Short-distance current correlators on the lattice

based on the work by the JLQCD collaboration:

Shoji Hashimoto (KEK, Sokendai)
@ CONF12,
Thessaloniki, Aug 29, 2016
JLQCD collaboration

• Members
  o Osaka: H. Fukaya, T. Onogi
  o Kyoto: S. Aoki
  o Tsukuba: Y. Taniguchi
  o RIKEN: N. Yamanaka
  o Columbia: X. Feng
  o Wuhan: A. Tomiya
  o Edinburgh: G. Cossu

• Machines @ KEK
  o Hitachi SR16000 M1
  o IBM Blue Gene /Q
Current correlator

$$\Pi(x) \equiv \langle 0 | T J(x) J(0) | 0 \rangle$$

**Long distances**
- Hadron spectrum from $\exp(-mt)$
- non-perturbative, main use of LQCD

**Short distances**
- perturbation theory + power corrections (= OPE)
- $\alpha_s$, quark mass, ...

LQCD can calculate the both.

HPQCD (2008~) achieved precision $m_c$ determination.
Lattice QCD

• Ab initio calculation of QCD
  → Use as an “experimental” facility of QCD

• Limitation due to finite lattice spacing
  o In order to match perturbation theory,
    \[ a \ll x \ll \Lambda_{QCD}^{-1} \]

To avoid large discretization effects
To avoid too large non-perturbative effects

Need sufficiently small \( a \). How small? See the data.
Observables at short distances

- Spectral function $\rho(s) \propto \text{Im} \tilde{\Pi}(s)$
  - available, e.g. from hadronic $\tau$ decays (ALEPH, 2013/14)

- Resonances at low energy; perturbative at high energy
- Testing QCD for the whole region? Yes, should be done.
On the coordinate space…

\[ \Pi_{V/A}(x) = \frac{3}{8\pi^4} \int_{0}^{\infty} ds \ s^{3/2} \rho_{V/A}(s) \frac{K_1(\sqrt{s}|x|)}{|x|} \]

Schafer, Shuryak, 2001
Lattice calculation

Tomii @ Lattice 2016
- on 2+1-flavor lattices with Mobius domain-wall fermion
- after taking the continuum limit with lattices at \( a = 0.08, 0.055, 0.044 \) fm.

V+A channel:

\[
R_{V+A}(x) = \frac{\Pi_V(x) + \Pi_A(x)}{2\Pi_V^{\text{free}}(x)\big|_{m_a=0}}
\]

Good agreement at \(|x| > 0.3\) fm.
Lattice calculation

Tomii @ Lattice 2016
- on 2+1-flavor lattices with Mobius domain-wall fermion
- after taking the continuum limit with lattices at \( a = 0.08, 0.055, 0.044 \) fm.

V–A channel:
- vanishes in the perturbative regime.

Good agreement at short distances.
Exp, Pert, Lattice

• Different sources of info:
  o Experimental data:
    • $e^+e^-$ scattering, $\tau$ decays, ...
  o Perturbation theory:
    • useful at short distances
    • available up to $O(\alpha_s^3)$ or even $O(\alpha_s^4)$
  o Lattice calculation:
    • useful for any distances, in principle.
    • continuum limit is now a common practice

• Consistency
  o provides a test of QCD/Lattice
  o determination of parameters of QCD
Charmonium correlator

- Available from $e^+e^-$

- Optical theorem:

$$ (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi_V^{(1)}(q^2) = i \int d^4 x \ e^{i p x} \langle 0 | T[j_\mu(x) j_\nu(0)] | 0 \rangle $$

$$ \frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n (\Pi(q^2))_{q^2=0} = \frac{1}{12\pi Q_f^2} \int ds \frac{1}{s^{n+1}} R(s) e^+ e^- \rightarrow \text{hadron} $$
Temporal moments on the lattice

- Coordinate space

\[ i \int dx \frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n e^{iqt} \rightarrow a^4 \sum_x t^{2n} \quad \text{(temporal moments)} \]

- On the Euclidean lattice,

\[ G_V(t) = a^6 \sum_x \langle 0 | j_k(x,t) j_k(0,0) | 0 \rangle, \quad G_{V,n} = \sum_t (t/a)^n G_V(t) \]

