\Upsilon(3)$ decays

Decay	Theor(keV)	Exp(keV)
$\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	3.51	2.70$\pm$0.57
$\Upsilon(3s) \to \gamma\chi_{b1}(2p)$	2.85	2.58$\pm$0.48
$\Upsilon(3s) \to \gamma\chi_{b0}(2p)$	1.52	1.21$\pm$0.23
$\Upsilon(3s) \to \gamma\eta_b(2s)$	0.006	$<$ 0.013
$\Upsilon(3s) \to \gamma\chi_{b2}(1p)$	0.149	0.204$\pm$0.045
$\Upsilon(3s) \to \gamma\chi_{b1}(1p)$	0.036	0.019$\pm$0.012
$\Upsilon(3s) \to \gamma\chi_{b0}(1p)$	0.032	0.056$\pm$0.013
$\Upsilon(3s) \to \gamma\eta_b(1s)$	0.009	0.011$\pm$0.003

Widths for $\Upsilon(2)$ and $\chi_b(2)$ decays

Decay	Theor(keV)	Exp(keV)
$\Upsilon(2s) \to \gamma\chi_{b2}(1p)$	2.13	2.30±0.20
$\Upsilon(2s) \to \gamma\chi_{b1}(1p)$	1.73	2.22±0.21
$\Upsilon(2s) \to \gamma\chi_{b0}(1p)$	0.87	1.22±0.15
$\Upsilon(2s) \to \gamma\eta_b(1s)$	0.013	0.013±0.04
$\chi_{b2}(2p) \to \gamma\eta_b(2s)$	18.77	13.60±1.87
$\chi_{b2}(2p) \to \gamma\eta_b(1s)$	10.27	7.44±1.02
$\chi_{b1}(2p) \to \gamma\eta_b(2s)$	16.80	17.50±2.92
$\chi_{b1}(2p) \to \gamma\eta_b(1s)$	7.68	8.00±1.33
$\chi_{b0}(2p) \to \gamma\eta_b(2s)$	11.77	-
$\chi_{b0}(2p) \to \gamma\eta_b(1s)$	1.49	-
$\chi_{b2}(1p) \to \gamma\eta_b(2s)$	33.73	-
$\chi_{b1}(1p) \to \gamma\eta_b(2s)$	29.48	-
$\chi_{b0}(1p) \to \gamma\eta_b(2s)$	19.65	-

Widths for other decays

Decay	Ours	[1]	[2]
$h_b(2p) \to \gamma\eta_b(2s)$	**20681**	**17600**	**16600**
$h_b(2p) \to \gamma\eta_b(1s)$	**16884**	**14900**	**17500**
$\Upsilon(2s) \to \gamma\eta_b(2s)$	**0.369**	**0.58**	**0.59**
$\eta_b(2s) \to \gamma\Upsilon(1s)$	**65.41**	**45**	**64**
$\chi_{b2}(1p) \to \gamma h_b(1p)$	**0.015**	**0.089**	
$\chi_{b2}(1p) \to \gamma\Upsilon(1s)$	**33731**	**39150**	**31800**
$h_b(1p) \to \gamma\chi_{b1}(1p)$	**0.050**	**0.012**	**0.0094**
$h_b(1p) \to \gamma\chi_{b0}(1p)$	**0.124**	**0.86**	**0.90**
$h_b(1p) \to \gamma\eta_b(1s)$	**39318**	**43660**	**35800**
$\Upsilon(1s) \to \gamma\eta_b(1s)$	**3.101***	**9.34**	**10**

Table: Comparison of the previsions of the theoretical widths of some radiative decays of χ_{b2}, h_b, χ_{b1}, χ_{b0} and Υ. Units are eV. The comparison data are from [1] Segovia J. et al, arXiv 1601.05093. [2] Wei-Jun Deng et al. arXiv 1607.04696. * This value is in agreement with N. Brambilla et al. Phys. Rev. D 73, 054005 (2006).


THANK YOU!

$c\bar{c}$ Mesons (1)

State	Exp	Num
$(1^1s_0)\ 0^+(0^{-+})\ \eta_c$	2978.40±1.2	2978.26
$(1^3s_1)\ 0^-(1^{--})\ J/\psi$	3096.916±.011	3097.91
$(1^3p_0)\ 0^+(0^{++})\ \chi_{c0}$	3414.75±.31	3423.88
$(1^3p_1)\ 0^+(1^{++})\ \chi_{c1}$	3510.66±.07	3502.83
$(1^1p_1)\ 0^-(1^{+-})\ h_c$	3525.41±.16	3523.67
$(1^3p_2)\ 0^+(2^{++})\ \chi_{c2}$	3556.20±.09	3555.84
$(2^1s_0)\ 0^+(0^{-+})\ \eta_c$	3637±4	3619.64
$(2^3s_1)\ 0^-(1^{--})\ \psi$	3686.09±.04	3692.91

$c\bar{c}$ Mesons (2)

State		Exp	Num
$(1^3d_1)\ 0^-(1^{--})$	ψ	3772.92±.35	3808.48
$0^+(?^{?+})$	X(3872)	3871.57± .25	
$(2^3p_1)\ 0^+(1^{++})$	χ_{c1}	-	3961.21
$0^+(?^{?+})$	X(3915)	3917.4± 2.7	
$(2^3p_2)\ 0^+(2^{++})$	χ_{c2}	3927±2.6	4003.93
$?^+(?^{??})$	X(3940)	3942± 13	
$(3^1s_0)\ 0^+(0^{-+})$	η_c	-	4064.21
$(3^3s_1)\ 0^-(1^{--})$	ψ	4039±1	4122.95
$(2^3d_1)\ 0^-(1^{--})$	ψ	4153±3	4200.51
$(4^3s_1)\ 0^-(1^{--})$	ψ	4421±4	4479.22

Table: The $c\bar{c}$ levels in MeV. σ=**1.111** GeV/fm, α=**0.435**, m_c=**1394.5** MeV. Experimental data from [PdG].

$s\bar{s}$ Mesons

State	Exp	Num
$(1^1s_0)\ 0^+(0^{-+})$	-	818.12
$(1^3s_1)\ 0^-(1^{--})\ \phi$	1019.455±.020	1019.44
$(1^3p_1)\ 0^+(1^{++})$ $\mathbf{f_1(1420)}$	1426.4±.9	1412.84
$(1^3p_2)\ 0^+(2^{++})$ $\mathbf{f'_2(1525)}$	1525±5	1525.60
$(2^3s_1)\ 0^-(1^{--})\ \phi$	1680±20	1698.41
$(1^3d_1)\ 0^-(1^{--})$ **X(1750)**	1753.5±3.8	1776.53
$(1^3d_3)\ 0^-(3^{--})$ ϕ_3**(1850)**	1854±7	1880.85
$(2^3p_2)\ 0^+(2^{++})$ $\mathbf{f_2(2010)}$	2011±70	2073.15
$(3^3s_1)\ 0^-(1^{--})\ \phi$	2175±15	2217.57

Table: The $s\bar{s}$ levels in MeV. σ=**1.34** GeV/fm, α=**0.6075**, m_s=**134.3** MeV. Experimental data from [PdG].

Bc, Bs, Ds, Mesons

State	Exp	Num
$(1^1s_0)\ 0(0^-)\ \mathbf{B_c^\pm}$	$6277\pm.006$	6277
$(1^1s_0)\ 0(0^-)\ \mathbf{B_s^0}$	$5366.77\pm.24$	5387.41
$(1^3s_1)\ 0(1^-)\ \mathbf{B_s^*}$	5415.4 ± 2.1	5434.34
$(1^3p_1)\ 0(1^+)\ \mathbf{B_{s1}(5830)^0}$	$5829.4\pm.7$	5817.80
$(1^3p_2)\ 0(2^+)\ \mathbf{B_{s2}(5840)^0}$	$5839.7\pm.6$	5829.33
$(1^1s_0)\ 0(0^-)\ \mathbf{D_s^\pm}$	$1968.49\pm.32$	1961.24
$(1^3s_1)\ 0(1^-)\ \mathbf{D_s^{*\pm}}$	$2112.3\pm.50$	2101.78
$(1^3p_0)\ 0(0^+)\ \mathbf{D_{s0}(2317)^\pm}$	$2317.8\pm.6$	2339.94
$(1^3p_1)\ 0(1^+)\ \mathbf{D_{s1}(2460)^\pm}$	$2459.6\pm.6$	2466.15
$(1^1p_1)\ 0(1^+)\ \mathbf{D_{s1}(2536)^\pm}$	$2535.12\pm.13$	2535.82
$(1^3p_2)\ 0(2^+)\ \mathbf{D_{s2}^*(2573)}$	$2571.9\pm.8$	2574.92

