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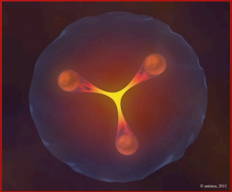

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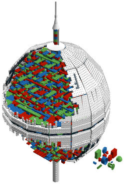
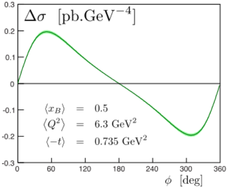
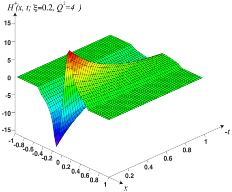
Concurrent approaches to Generalized Parton Distribution modeling: the pion's case







Confinement 12 | Hervé MOUTARDE

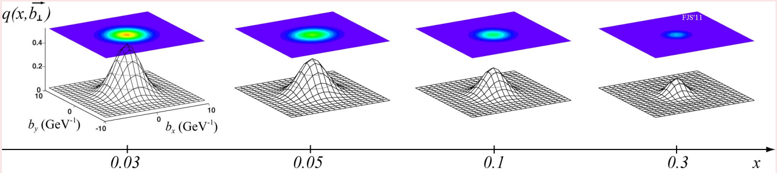
Aug. 30$^{\text{th}}$, 2016

Motivations.

3D imaging of nucleon's partonic content but also...

- Correlation of the **longitudinal momentum** and the **transverse position** of a parton in the nucleon.
- Insights on:
  - **Spin** structure,
  - **Energy-momentum** structure.
- **Probabilistic interpretation** of Fourier transform of GPD$(x, \xi = 0, t)$ in **transverse plane**.

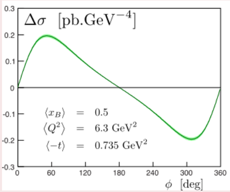
Transverse plane density (Goloskokov and Kroll model)



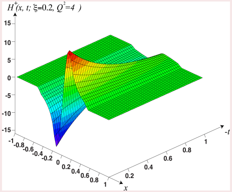
Imaging the nucleon. How?

Extracting GPDs is not enough...Need to extrapolate!

1. Experimental data fits



2. GPD extraction



3. Nucleon imaging

Images from Guidal et al.,
Rept. Prog. Phys. 76 (2013) 066202

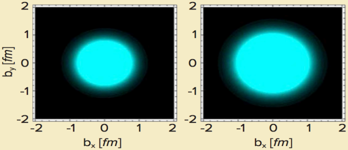
Reaching for the Horizon

The 2015 Long Range Plan for Nuclear Science

Sidebar 2.2: The First 3D Pictures of the Nucleon

A computed tomography (CT) scan can help physicians pinpoint minute cancer tumors, diagnose tiny broken bones, and spot the early signs of osteoporosis. Now physicists are using the principles behind the procedure to peer at the inner workings of the proton. This breakthrough is made possible by a relatively new concept in nuclear physics called generalized parton distributions.

An intense beam of high-energy electrons can be used



1. **Extract** $H(x, \xi, t, \mu_F^{\mathrm{ref}})$ from experimental data.
2. **Extrapolate** to vanishing skewness $H(x, 0, t, \mu_F^{\mathrm{ref}})$.
3. **Extrapolate** $H(x, 0, t, \mu_F^{\mathrm{ref}})$ up to infinite t.
4. **Compute** 2D Fourier transform in transverse plane:

$$H(x, b_\perp) = \int_0^{+\infty} \frac{\mathrm{d}\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp) H(x, 0, -\Delta_\perp^2)$$

5. **Propagate** uncertainties.
6. **Control** extrapolations with an accuracy matching that of experimental data with **sound** GPD models.

- Important topic for several **past, existing and future** experiments: H1, ZEUS, HERMES, CLAS, CLAS12, JLab Hall A, COMPASS, EIC, ...
- GPD modeling / parameterizing is an essential ingredient for the interpretation of experimental data.
- **Recent applications** of the (symmetry-preserving) Dyson-Schwinger framework to **hadron structure studies**.

Overview.

GPDs as a scalpel-like probe of (nonperturbative) hadron structure.

- Important topic for several **past, existing and future** experiments: H1, ZEUS, HERMES, CLAS, CLAS12, JLab Hall A, COMPASS, EIC, ...
- GPD modeling / parameterizing is an essential ingredient for the interpretation of experimental data.
- **Recent applications** of the (symmetry-preserving) Dyson-Schwinger framework to **hadron structure studies**.
- Here develop **pion GPD model** for simplicity.
- No planned experiment on pion GPDs but existing proposal of DVCS on a virtual pion.

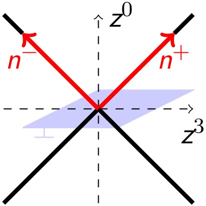
Amrath *et al.*, Eur. Phys. J. **C58**, 179 (2008)

- Important topic for several **past, existing and future** experiments: H1, ZEUS, HERMES, CLAS, CLAS12, JLab Hall A, COMPASS, EIC, ...
- GPD modeling / parameterizing is an essential ingredient for the interpretation of experimental data.
- **Recent applications** of the (symmetry-preserving) Dyson-Schwinger framework to **hadron structure studies**.