- \( G_V(t) \) represents a J/ψ correlator, \( \sim \exp(-m_{J/ψ} t) \), plus its excited states, continuum, etc.
Sum to define the moment

\[ G_n = \sum_t \left( \frac{t}{a} \right)^n G(t) \]
Sum to define the moment

\[ \Lambda_{QCD} \sim 300 \text{ MeV} \]

\[ a m_{\eta_c} \sim 0.6636 \]
Lattices with domain-wall

New set of lattice ensembles by JLQCD:

- 2+1 flavors of dynamical quarks described by domain-wall fermions
  - Good chiral symmetry on the lattice, $m_{\text{res}} \sim 1$ MeV or less.

- Fine lattices: $1/a \sim 2.4, 3.6, 4.6$ GeV
  - Enough resolution to treat charm

- Sea quark mass coverage
  - $m_\pi = 230 \sim 500$ MeV
  - sandwich $m_s$
# Lattices with domain-wall

<table>
<thead>
<tr>
<th>$\beta = 4.17$, $1/a \sim 2.4$ GeV, $32^3 \times 64$ (x12)</th>
<th>$\beta = 4.35$, $1/a \sim 3.6$ GeV, $48^3 \times 96$ (x8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{ud}$</td>
<td>$m_{ud}$</td>
</tr>
<tr>
<td>$m_{\pi}$</td>
<td>$m_{\pi}$</td>
</tr>
<tr>
<td>[MeV]</td>
<td>[MeV]</td>
</tr>
<tr>
<td>$m_s = 0.030$</td>
<td></td>
</tr>
<tr>
<td>0.007</td>
<td>310</td>
</tr>
<tr>
<td>0.012</td>
<td>410</td>
</tr>
<tr>
<td>0.019</td>
<td>510</td>
</tr>
<tr>
<td>$m_s = 0.040$</td>
<td></td>
</tr>
<tr>
<td>0.0035</td>
<td>230</td>
</tr>
<tr>
<td>0.0035</td>
<td>230</td>
</tr>
<tr>
<td>$(48^3 \times 96)$</td>
<td></td>
</tr>
<tr>
<td>0.007</td>
<td>320</td>
</tr>
<tr>
<td>0.012</td>
<td>410</td>
</tr>
<tr>
<td>0.019</td>
<td>510</td>
</tr>
<tr>
<td>$\beta = 4.47$, $1/a \sim 4.6$ GeV, $64^3 \times 128$ (x8)</td>
<td></td>
</tr>
<tr>
<td>$m_s = 0.030$</td>
<td>$\sim 300$</td>
</tr>
</tbody>
</table>

S. Hashimoto (KEK)
Lattice versus “exp+pheno”

Vector channel:

“exp” data from
- Dehnadi et al. (2015)
- Kuhn et al. (2007)

Lattice data after (perturbative) corrections of charm quark loop.

\[
R_{2l+2} = m_{J/\psi} \left( \frac{M_l}{g_{2l+2}} \right)^{1/2l};
\]

\[
n = 2l + 2
\]
Determination of $m_c$ and $\alpha_s$

- Following the method of the pioneering work by the HPQCD + Karlsruhe group (2008~).
  - Allison et al., PRD78, 054513 (2008); McNeile et al., PRD82, 034512 (2010); Chakraborty et al., PRD91, 054508 (2015).

Lattice data after continuum extrap

Continuum perturbation theory known to $\alpha_s^3$.

$$R_n = \frac{am_n^{(exp)}}{2am_c(\mu)} r_n(\mu; m_c(\mu), \alpha_s(\mu))$$

- Solve equations with different $n$’s ($n = 6, 8, 10$, in this work).
- Use the pseudo-scalar density correlator for both lattice and pert. Perturbative coefficients known to $\alpha_s^3$. 
LHS: Continuum extrapolation of the lattice data

Sea quark mass dependence is invisible in this scale.
Fit with a linear function in $m_u + m_d + m_s$. 
LHS: Continuum extrapolation of the lattice data

Larger slope than those of HPQCD with Highly Improved Staggered Quark (HISQ). Fine lattices are crucial.
RHS: Estimate of the perturbative truncation error.

\[
\frac{r_n(\mu)}{m_c^{\overline{MS}}(\mu)}
\]

\(r_n(\mu)\): should be independent of \(\mu\).