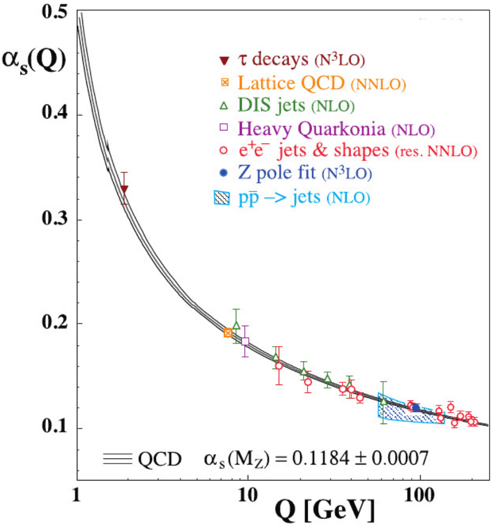
Table: The **Bc**, **Bs** and **Ds** levels in MeV. σ=**1.111**, **1.111**, **1.227** GeV/fm and α=**0.3591**, **0.3975**, **0.5348** respectively.

Light Mesons

State	Exp	Num
$(1^3s_1)\ 1^+(1^{--})\ \rho(770)$	775.49±.39	826.14
$(1^3p_0)\ 1^-(0^{++})\ a_0(980)$	980.±20	970.34
$(1^3p_1)\ 1^-(1^{++})\ a_1(1260)$	1230.±.40	1204.66
$(1^1p_1)\ 1^+(1^{+-})\ b_1(1235)$	1229.5±3.2	1274.76
$(1^3p_2)\ 1^-(2^{++})\ a_2(1320)$	1318.3±.6	1325.40
$(2^1s_0)\ 1^-(0^{-+})\ \pi(1300)$	1300±100	1337.36
$(2^3s_1)\ 1^+(1^{--})\ \rho(1450)$	1465±25	1497.63
$(1^3d_1)\ 1^+(1^{--})\ \rho(1570)$	1570$^{(*)}$	1565.42
$(3^1s_0)\ 1^-(0^{-+})\ \pi(1800)$	1812±12	1882.30
$(3^3s_1)\ 1^+(1^{--})\ \rho(1900)$	1900$^{(*)}$	2016.35
$(2^3d_1)\ 1^+(1^{--})\ \rho(2150)$	2149±17	2064.36

Table: The $u\bar{d}$ levels in MeV. σ=**1.34** GeV/fm, α=**0.656**, m_d=**6.1** MeV, m_u=**2.94** MeV. $^{(*)}$Meson Summary Table, [PdG].

The running coupling constant α_{QCD}



Ratios of α_{num} and ratios of α_S

Ratios α_{num}	Ratios α_S
$\frac{\alpha_{b\bar{b}}}{\alpha_{c\bar{c}}} = \mathbf{0.752}$	$\frac{\alpha_S\left(\chi_{b1,(1P)}\right)}{\alpha_S\left(\chi_{c0,(1P)}\right)} = \mathbf{0.754}$
$\frac{\alpha_{b\bar{b}}}{\alpha_{b\bar{c}}} = \mathbf{0.911}$	$\frac{\alpha_S\left(\chi_{b1,(1P)}\right)}{\alpha_S\left(B_c^{\pm}\right)} = \mathbf{0.914}$
$\frac{\alpha_{b\bar{c}}}{\alpha_{b\bar{s}}} = \mathbf{0.903}$	$\frac{\alpha_S\left(B_c^{\pm}\right)}{\alpha_S\left(B_s^{*}\right)} = \mathbf{0.955}$
$\frac{\alpha_{b\bar{c}}}{\alpha_{c\bar{s}}} = \mathbf{0.672}$	$\frac{\alpha_S\left(B_c^{\pm}\right)}{\alpha_S\left(D_c^{*\pm}\right)} = \mathbf{0.686}$
$\frac{\alpha_{c\bar{c}}}{\alpha_{s\bar{s}}} = \mathbf{0.716}$	$\frac{\alpha_S\left(\chi_{c0,(1P)}\right)}{\alpha_S\left(f_{1,(1P)}\right)} = \mathbf{0.714}$
$\frac{\alpha_{s\bar{s}}}{\alpha_{u\bar{d}}} = \mathbf{0.926}$	$\frac{\alpha_S\left(f_{1,(1P)}\right)}{\alpha_S\left(a_{1,(1P)}\right)} = \mathbf{0.933}$

Table: Behavior of α_{num} *vs.* α_S for average values $\Lambda_S = \mathbf{0.221}, \mathbf{0.296}, \mathbf{0.349}$ GeV for $n_f = \mathbf{5}, \mathbf{4}, \mathbf{3}$ [PdG].