GPDs: Theoretical Framework

$$H^q_\pi(x,\xi,t) =$$
$$\frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\middle|\, \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle|\pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**

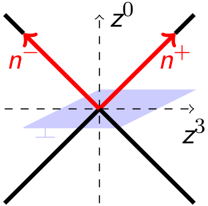
$$H^q(x,0,0) = q(x)$$

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**

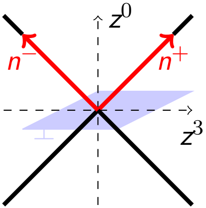
$$\int_{-1}^{+1} dx\, H^q(x,\xi,t) = F_1^q(t)$$

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H_\pi^q(x, \xi, t) \quad = \quad \frac{1}{2} \int \frac{\mathrm{d}z^-}{2\pi} \, e^{ixP^+ z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+ = 0 \\ z_\perp = 0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

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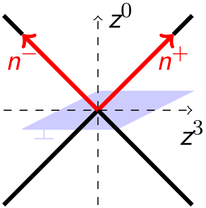


References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.
- H^q is **real** from hermiticity and time-reversal invariance.

Polynomiality

$$\int_{-1}^{+1} dx\, x^n H^q(x, \xi, t) = \text{polynomial in } \xi$$

Polynomiality

Lorentz covariance

- **Polynomiality**

Lorentz covariance

- **Positivity**

$$H^q(x, \xi, t) \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

Spin-0 Generalized Parton Distribution.

Not so simple properties.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

Relativistic quantum mechanics

Spin-0 Generalized Parton Distribution.

Not so simple properties.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

$$H^q(x, \xi = 1, t = 0) = \frac{1}{2}\phi^q_\pi\left(\frac{1+x}{2}\right)$$

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

Dynamical chiral symmetry breaking

Spin-0 Generalized Parton Distribution.

Not so simple properties.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

How can we implement *a priori* these theoretical constraints?

- There is no known GPD parameterization **relying only on first principles**.
- In the following, focus on **polynomiality** and **positivity**.

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x\, x^m H^q(x, \xi, t) = \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

$$\text{linear combination of } (P^+)^{m+1-k}(\Delta^+)^k \text{ for } 0 \leq k \leq m+1$$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x, \xi, t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) \,.$$

Double Distributions.

A convenient tool to encode GPD properties.

GPD models: Concurrent approaches

Introduction

Theoretical framework

Definition

Polynomiality

Positivity

Schwinger Dyson GPD

Diagrams

Preserving symmetries

Extension

Radon transform

Covariant extension

PARTONS

Computing chain

Example

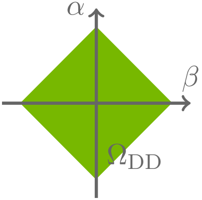
Conclusions

Appendix

- Define Double Distributions F^q and G^q as matrix elements of **twist-2 quark operators**:

$$\left\langle P+\frac{\Delta}{2}\left|\bar{q}(0)\gamma^{\{\mu}i\overleftrightarrow{D}^{\mu_1}\dots i\overleftrightarrow{D}^{\mu_m\}}q(0)\right|P-\frac{\Delta}{2}\right\rangle = \sum_{k=0}^{m}\binom{m}{k}$$

$$\left[F^q_{mk}(t)2P^{\{\mu}-G^q_{mk}(t)\Delta^{\{\mu}\right]P^{\mu_1}\dots P^{\mu_{m-k}}\left(-\frac{\Delta}{2}\right)^{\mu_{m-k+1}}\dots\left(-\frac{\Delta}{2}\right)^{\mu_m\}}$$



with

$$F^q_{mk} = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}F^q(\beta,\alpha)$$

$$G^q_{mk} = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}G^q(\beta,\alpha)$$

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Rev. **D59**, 014030 (1999)
Radyushkin, Phys. Lett. **B449**, 81 (1999)

Double Distributions.

Relation to Generalized Parton Distributions.

- Representation of GPD:

$$H^q(x,\xi,t) = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\big(F^q(\beta,\alpha,t) + \xi\, G^q(\beta,\alpha,t)\big)$$

- Support property: $x \in [-1,+1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.
- **Gauge**: any representation (F^q, G^q) can be recast in one representation with a single DD f^q:

$$H^q(x,\xi,t) = x \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha\, f^q_{\mathrm{BMKS}}(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Belitsky *et al.*, Phys. Rev. **D64**, 116002 (2001)

Positivity.

A consequence of the positivity of the nom in a Hilbert space.

- Identify the matrix element defining a GPD as an **inner product** of two different states.
- Apply Cauchy-Schwartz inequality, and identify PDFs at specific kinematic points, *e.g.*:

$$|H^q(x,\xi,t)| \leq \sqrt{\frac{1}{1-\xi^2} q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- This procedures yields **infinitely many inequalities** stable under LO evolution.

Pobylitsa, Phys. Rev. **D66**, 094002 (2002)

- The **overlap representation** guarantees *a priori* the fulfillment of positivity constraints.

Overlap representation.

A first-principle connection with Light Front Wave Functions.

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x \mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta, j} \int [\mathrm{d}\bar{x} \mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \big(\psi_N^{(\beta,\lambda)}\big)^* (\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)}(\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

  with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

  Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWFs.

Overlap representation.

Advantages and drawbacks.

- Physical picture.
- Positivity relations are fulfilled **by construction**.
- Implementation of **symmetries of N-body problems**.

What is not obvious anymore

What is *not* obvious to see from the wave function representation is however the **continuity of GPDs at** $x = \pm\xi$ and the **polynomiality** condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies **nontrivial relations between the wave functions** for the different Fock states relevant in the two regions. An *ad hoc* Ansatz for the wave functions would **almost certainly lead** to GPDs that **violate the above requirements**.

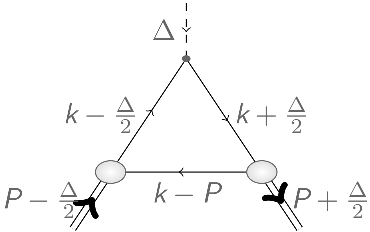
Diehl, Phys. Rept. **388**, 41 (2003)

GPDs in the Dyson-Schwinger and Bethe-Salpeter Approach

GPDs in the rainbow ladder approximation.

Evaluation of triangle diagrams.

$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle$$

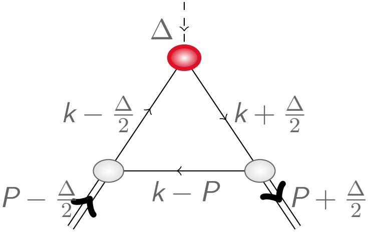


- Compute **Mellin moments** of the pion GPD H.

GPDs in the rainbow ladder approximation.

Evaluation of triangle diagrams.

$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$

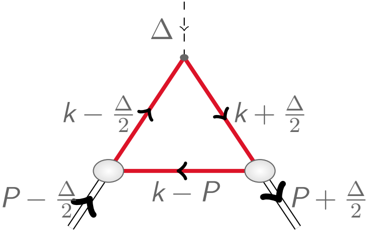


- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.

GPDs in the rainbow ladder approximation.

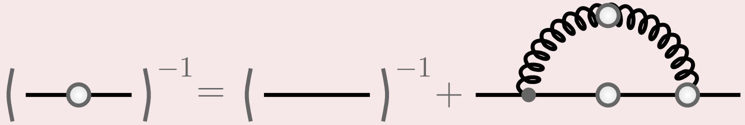
Evaluation of triangle diagrams.

$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i \overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

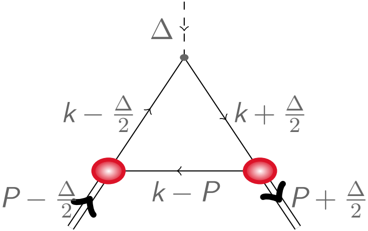
Dyson - Schwinger equation



GPDs in the rainbow ladder approximation.

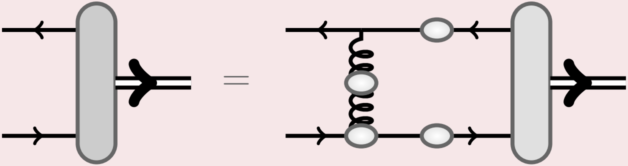
Evaluation of triangle diagrams.

$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

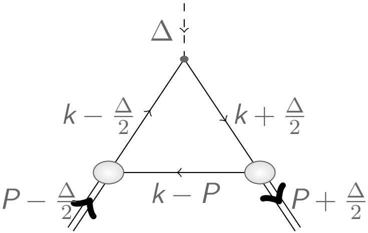
Bethe - Salpeter equation



GPDs in the rainbow ladder approximation.

Evaluation of triangle diagrams.

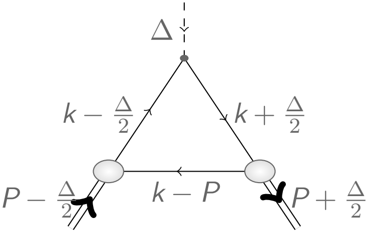
$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.
- **Nonperturbative** modeling.

- Most GPD properties **satisfied by construction**.

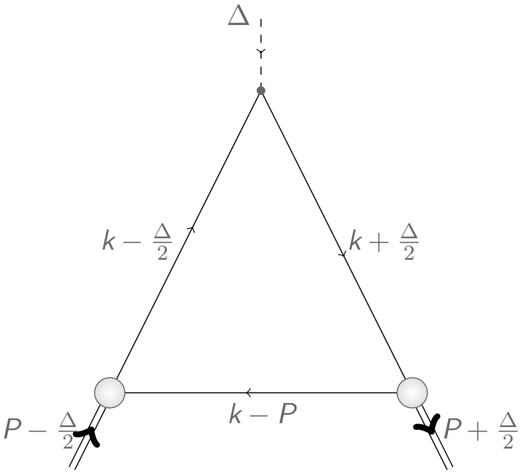
$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$

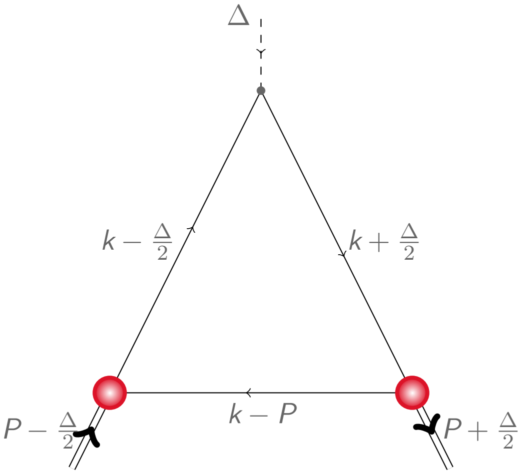


- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.
- **Nonperturbative** modeling.

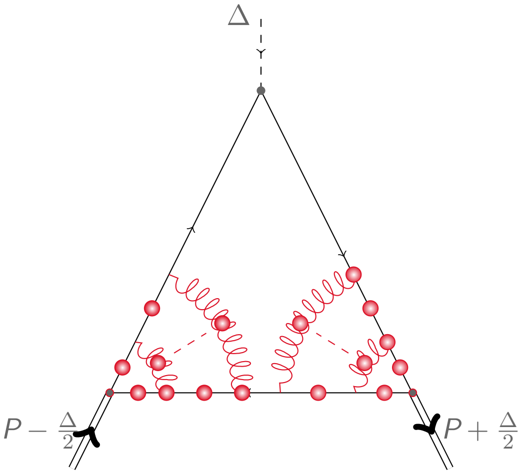
- Most GPD properties **satisfied by construction**.
- Also compute crossed triangle diagram.

Mezrag *et al.*, arXiv:1406.7425 [hep-ph] and Phys. Lett. **B741**, 190 (2015)

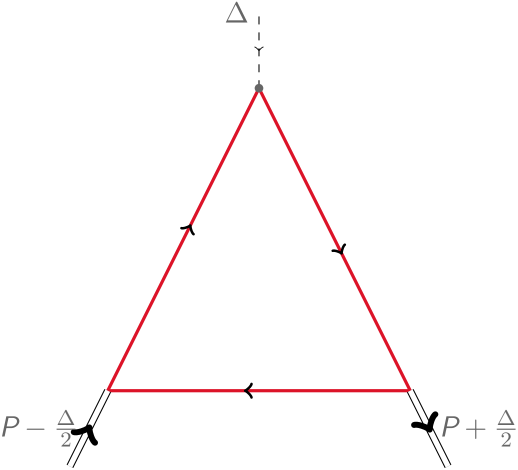

Δ
$k - \frac{\Delta}{2}$
$k + \frac{\Delta}{2}$
$k - P$
$P - \frac{\Delta}{2}$
$P + \frac{\Delta}{2}$



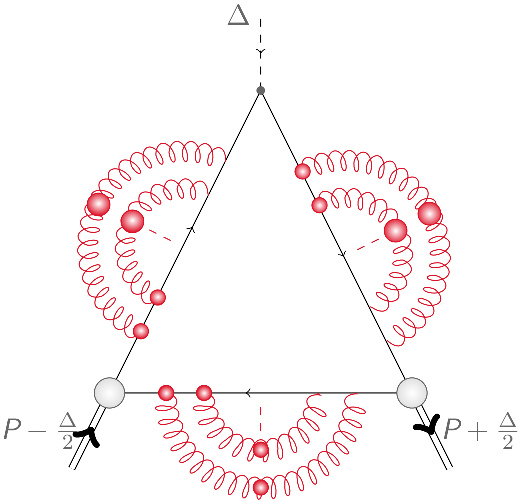
- Bethe-Salpeter vertex.



- Bethe-Salpeter vertex.



- Bethe-Salpeter vertex.
- Dressed quark propagator.



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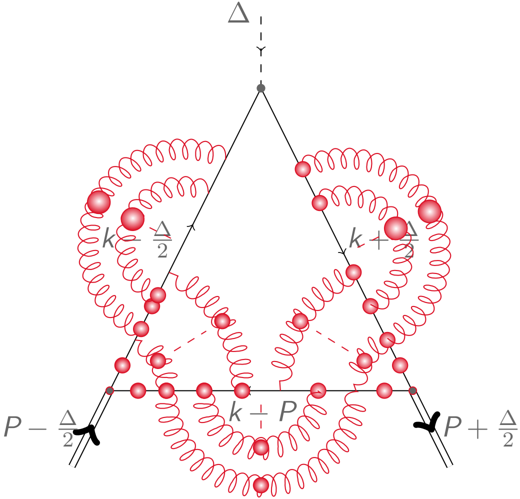

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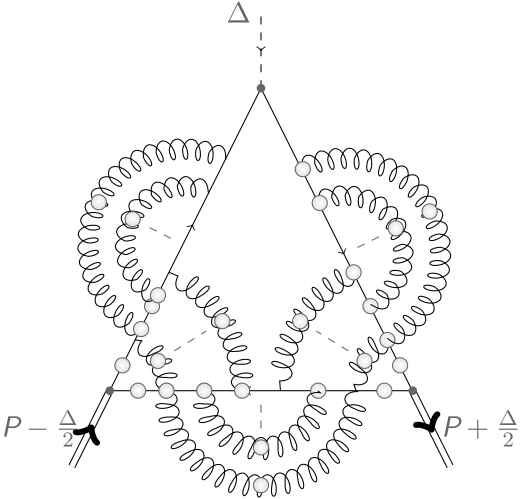

- Bethe-Salpeter vertex.
- Dressed quark propagator.
- Much more than tree level perturbative diagram!
- Enable description of **non perturbative phenomena**.

- **Polynomiality** from Poincaré covariance.

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Mezrag *et al.*, Phys. Lett. **B741**, 190 (2015)

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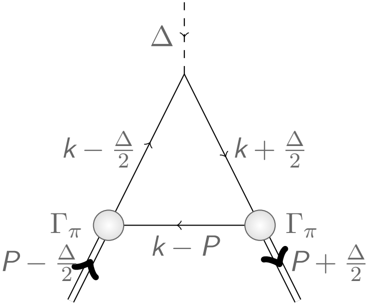
Mezrag *et al.*, Phys. Lett. **B741**, 190 (2015)

- Mellin moments.



Symmetry-preserving truncation.

Most of the GPD properties are obtained *a priori*.

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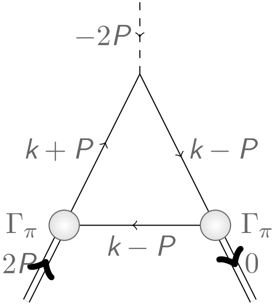
Mezrag *et al.*, Phys. Lett. **B741**, 190 (2015)



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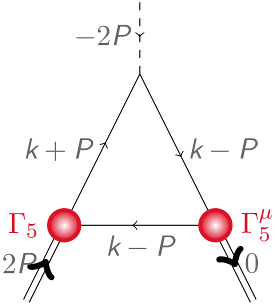

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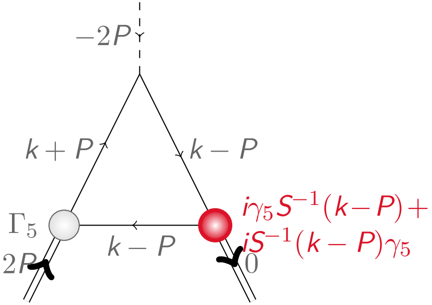

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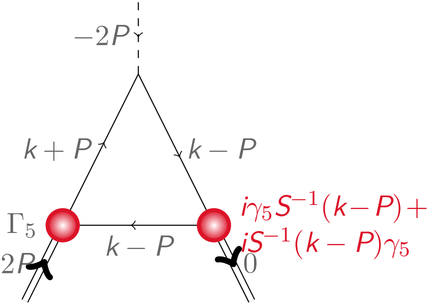
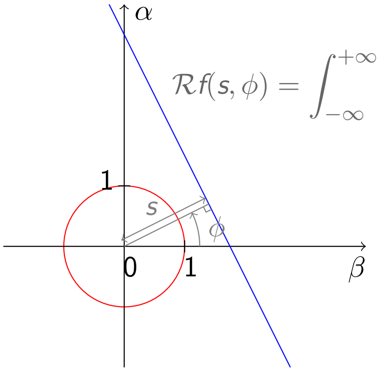


- Mellin moments.
- Soft pion kinematics.
- Axial and axial vector vertices Γ_5, Γ_5^μ in chiral limit.
- Axial-vector Ward identity.
- Recover pion DA Mellin moments.

Extension: Implementing Positivity and Polynomiality



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s,\phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha\, f(\beta,\alpha)\delta(s - \beta\cos\phi - \alpha\sin\phi)$$

and:

$$\mathcal{R}f(-s,\phi) = \mathcal{R}f(s,\phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Belistky *et al.* gauge

$$\frac{\sqrt{1+\xi^2}}{x} H(x,\xi) = \mathcal{R}f_{\mathrm{BMKS}}(s,\phi)$$

The range of the Radon transform.

The polynomiality property a.k.a. the Ludwig-Helgason condition.

- The Mellin moments of a Radon transform are **homogeneous polynomials** in $\omega = (\sin\phi, \cos\phi)$.
- The converse is also true:

Theorem (Hertle, 1983)

*Let $g(s, \omega)$ an even compactly-supported distribution. Then g is itself the Radon transform of a compactly-supported distribution if and only if the **Ludwig-Helgason consistency condition** hold:*

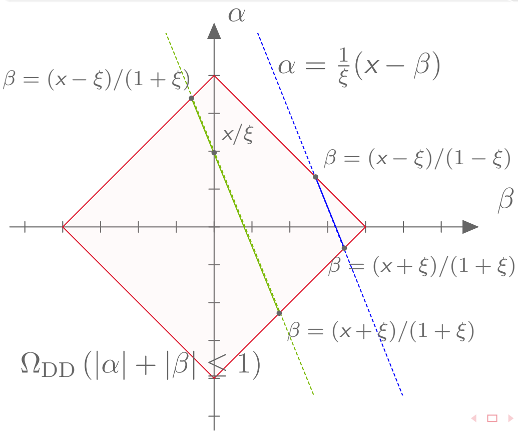
(i) *g is C^∞ in ω,*

(ii) *$\int \mathrm{d}s\, s^m g(s, \omega)$ is a homogeneous polynomial of degree m for all integer $m \geq 0$.*

- Double Distributions and the Radon transform are the **natural solution** of the polynomiality condition.

DGLAP and ERBL regions

$$(x, \xi) \in \text{DGLAP} \Leftrightarrow |s| \geq |\sin\phi| \,,$$
$$(x, \xi) \in \text{ERBL} \Leftrightarrow |s| \leq |\sin\phi| \,.$$

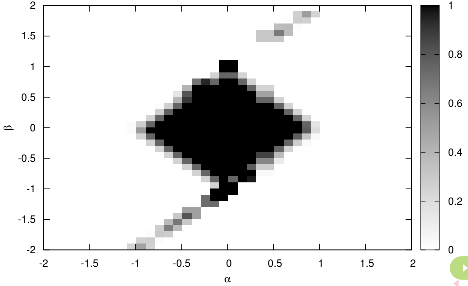


- Each point (β, α) with $\beta \neq 0$ contributes to **both** DGLAP and ERBL regions.
- Expressed in **support theorem**.

Ill-posedness in the sense of Hadamard.

A first glimpse at the inverse Radon transform.

- Numerical evaluation ***almost unavoidable*** (polar vs cartesian coordinates).
- Ill-posedness by **lack of continuity**.
- The **unlimited** Radon inverse problem is **mildly** ill-posed while the **limited** one is **severely** ill-posed.
- Careful selection of **algorithms** and **numerical methods**.



Mezrag PhD dissertation

See more on inverse Radon transform.

PARTONS Project

PARtonic
Tomography
Of
Nucleon
Software

Experimental data and phenomenology	Full processes
Computation of amplitudes	Small distance contributions
First principles and fundamental parameters	Large distance contributions

Computing chain design.

Differential studies: physical models and numerical methods.

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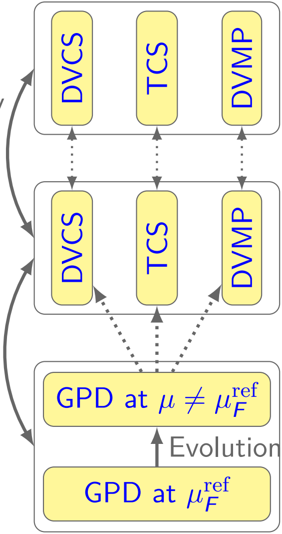
Experimental data and phenomenology

Computation of amplitudes

First principles and fundamental parameters



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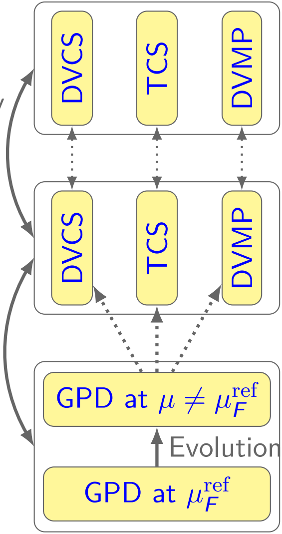

- Many observables.
- Kinematic reach.

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Need for modularity

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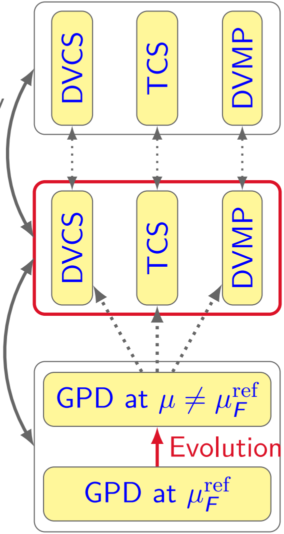
First principles and fundamental parameters



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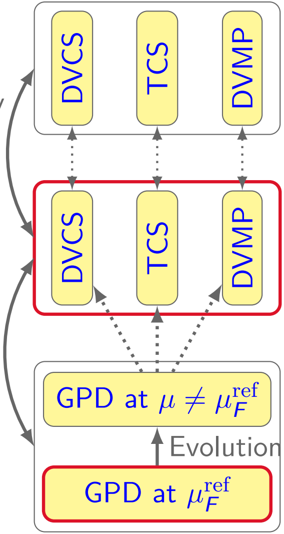

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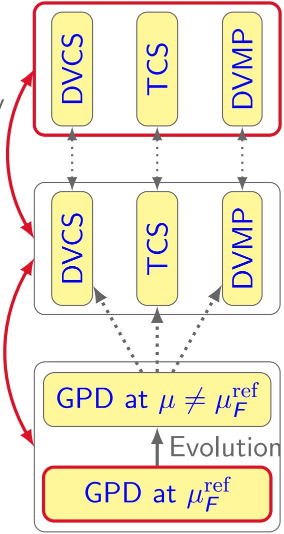

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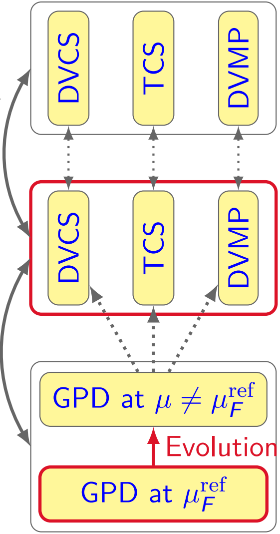

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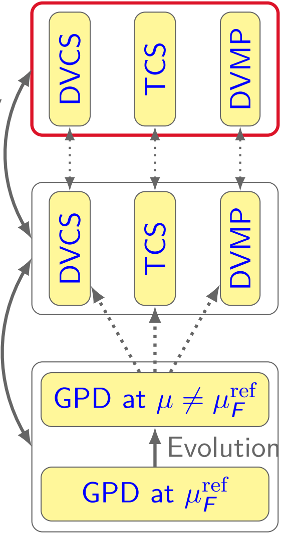

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Towards the first release.

Currently: tests, benchmarking, documentation, tutorials.

- 3 stages:
  1. Design.
  2. Integration and validation.
  3. Benchmarking and production.
- Flexible software architecture.

  B. Berthou *et al.*, *PARTONS: a computing platform for the phenomenology of Generalized Parton Distributions*

  arXiv:1512.06174
- 1 new physical development = 1 new module.
- *Aggregate* **knowledge** and **know-how**:
  - Models
  - Measurements
  - Numerical techniques
  - Validation
- What *can* be automated *will be* automated.

CFF computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneCFF.xml

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario date="2016-08-28"
    description="Computing␣a␣convol␣coeff␣function␣from␣a␣GPD␣model">
    <task service="ConvolCoeffFunctionService" method="
computeWithGPDModel">
        <kinematics type="DVCSConvolCoeffFunctionKinematic">
            <param name="xi" value="0.5" />
            <param name="t" value="-0.1346" />
            <param name="Q2" value="1.5557" />
            <param name="MuF2" value="4" />
            <param name="MuR2" value="4" />
        </kinematics>
        <computation_configuration>
            <module type="GPDModule">
                <param name="className" value="GK11Model" />
            </module>
            <module type="DVCSConvolCoeffFunctionModule">
                <param name="className" value="DVCSCFFModel" />
                <param name="qcd_order_type" value="LO" />
            </module>
        </computation_configuration>
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        </kinematics>
        <computation_configuration>
            <module type="GPDModule">
                <param name="className" value="GK11Model" />
            </module>
            <module type="DV
                <param name=
                <param name=
            </module>
        </computation_configu
    </task>
```

$$
\begin{aligned}
\mathcal{H} &= 1.47722 + 1.76698\ i \\
\mathcal{E} &= 0.12279 + 0.512312\ i \\
\widetilde{\mathcal{H}} &= 1.54911 + 0.953728\ i \\
\widetilde{\mathcal{E}} &= 18.8776 + 3.75275\ i
\end{aligned}
$$

Conclusions

- **Nonperturbative** computation of GPDs, DDs, LFWFs,...from Dyson-Schwinger equations.
- **Explicit check** of several theoretical constraints, including polynomiality, support property and soft pion theorem.
- **Systematic** procedure to construct GPD models from any "reasonable" Ansatz of LFWFs.
- **Characterization** of the **existence** and **uniqueness** of the extension from the DGLAP to the ERBL region.
- Development of the platform PARTONS for **phenemonology** and **theory** purposes.
- Numerical tests *in progress*.
- **First release** of PARTONS in Fall 2016!

Appendix

Implementing Lorentz covariance.

Extend an overlap in the DGLAP region to the whole GPD domain.

For **any model of LFWF**, one has to address the following three questions:

1. Does the extension exist?
2. If it exists, is it unique?
3. How can we compute this extension?

Work in progress!

Theorem

Let f be a compactly-supported locally summable function defined on $\mathbb{R}^2$ and $\mathcal{R}f$ its Radon transform.
Let $(s_0, \omega_0) \in \mathbb{R} \times S^1$ and U_0 an open neighborhood of ω_0 such that:

$$\textit{for all } s > s_0 \textit{ and } \omega \in U_0 \quad \mathcal{R}f(s, \omega) = 0 \; .$$

Then $f(\aleph) = 0$ on the half-plane $\langle \aleph \,|\omega_0 \rangle > s_0$ of $\mathbb{R}^2$.

Consider a GPD H being zero on the DGLAP region.

- Take ϕ_0 and s_0 *s.t.* $\cos\phi_0 \neq 0$ and $|s_0| > |\sin\phi_0|$.
- Neighborhood U_0 of ϕ_0 *s.t.* $\forall \phi \in U_0 \quad |\sin\phi| < |s_0|$.
- The underlying DD f has a zero Radon transform for all $\phi \in U_0$ and $s > s_0$ (DGLAP).
- Then $f(\beta, \alpha) = 0$ for all $(\beta, \alpha) \in \Omega_{\mathrm{DD}}$ with $\beta \neq 0$.
- Extension **unique** up to adding a **D-term**: $\delta(\beta) D(\alpha)$.

A discretized problem

Consider $N+1$ Hilbert spaces H, H_1, ..., H_N, and a family of continuous surjective operators $R_n : H \to H_n$ for $1 \leq n \leq N$.
Being given $g_1 \in H_1$, ..., $g_n \in H_n$, we search f solving the following system of equations:

$$R_n f = g_n \quad \text{for } 1 \leq n \leq N$$

Fully discrete case

Assume f piecewise-constant with values f_m for $1 \leq m \leq M$.
For a collection of lines $(L_n)_{1 \leq n \leq N}$ crossing Ω_{DD}, the Radon transform writes:

$$g_n = \mathcal{R}f = \int_{L_n} f = \sum_{m=1}^{M} f_m \times \mathrm{Measure}(L_n \cap C_m) \quad \text{for } 1 \leq n \leq N$$

Kaczmarz algorithm

Denote P_n the orthogonal projection on the *affine* subspace $R_n f = g_n$. Starting from $f^0 \in H$, the sequence defined iteratively by:

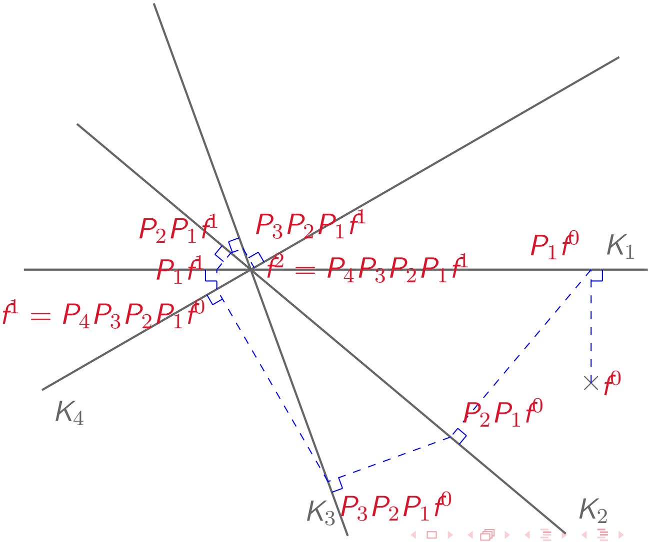
$$f^{k+1} = P_N P_{N-1} \dots P_1 f^k$$

converges to the solution of the system.
The convergence is exponential if the projections are randomly ordered.

Strohmer and Vershynin, Jour. Four. Analysis and Appl. **15**, 437 (2009)

Computation of the extension.

Numerical evaluation of the inverse Radon transform (2/3).



And if the input data are inconsistent?

- Instead of solving $g = \mathcal{R}f$, find f such that $\|g - \mathcal{R}f\|_2$ is **minimum**.
- The solution **always exists**.
- The input data are **inconsistent** if $\|g - \mathcal{R}f\|_2 > 0$.

Relaxed Kaczmarz algorithm

Let $\omega \in]0, 2[$ and:

$$P_n^\omega = (1 - \omega)\,\mathrm{Id}_H + \omega P_n \qquad \text{for } 1 \leq n \leq N$$

Write:

$$RR^\dagger = (R_i R_j^\dagger)_{1 \leq i,j \leq N} = D + L + L^\dagger$$

where D is diagonal, and L is lower-triangular with zeros on the diagonal.

Theorem

Let $0 < \omega < 2$. *For* $f^0 \in \mathrm{Ran}\ R^\dagger$ *(e.g.* $f^0 = 0$*), the Kaczmarz method with relaxation converges to the unique solution* $f^\omega \in \mathrm{Ran}\ R^\dagger$ *of:*

$$R^\dagger (D + \omega L)^{-1} (g - R f^\omega) = 0 \ ,$$

where the matrix D and L appear in the decomposition of $RR^\dagger$. If $g = \mathcal{R}f$ has a solution, then f^ω is its solution of minimal norm. Otherwise:

$$f^\omega = f_{MP} + \mathcal{O}(\omega) \ ,$$

where f_{MP} is the minimizer in H of:

$$\langle g - \mathcal{R}f | g - \mathcal{R}f \rangle_D \ ,$$

the inner product being defined by:

$$\langle h | k \rangle_D = \langle D^{-1} h | k \rangle \ .$$

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