Included up to \(O(\alpha_s^3)\). Also included the variation with \(\mu_m \neq \mu_\alpha\).

\[
(r_6)^2 = 1 + \left(3.9 + 2.0 \log \frac{m_c(\mu)^2}{\mu^2}\right) \frac{\alpha_s}{\pi} + \left(13.6 + 3.0 \log \frac{m_c(\mu)^2}{\mu^2} - 0.08 \left(\log \frac{m_c(\mu)^2}{\mu^2}\right)^2\right) \left(\frac{\alpha_s}{\pi}\right)^2 + \left(13.2 + 14.2 \log \frac{m_c(\mu)^2}{\mu^2} + 1.03 \left(\log \frac{m_c(\mu)^2}{\mu^2}\right)^2 + 0.06 \left(\log \frac{m_c(\mu)^2}{\mu^2}\right)^3\right) \left(\frac{\alpha_s}{\pi}\right)^3
\]
\[ R_n = \frac{m_{\eta_c}^{\text{exp}}}{2m_c^{\overline{\text{MS}}}} r_n(m_c^{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \]

Determine \( m_c(\mu), \alpha_s(\mu), \langle G_{\mu\nu}^2 \rangle \) with 3 moments.

\( \alpha_s(\mu) \) is consistent, but its error is not competitive.

\( m_c(3 \text{ GeV}) = 1.003(10) \text{ GeV} \)
## Error budgets

<table>
<thead>
<tr>
<th>$m_c(3\text{GeV})$ [GeV]</th>
<th>pert</th>
<th>$t_0^{1/2}$</th>
<th>stat</th>
<th>$O(a^4)$</th>
<th>vol</th>
<th>$m_{\eta_c}^{\text{exp}}$</th>
<th>disc</th>
<th>EM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0033(96)</td>
<td>(77)</td>
<td>(49)</td>
<td>(4)</td>
<td>(30)</td>
<td>(4)</td>
<td>(3)</td>
<td>(4)</td>
<td>(6)</td>
</tr>
<tr>
<td>$\alpha_s(3\text{GeV})$</td>
<td>(120)</td>
<td>(32)</td>
<td>(2)</td>
<td>(26)</td>
<td>(1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1)</td>
</tr>
</tbody>
</table>

### Notes:
- Dominant = truncation of perturbative expansion
- Lattice scale $\Delta a \sim 1\%$
- Discretization effect

---

S. Hashimoto (KEK)
Comparison

\begin{figure}
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{Comparison of different experimental and theoretical results for the mass of the charm quark.}
\end{figure}

\begin{itemize}
\item HPQCD 14
\item ETM a 14
\item ETM b 14
\item This work
\item Maezawa et al 16
\item HPQCD 10
\item HPQCD 8
\item \(\chi QCD\) 14
\item ALPHA 13
\item ETM a 11
\item ETM b 11
\item PDG
\end{itemize}
Summary

• Test of LQCD at short distances
  o For light-hadron and charmonium correlators.
  o Fine lattices ($1/a \sim 3-4$ GeV) are necessary to keep the discretization errors under control.
  o Lattice results are consistent with the experimental data, also at short distances.

• Determination of $m_c(\mu)$
  o LQCD may provide an extra input (pseudo-scalar channel) to feed in the perturbative analysis.
  o $m_c$ and $\alpha_s$ obtained with similar size of error as HPQCD.
backup
LHS: Test of the discretization effect

Hyperfine splitting $J/\psi - \eta_c$ is sensitive to the discretization effect.

exp: 111(2) MeV

consistent after extrapolation
RHS: Non-perturbative correction

\[ r_{n}^{n-4} = \frac{1}{C_{n/2-1}^{(0)}} \left( C_{n/2-1} + \frac{16\pi^2}{3} \frac{\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle}{(2m_{OS})^4} \left( a_{n/2} + \frac{\alpha_s}{\pi} c_{n/2} \right) \right) \]

perturbative expansion

 gluon condensate from OPE
 No precise determination available.

\[ \langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle \sim 0.00-0.02 \text{ GeV}^4 \]

Keep it as a free parameter and determine with the charmonium moments. Result:

\[ \langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle \sim 0.00(1) \text{ GeV}^4 \]
From the vector channel: