Lectures on Non-Relativistic EFTs

(for near threshold states)

Antonio Vairo

Technische Universität München

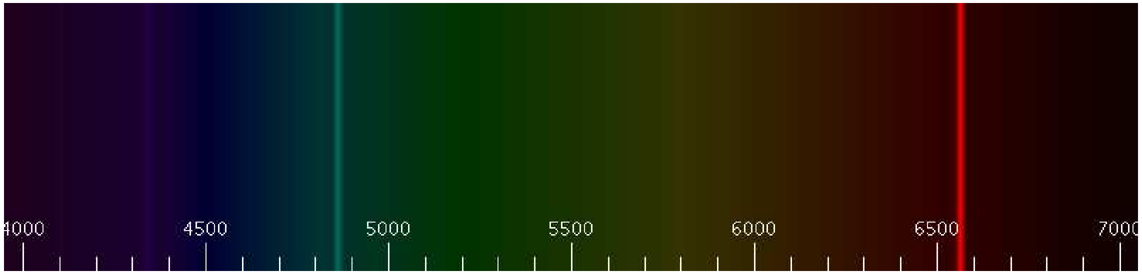
Outline

- Non-Relativistic EFTs
- Hydrogen atom
- NRQCD/pNRQCD
- The QCD potential at short distances: $\alpha_{\rm s}$
- Electromagnetic transitions
- Conclusions

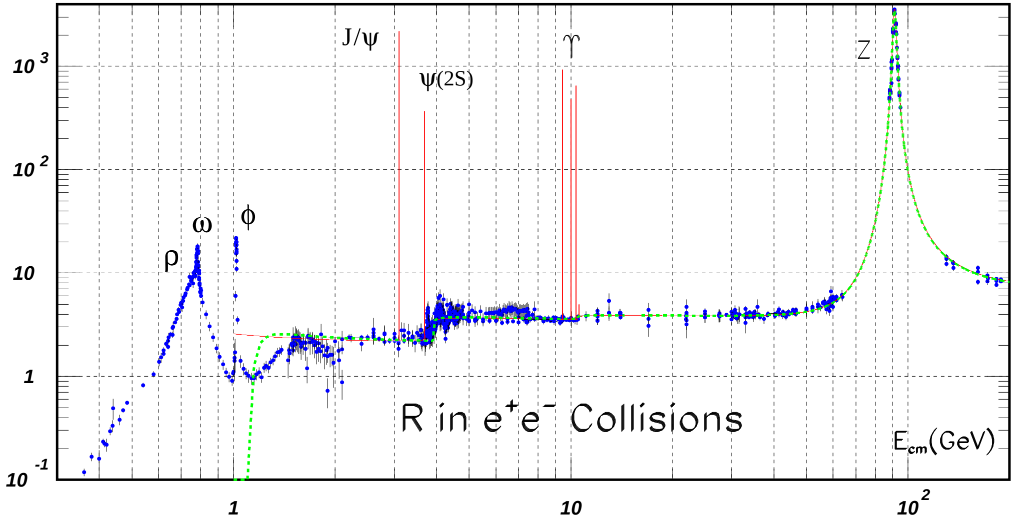
Non-Relativistic EFTs

Matter is (mostly) made of bound states

- Electromagnetic bound states: atoms, molecules, ...



- Strong-interaction bound states: hadrons, nuclei, ...
  (At low T and ρ, confinement only allows for bound states!)



... many of them non-relativistic

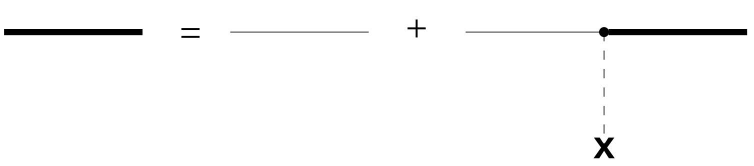
- atoms, molecules, ...
- baryonium, pionium, ...
- quarkonium (charmonium, bottomonium, top-antitop pairs, ...)

Non-relativistic quantum theory of bound states

Non-relativistic bound states accompanied the history of the quantum theory from its inception to the establishing of the quantum theory of fields:

- 1926 Schrödinger equation: $\left(\frac{\mathbf{p}^2}{2m} + V\right)\phi = E\phi$

$$\begin{cases} g = g_0 + g_0(-iV)g \\ g_0 = \frac{i}{E-\mathbf{p}^2/(2m)} \end{cases}$$



- 1927 Pauli equation: $\left(\frac{(\mathbf{p}-e\mathbf{A})^2}{2m} + V - \frac{\boldsymbol{\sigma}\cdot e\mathbf{B}}{2m}\right)\phi = E\phi$

The relevant scales of the non-relativistic bound state dynamics are

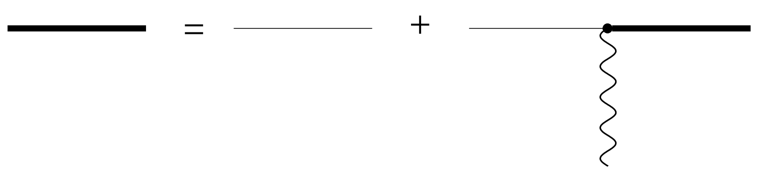
- $E \sim \frac{\mathbf{p}^2}{2m} \sim V \sim mv^2$,
- $p \sim 1/r \sim mv$;

a crucial observation: if v(elocity) $(\sim e^2) \ll 1$, then $m \gg mv \gg mv^2$.

Relativistic quantum theory of bound states

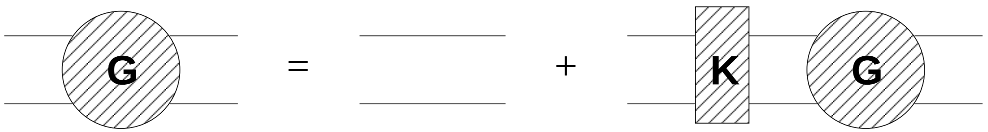
- 1928 Dirac equation: $(i\not{D} - m)\,\psi = 0$

$$\begin{cases} g^D = g_0^D + g_0^D(-ie\not{A})g^D \\ g_0^D = \frac{i}{\not{p} - m} \end{cases}$$

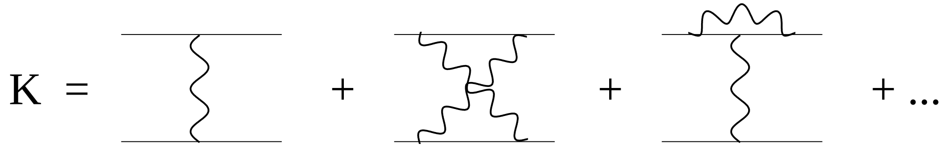


- 1951 Bethe–Salpeter equation:

$$\begin{cases} G = G_0 + G_0 K G \\ G_0 = g_0^D \otimes g_0^D \end{cases}$$



All the complexity of the field theory is in the kernel

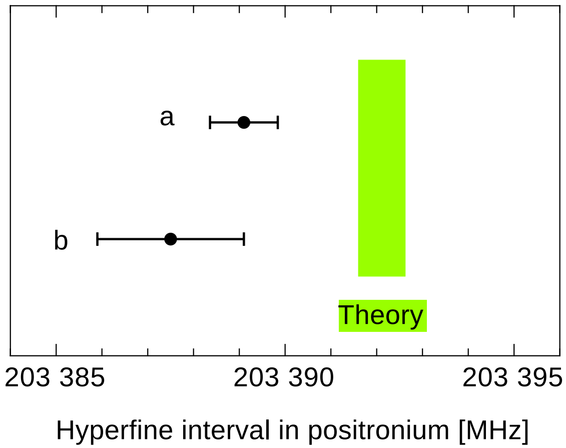


which only in the non-relativistic limit reduces to the Coulomb potential, but, in general, keeps entangled all bound-state scales.

Disentangling the bound-state scales at the Lagrangian level has advantages.

(I) It facilitates higher-order perturbative calculations.

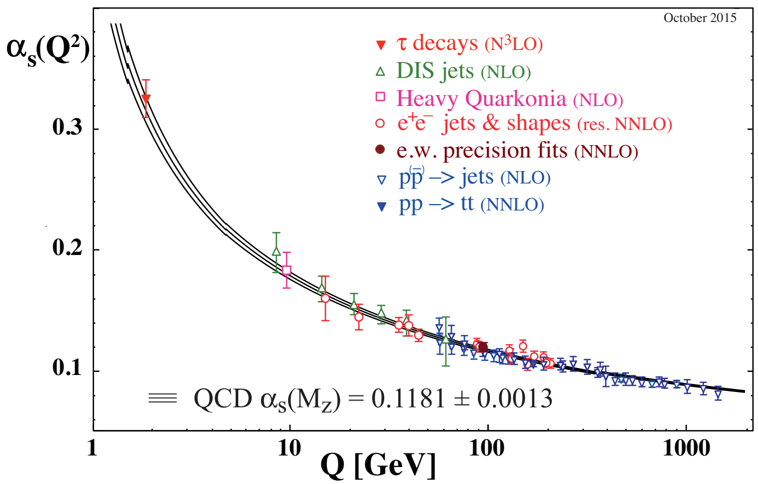
E.g. it took twenty-five years to go from the calculation of the $m\alpha^5$ correction in the hyperfine splitting of the positronium ground state to the $m\alpha^6 \ln \alpha$ term!

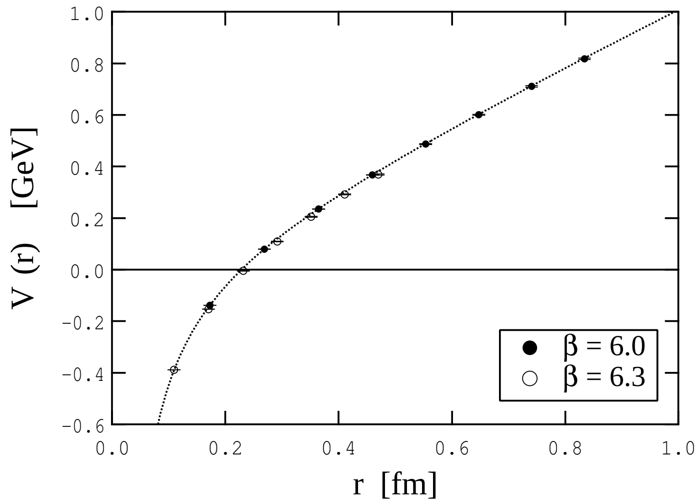


Relevant for

- atomic physics: Hydrogen atom (e.g. proton radius), positronium (e.g. width, hfs), ...
- $t\bar{t}$ threshold production, ...
- ...

(II) In QCD, it factorizes automatically high-energy (perturbative) contributions from low-energy (non-perturbative, thermal, ...) ones.





Relevant for

- pionium and precision chiral dynamics, ...
- nucleon-nucleon systems, ...
- quarkonia and new quarkonium states
- confinement and lattice calculations, ...
- quarkonium in heavy ion collisions: factorization of thermal contributions.

(III) More conceptually: it provides a field theoretical foundation of the Schrödinger equation:

$$\mathcal{L}_{\mathrm{EFT}} = \phi^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V \right) \phi + \Delta\mathcal{L}$$

The Lagrangian $\mathcal{L}_{\mathrm{EFT}}$, which factorizes the dynamics of the two-particle field ϕ, from the low-energy dynamics encoded in $\Delta\mathcal{L}$ defines an effective field theory.

Effective Field Theories

Whenever a system H, described by a Lagrangian $\mathcal{L}$, is characterized by 2 scales $\Lambda \gg \lambda$, observables may be calculated by expanding one scale with respect to the other. An effective field theory makes the expansion in λ/Λ explicit at the Lagrangian level.

The EFT Lagrangian, $\mathcal{L}_{\rm EFT}$, suitable to describe H at scales lower than Λ is defined by
(1) a cut off $\Lambda \gg \mu \gg \lambda$;
(2) by some degrees of freedom that exist at scales lower than μ

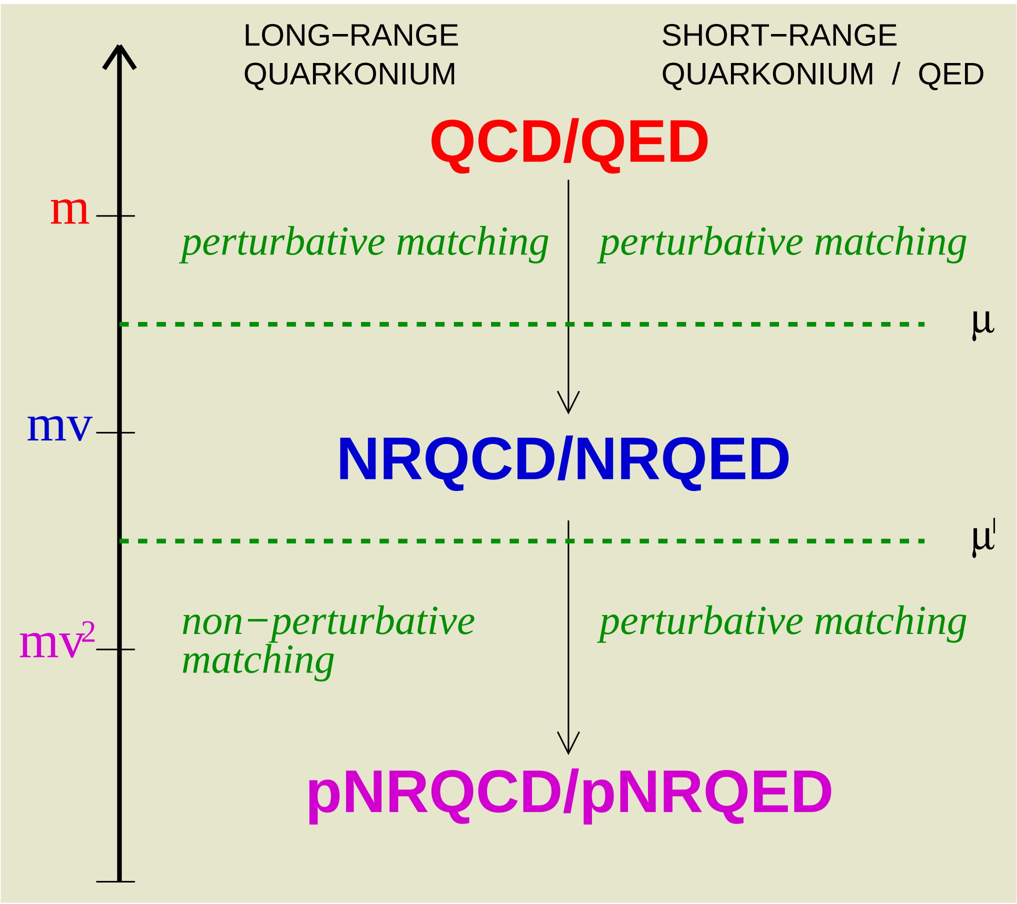
$\Rightarrow$ $\mathcal{L}_{\rm EFT}$ is made of all operators O_n that may be built from the effective degrees of freedom and are consistent with the symmetries of $\mathcal{L}$.

Effective Field Theories

$$\mathcal{L}_{\rm EFT} = \sum_n c_n(\Lambda/\mu) \frac{O_n(\mu, \lambda)}{\Lambda^n}$$

- Since at $\mu \sim \lambda$, $\langle O_n \rangle \sim \lambda^n$, the EFT is organized as an expansion in λ/Λ.
- The EFT is renormalizable order by order in λ/Λ.
- The matching coefficients $c_n(\Lambda/\mu)$ encode the non-analytic behaviour in Λ. They are calculated by imposing that $\mathcal{L}_{\rm EFT}$ and $\mathcal{L}$ describe the same physics at any finite order in the expansion: matching procedure.
- In QCD, if $\Lambda \gg \Lambda_{\rm QCD}$ then $c_n(\Lambda/\mu)$ may be calculated in perturbation theory.

Non-Relativistic EFTs (for near threshold states)



- Caswell Lepage PLB 167(86)437
- Lepage Thacker NP PS 4(88)199
- Bodwin et al PRD 51(95)1125, ...

- Pineda Soto PLB 420(98)391
- Pineda Soto NP PS 64(98)428
- Brambilla et al PRD 60(99)091502
- Brambilla et al NPB 566(00)275
- Kniehl et al NPB 563(99)200
- Luke Manohar PRD 55(97)4129
- Luke Savage PRD 57(98)413
- Grinstein Rothstein PRD 57(98)78
- Labelle PRD 58(98)093013
- Griesshammer NPB 579(00)313
- Luke et al PRD 61(00)074025
- Hoang Stewart PRD 67(03)114020, ...

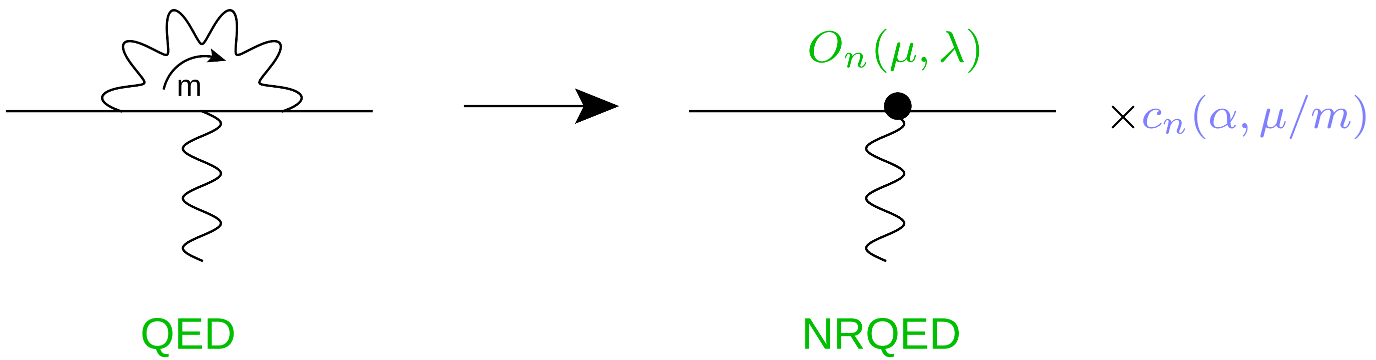
- They exploit the expansion in v/ factorization of low and high energy contributions.
- They are renormalizable order by order in v.
- In perturbation theory, RG techniques provide resummation of large logs.

The Hydrogen Atom

NRQED

NRQED is obtained by integrating out modes associated with the scale m

E.g.



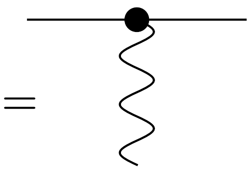
- The EFT has a cut-off $m > \mu > m\alpha$.
- The degrees of freedom are photons, electrons ψ, protons N (static if no recoil).
- The matching is perturbative.
- The Lagrangian is organized as an expansion in $1/m$ and α:

$$\mathcal{L}_{\rm NRQED} = \sum_n \frac{1}{m^n} \times c_n(\alpha, \mu/m) \times O_n(\mu, \lambda)$$

NRQED: matching

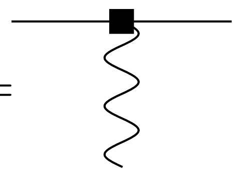
The matching at order $1/m^2$ gives:

$$c_2^{(a)} \frac{O_2^{(a)}}{m^2} =$$



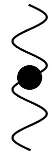
$$= \psi^\dagger e \frac{\nabla E}{8m^2} \psi \qquad \times \left(1 + \frac{8}{3} \frac{\alpha}{\pi} \ln \frac{m}{\mu} + \dots \right)$$

$$c_2^{(b)} \frac{O_2^{(b)}}{m^2} =$$



$$= ie\sigma \cdot \psi^\dagger \frac{\nabla \times E - E \times \nabla}{8m^2} \psi \qquad \times (1 + \dots)$$

$$c_2^{(c)} \frac{O_2^{(c)}}{m^2} =$$



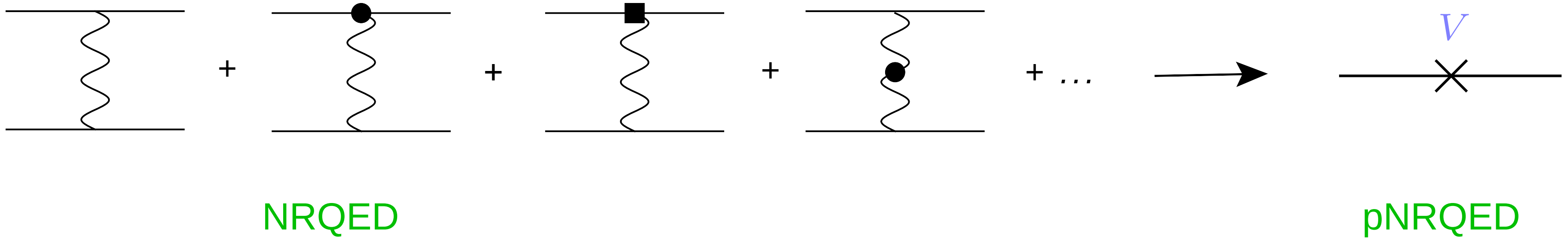
$$= \frac{F_{\mu\nu} \partial^2 F^{\mu\nu}}{m^2} \qquad \times \left(\frac{\alpha}{60\pi} + \dots \right)$$

If the recoil of the proton is considered, the radius of the proton is encoded in $c_2^{(a)\,\mathrm{proton}}$.

○ Caswell Lepage PLB 167 (1986) 437

pNRQED

pNRQED is obtained by integrating out modes associated with the scale $\frac{1}{r} \sim m\alpha$



- The EFT has a cut-off $m\alpha > \mu' > m\alpha^2$.
- The degrees of freedom are photons, atoms ϕ.
- The matching is perturbative.
- The Lagrangian is organized as an expansion in $1/m$, r, and α:

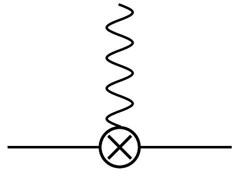
$$\mathcal{L}_{\text{pNRQED}} = \sum_k \sum_n \frac{1}{m^k} \times c_k(\alpha, \mu/m) \times V_n(r\mu', r\mu) \times r^n \times O_n(\mu', \lambda)$$

pNRQED: matching

$$\mathcal{L}_{\rm pNRQED} \quad = \quad \int d^3r\, \phi^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} + \cdots - V \right) \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Delta\mathcal{L}$$

- $V = -\frac{\alpha}{r} + \pi\alpha \frac{\delta^3(r)}{m^2} \left(\frac{c_2^{(a)}}{2} - 16 c_2^{(c)} \right) + \frac{\alpha}{4m^2} \frac{\mathbf{L}\cdot\boldsymbol{\sigma}}{r^3} c_2^{(b)} + \cdots$
- At leading order the bound-state field ϕ just satisfies the Schrödinger equation.
- $\Delta\mathcal{L}$ contains terms suppressed by powers of r:

$$\Delta\mathcal{L} = \int d^3r\, \phi^\dagger \mathbf{r} \cdot e\mathbf{E} \phi + \cdots$$



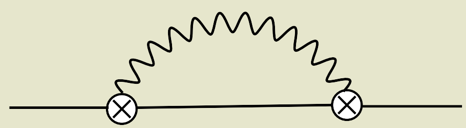
dipole interaction

The Hydrogen spectrum

The Hydrogen spectrum at order $m\alpha^5$ (responsible for the Lamb shift) reads

$$E_n = \langle n| \frac{\mathbf{p}^2}{m} + V |n\rangle + \Delta E_n$$

$$\Delta E_n = \langle n|$$



$$|n\rangle$$

$$= -\frac{\alpha}{3\pi}\left(\frac{2\pi\alpha}{m^2}\right)|\phi_n(0)|^2\left(-\frac{5}{3}+2\ln 2+\ln\frac{m^2}{\mu^2}\right)$$

$$-\frac{\alpha}{3\pi}\sum_i \left|\langle n|\frac{\mathbf{p}}{m}|i\rangle\right|^2 (E_n - E_i)\ln\left(\frac{E_n - E_i}{m}\right)^2$$

- The μ dependence cancels against the μ dependence of $c_2^{(a)}$ in the potential.
- The Bethe logarithm follows from the one-loop diagram in the EFT.

∘ Pineda Soto PLB 420 (1998) 391

NRQCD/pNRQCD

NRQCD

NRQCD is obtained by integrating out modes associated with the scale m

- The Lagrangian is organized as an expansion in $1/m$:

$$\mathcal{L}_{\rm NRQCD} = \psi^\dagger \left(iD_0 + \frac{\mathbf{D}^2}{2m} + \dots \right) \psi + \chi^\dagger \left(iD_0 - \frac{\mathbf{D}^2}{2m} + \dots \right) \chi + \dots$$
$$- \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{i=1}^{n_f} \bar{q}_i \, i \not{D} q_i$$

  ψ (χ) is the field that annihilates (creates) the (anti)fermion.

- The relevant dynamical scales of NRQCD are: mv, mv^2, ...

- Low-energy scales may be set to zero while matching.

pNRQCD (weak coupling)

pNRQCD is obtained by integrating out modes associated with the scale $\frac{1}{r} \sim mv \sim m\alpha_{\rm s}$

- The degrees of freedom of pNRQCD are quark-antiquark states (color singlet S, color octet O (weak coupling)), low energy gluons (weak coupling) and light quarks.
- The Lagrangian is organized as an expansion in $1/m$ and r:

$$\begin{aligned}\mathcal{L}_{\rm pNRQCD} &= \int d^3r \; {\rm Tr} \left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} + \cdots - V_s \right) {\rm S} \right. \\ &\qquad \left. + {\rm O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} + \cdots - V_o \right) {\rm O} \right\} \\ &\qquad -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \sum_{i=1}^{n_f} \bar{q}_i \, i\not{D} q_i + \Delta\mathcal{L}\end{aligned}$$

$$\Delta\mathcal{L} = \int d^3r \; V_A {\rm Tr} \left\{ {\rm O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm S} + {\rm H.c.} \right\} + \frac{V_B}{2} {\rm Tr} \left\{ {\rm O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm O} + {\rm c.c.} \right\} + \cdots$$

- At leading order in r, the singlet S satisfies the QCD Schrödinger equation.

pNRQCD (weak coupling): matching

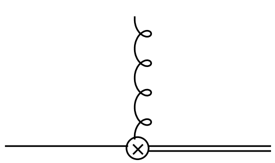
- Low-energy scales may be set to zero while matching.
- The (weak coupling) static potential is the Coulomb potential:

$$V_s(r) = -C_F \frac{\alpha_s}{r} + \dots, \qquad V_o(r) = \frac{1}{2N}\frac{\alpha_s}{r} + \dots, \qquad N = 3,\ C_F = \frac{4}{3}$$

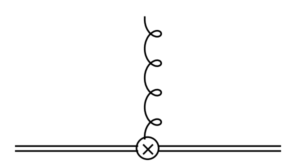
- $V_A = V_B = 1 + \mathcal{O}(\alpha_s^2)$
- Feynman rules:

$$= \theta(t)\, e^{-itH_s}$$

$$= \theta(t)\, e^{-itH_o} \left(e^{-i\int dt\, A^{\mathrm{adj}}}\right)$$



$$= \mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\, \mathrm{S}$$



$$= \mathrm{O}^\dagger \{\mathbf{r}\cdot g\mathbf{E},\, \mathrm{O}\}$$

∘ Pineda Soto NP PS 64 (1998) 428
Brambilla Pineda Soto Vairo NPB 566 (2000) 275, RMP 77 (2005) 1423

The QCD potential at short distances: α_s

Static energy

$$E_0(r) \;=\; \lim_{T\to\infty} \frac{i}{T} \ln \langle \square \rangle; \qquad \square = \exp\left\{ ig \oint dz^\mu A_\mu \right\}$$

Perturbation theory describes $E_0(r)$ in the short range ($r\Lambda \ll 1$, $\alpha_s(1/r) < 1$):

$$E_0(r) = \Lambda_s - \frac{C_F \alpha_s}{r}\left(1 + \#\alpha_s + \#\alpha_s^2 + \#\alpha_s^3 + \#\alpha_s^3 \ln\alpha_s + \#\alpha_s^4 \ln^2\alpha_s + \#\alpha_s^4 \ln\alpha_s + \dots\right)$$

- $E_0(r)$ is known at three loops.
- $\ln\alpha_s$ signals the cancellation of contributions coming from different energy scales:

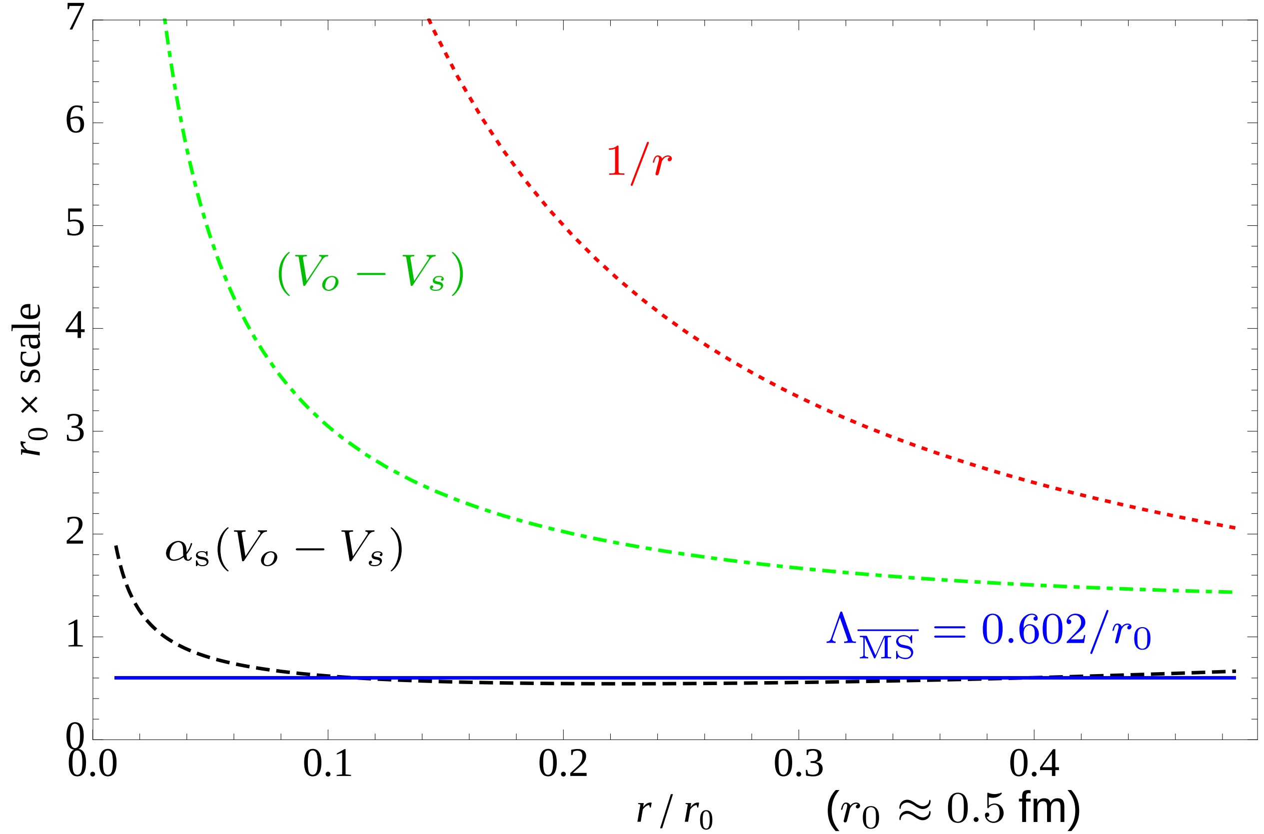
$$\ln\alpha_s = \ln\frac{\mu}{1/r} + \ln\frac{\alpha_s/r}{\mu}$$

  - Brambilla Pineda Soto Vairo PRD 60 (1999) 091502

Energy scales

In the short range the static Wilson loop is characterized by a hierarchy of energy scales:

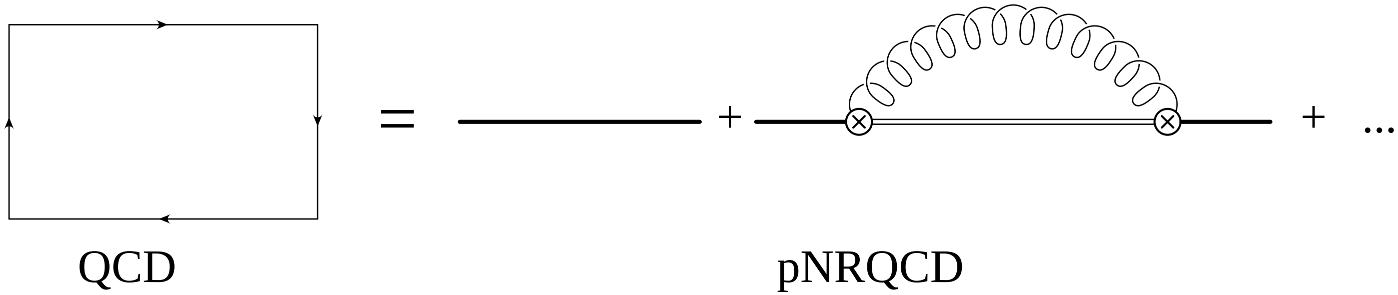
$$\frac{1}{r} \gg V_o - V_s \gg \Lambda; \qquad V_s \approx -C_F \frac{\alpha_s}{r}, \quad V_o \approx \frac{1}{2N} \frac{\alpha_s}{r}$$



$(r_0 \approx 0.5$ fm$)$

The static energy in pNRQCD

pNRQCD allows the factorization of contributions from different energy scales.



$$E_0(r) = \Lambda_s + V_s(r,\mu) - i\frac{g^2}{N}V_A^2\int_0^\infty dt\, e^{-it(V_o-V_s)}\langle \mathrm{Tr}\,\mathbf{r}\cdot\mathbf{E}(t)\,\mathbf{r}\cdot\mathbf{E}(0)\rangle(\mu) + \dots$$

res. mass potential ultrasoft contribution

○ Brambilla Pineda Soto Vairo NPB 566 (2000) 275

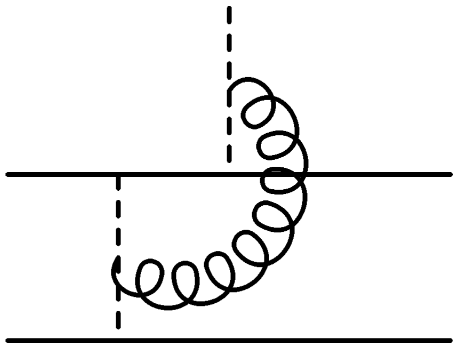
The μ dependence cancels between

$V_s \sim \ln r\mu, \ln^2 r\mu$, ...

ultrasoft contribution $\sim \ln(V_o - V_s)/\mu, \ln^2(V_o - V_s)/\mu, \dots \ln r\mu, \ln^2 r\mu, \dots$

V_A

The first contributing diagrams are of the type:

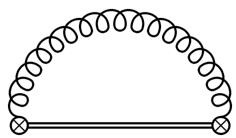


Therefore

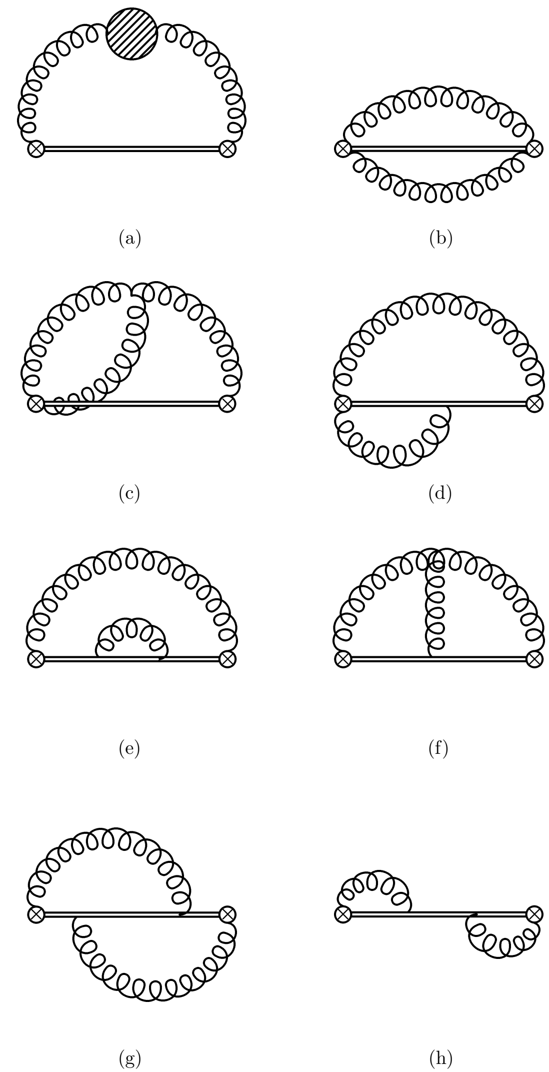
$$V_A(r,\mu) = 1 + \mathcal{O}(\alpha_s^2)$$

Chromoelectric field correlator: $\langle E(t)E(0)\rangle$

Is known at two loops.



LO



NLO

- Eidemüller Jamin PLB 416 (1998) 415

Static octet potential

$$\lim_{T\to\infty} \frac{i}{T} \ln \frac{\langle \square \rangle}{\langle \phi_{ab}^{\mathrm{adj}} \rangle} = \frac{1}{2N} \frac{\alpha_{\mathrm{s}}}{r} (1 + \#\alpha_{\mathrm{s}} + \#\alpha_{\mathrm{s}}^2 + \#\alpha_{\mathrm{s}}^3 + \#\alpha_{\mathrm{s}}^3 \ln \mu r + \dots)$$

Is known at three loops.

- Anzai Prausa A.Smirnov V.Smirnov Steinhauser PRD 88 (2013) 054030

Static singlet potential at N^{4}LO

$$
\begin{aligned}
V_s(r,\mu) \;=\; & -C_F \frac{\alpha_s(1/r)}{r}\left\{1 + \frac{\alpha_s(1/r)}{4\pi} a_1 + \left(\frac{\alpha_s(1/r)}{4\pi}\right)^2 a_2 \right. \\
& + \left(\frac{\alpha_s(1/r)}{4\pi}\right)^3 \left[\frac{16\,\pi^2}{3} C_A^3 \,\ln r\mu + a_3\right] \\
& + \left(\frac{\alpha_s(1/r)}{4\pi}\right)^4 \left[a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2\, C_A^3 \beta_0(-5 + 6\ln 2)\right) \ln r\mu + \dots\right] \\
& \left. + \qquad \cdots \qquad \right\}
\end{aligned}
$$

- Anzai Kiyo Sumino PRL 104 (2010) 112003
  A.Smirnov V.Smirnov Steinhauser PRL 104 (2010) 112002

Static energy at N^{4}LO

$$
\begin{aligned}
E_0(r) \quad = \quad & \Lambda_s - \frac{C_F \alpha_s(1/r)}{r} \left\{ 1 + \frac{\alpha_s(1/r)}{4\pi} \left[a_1 + 2\gamma_E \beta_0 \right] \right. \\
& + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \left[a_2 + \left(\frac{\pi^2}{3} + 4\gamma_E^2 \right) \beta_0^2 + \gamma_E \left(4a_1\beta_0 + 2\beta_1 \right) \right] \\
& + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \left[\frac{16\pi^2}{3} C_A^3 \ln \frac{C_A \alpha_s(1/r)}{2} + \tilde{a}_3 \right] \\
& + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \left[a_4^{L2} \ln^2 \frac{C_A \alpha_s(1/r)}{2} + a_4^L \ln \frac{C_A \alpha_s(1/r)}{2} + \ldots \right] \\
& \left. + \qquad \ldots \qquad \right\}
\end{aligned}
$$

- Brambilla Garcia Soto Vairo PLB 647 (2007) 185

Renormalization group equations

$$
\begin{cases}
\mu \dfrac{d}{d\mu} V_s = -\dfrac{2}{3} C_F \dfrac{\alpha_s}{\pi} r^2 V_A^2 \, [V_o - V_s]^{\,3} \left(1 + \dfrac{\alpha_s}{\pi}\, c\right) \\[2ex]
\mu \dfrac{d}{d\mu} V_o = \dfrac{1}{N} \dfrac{\alpha_s}{\pi} r^2 V_A^2 \, [V_o - V_s]^{\,3} \left(1 + \dfrac{\alpha_s}{\pi}\, c\right) \\[2ex]
\mu \dfrac{d}{d\mu} V_A = 0 \\[2ex]
\mu \dfrac{d}{d\mu} \alpha_s = \alpha_s \beta(\alpha_s); \qquad c = \dfrac{-5 n_f + C_A (6\pi^2 + 47)}{108}
\end{cases}
$$

- Brambilla Garcia Soto Vairo PRD 80 (2009) 034016

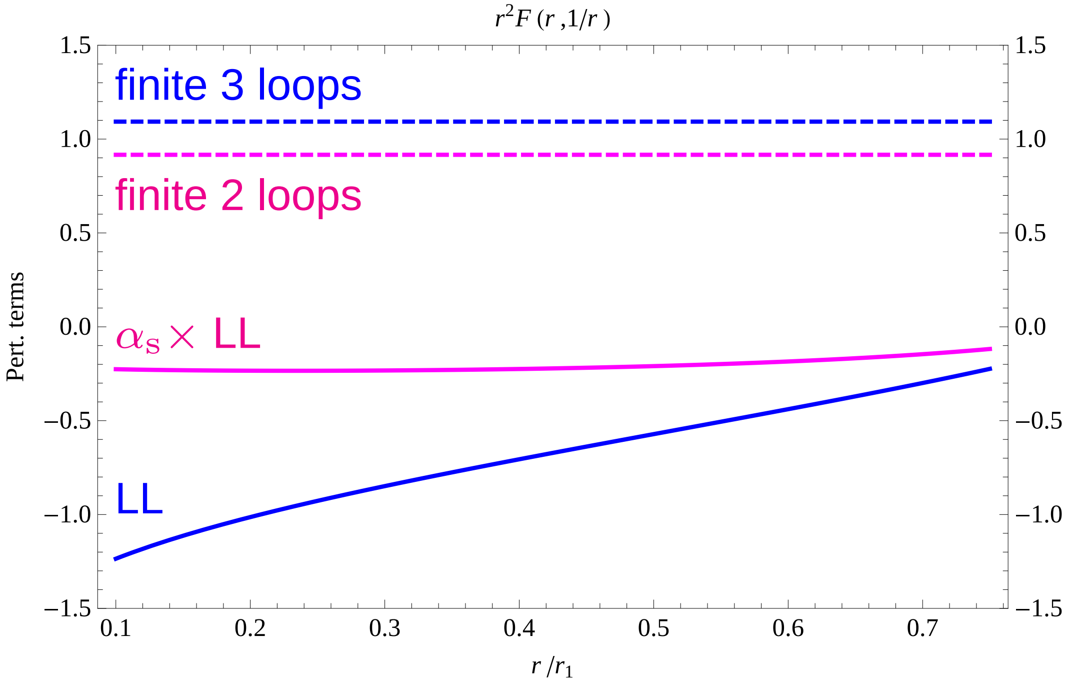
Static singlet potential and energy at N^3LL

$$V_s(r,\mu) = V_s(r,1/r) - \frac{C_F C_A^3}{6\beta_0} \frac{\alpha_s^3(1/r)}{r} \left\{ \left(1 + \frac{3}{4}\frac{\alpha_s(1/r)}{\pi} a_1\right) \ln \frac{\alpha_s(1/r)}{\alpha_s(\mu)} \left(\frac{\beta_1}{4\beta_0} - 6c\right) \left[\frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(1/r)}{\pi}\right] \right\}$$

Summed to the ultrasoft contribution at two loops, it provides the static energy at N^3LL.

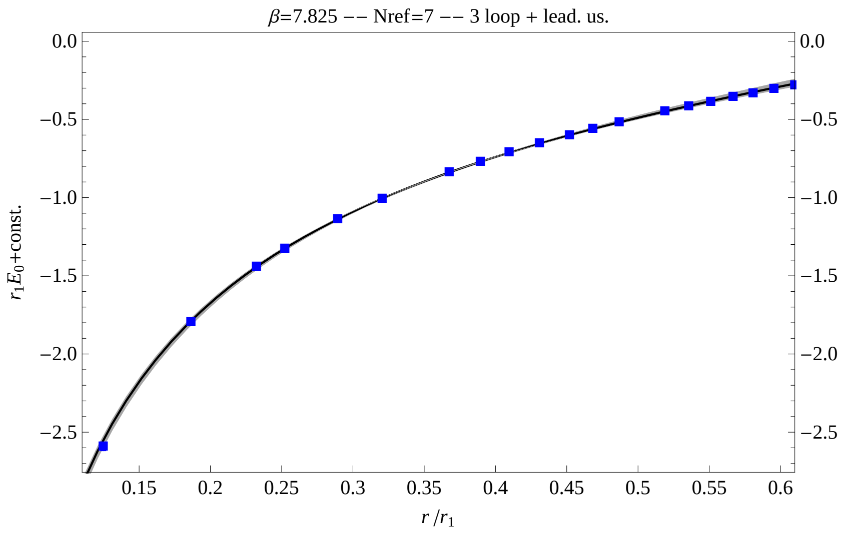
○ Brambilla Garcia Soto Vairo PRD 80 (2009) 034016

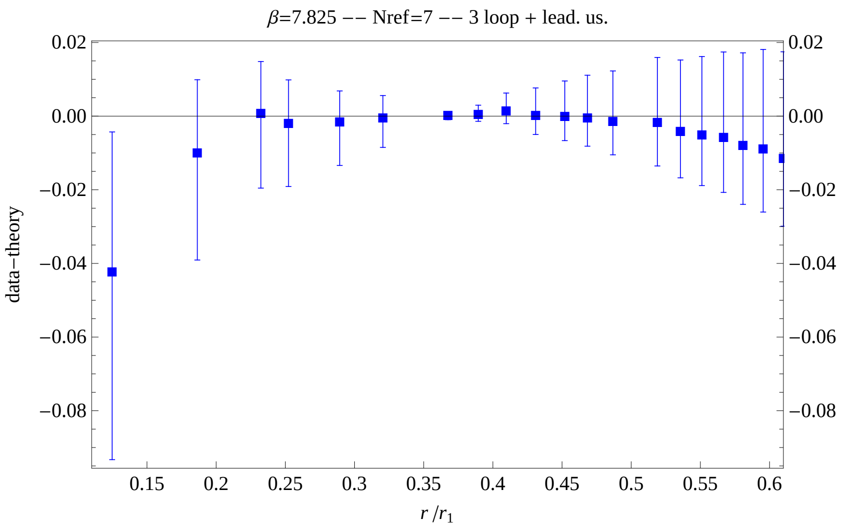
The counting of the ultrasoft contributions



The lattice scale is $r_1 = 0.3106 \pm 0.0017$ fm.
We chose $\mu = 1.26 r_1^{-1} \sim 0.8$ GeV, for the ultrasoft factorization scale.
Variations of μ only produce small effects on the results.

Static energy vs lattice data





The fit gives

$$r_1\Lambda_{\overline{\rm MS}} = 0.495^{+0.028}_{-0.018} \quad \text{which converts to} \quad \Lambda_{\overline{\rm MS}} = 315^{+18}_{-12}\ \text{MeV}$$

Perturbation theory agrees with lattice data up to about 0.2 fm.

- Bazavov Brambilla Garcia Petreczky Soto Vairo PRD 90 (2014) 074038

$\alpha_{\rm s}$

$$\alpha_{\rm s}(1.5 \text{ GeV}, n_f = 3) = 0.336^{+0.012}_{-0.008}$$

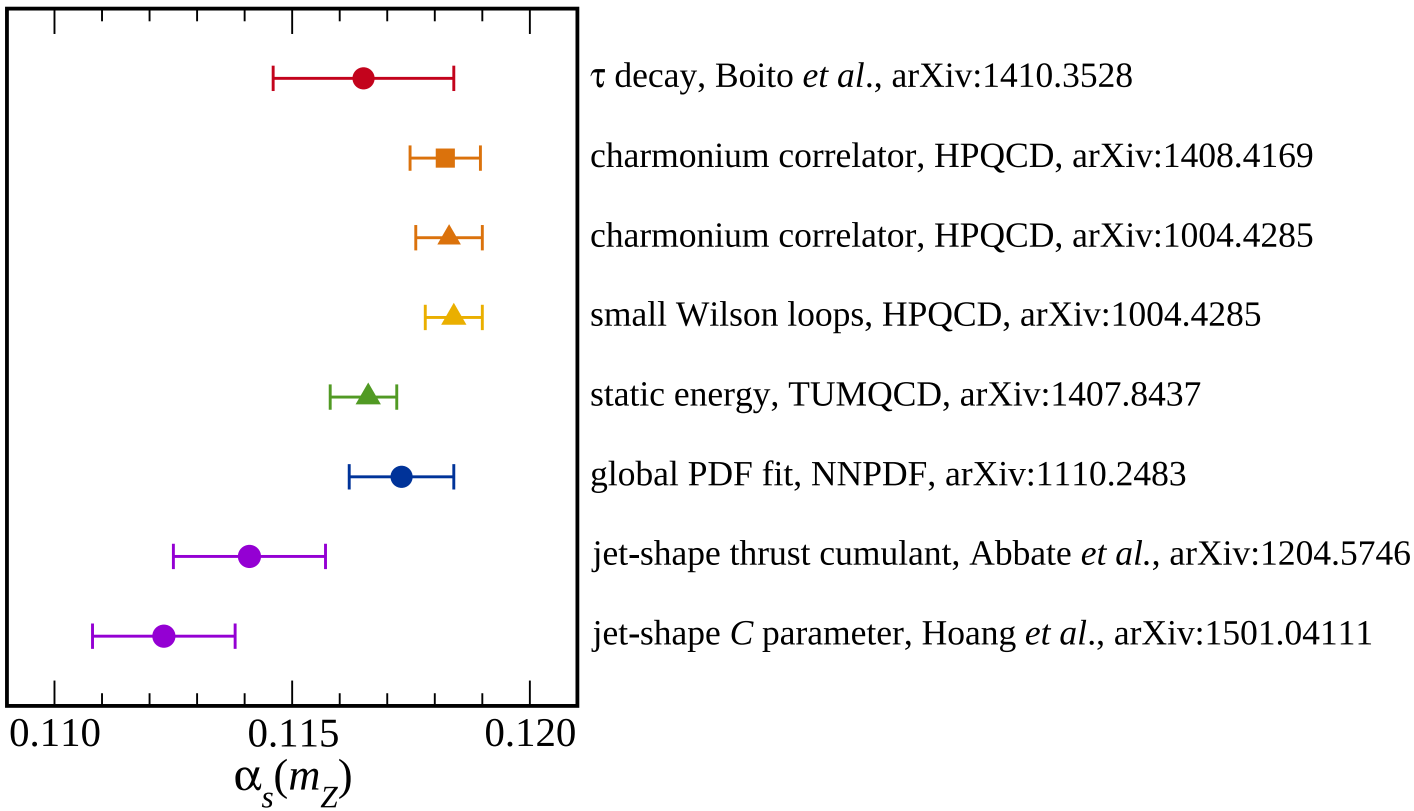
which corresponds to

$$\alpha_{\rm s}(M_Z, n_f = 5) = 0.1166^{+0.0012}_{-0.0008}$$

from four-loop running, $m_c = 1.6$ GeV and $m_b = 4.7$ GeV.

- Bazavov Brambilla Garcia Petreczky Soto Vairo PRD 90 (2014) 074038

Comparison with other determinations



- Andreas Kronfeld 2016

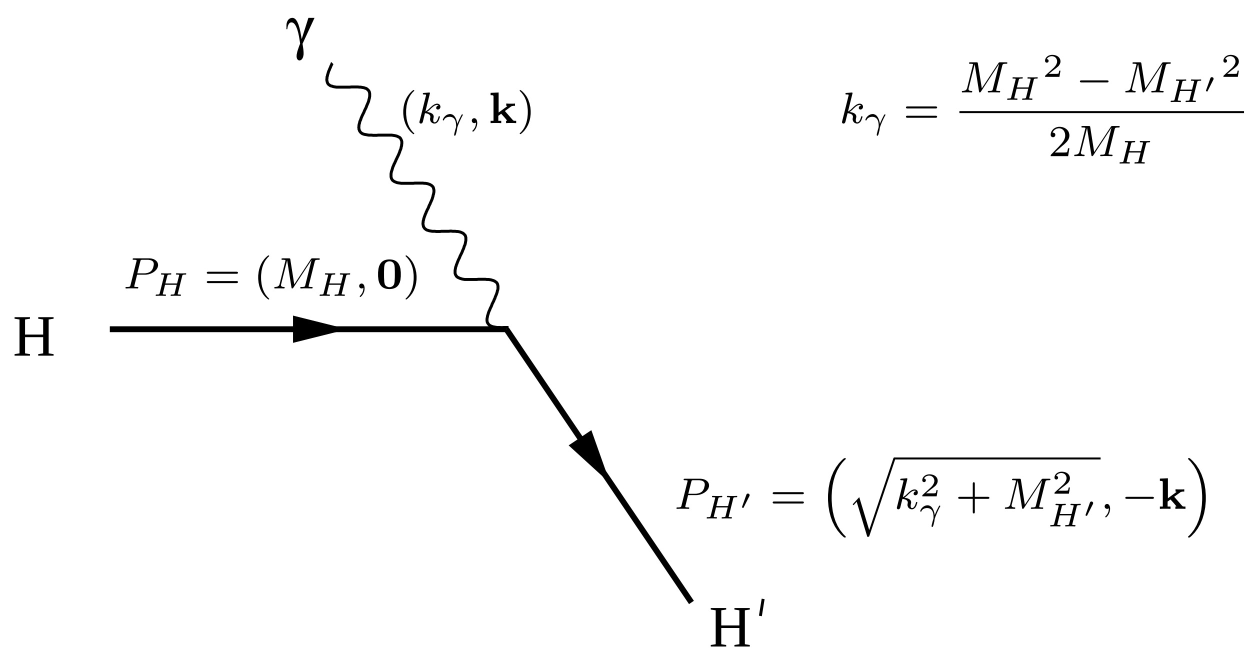
Electromagnetic transitions

Radiative transitions: basics

Two dominant single-photon-transition processes:

(1) magnetic dipole transitions (M1)

(2) electric dipole transitions (E1)



∘ QWG coll CERN-2005-005 Yellow Book hep-ph/0412158

M1 transitions in the non-relativistic limit

(1) M1 transitions in the non-relativistic limit:

$$\Gamma^{\mathrm{M1}}_{n^3S_1 \to n'^1S_0\,\gamma} = \frac{4}{3}\,\alpha\,e_Q^2\,\frac{k_\gamma^3}{m^2}\,\left|\int_0^\infty dr\,r^2\,R_{n'0}(r)\,R_{n0}(r)\,j_0\left(\frac{k_\gamma r}{2}\right)\right|^2$$

If $k_\gamma \langle r \rangle \ll 1$ $\quad j_0(k_\gamma r/2) = 1 - (k_\gamma r)^2/24 + \ldots$

- $n = n'$ allowed transitions
- $n \neq n'$ hindered transitions

$\Gamma_{J/\psi \to \eta_c \gamma}$

At leading order in the multipole expansion, M1 (allowed) transition rates are independent from the low-energy dynamics (i.e. the quarkonium wave-function).

As an example consider

$$\Gamma_{J/\psi \to \eta_c \gamma} = \frac{16}{27} \alpha \frac{k_\gamma^3}{m_c^2} \approx 2.83 \, \text{keV}$$

* from $M_{J/\psi} \approx 3097$ MeV and $M_{\eta_c} \approx 2984$ MeV ($k_\gamma \approx 111$ MeV).

To be compared with the PDG value $\Gamma_{J/\psi \to \eta_c \gamma} = 1.6 \pm 0.4 \, \text{keV}$.

E1 transitions in the non-relativistic limit

(2) E1 transitions in the non-relativistic limit:

$$\Gamma^{\mathrm{E1}}_{n^{2S+1}L_J \to n'^{2S+1}L'_{J'}\,\gamma} = \frac{4}{3}\,\alpha e_Q^2\, k_\gamma^3\, [\mathcal{E}(nL \to n'L')]^2\,(2J'+1)\,\mathrm{max}_{\{L,L'\}} \begin{Bmatrix} J & 1 & J' \\ L' & S & L \end{Bmatrix}^2$$

where

$$\begin{aligned} \mathcal{E}(nL \to n'L') &= \int_0^\infty dr\, r^2\, R_{n'L'}(r)\, R_{nL}(r) \left[\frac{k_\gamma r}{2} j_0\left(\frac{k_\gamma r}{2}\right) - j_1\left(\frac{k_\gamma r}{2}\right)\right] \\ &\approx I_3(nL \to n'L') \times \left[1 + \mathcal{O}\left((k_\gamma r)^2\right)\right] \quad \text{if } k_\gamma \langle r \rangle \ll 1 \\ I_N(nL \to n'L') &= \int_0^\infty dr\, r^N\, R_{n'L'}(r)\, R_{nL}(r) \end{aligned}$$

Note that, for equal energies and masses, M1 transitions are suppressed by a factor $1/(m\langle r \rangle)^2 \sim v^2$ with respect to E1 transitions, which are much more common.

Relativistic corrections

- Relativistic corrections may be sizeable:
  about 30% for charmonium ($v_c^2 \approx 0.3$) and 10% for bottomonium ($v_b^2 \approx 0.1$).

- For quarkonium radiative transitions, essentially one model-dependent calculation has been used for over twenty years to account for relativistic corrections, based upon:

  relativistic equation with scalar and vector potentials;

  non-relativistic reduction;

  a somewhat imposed relativistic invariance to calculate recoil corrections.

- Grotch Owen Sebastian PR D30 (1984) 1924
  see also QWG CERN Yellow Book CERN-2005-005, hep-ph/0412158

Relativistic corrections and EFTs

Non-Relativistic EFTs for quarkonium allow

- to derive expressions for radiative transitions directly from QCD;
- with a well specified range of applicability;
- to determine a reliable error associated with the theoretical determinations;
- to improve the theoretical determinations in a systematic way.

Scales

- $p \sim \frac{1}{r} \sim mv$, $E \sim mv^2$; in a non-relativistic system $mv \gg mv^2$
- $\Lambda_{\rm QCD}$
- k_γ

$mv \gg \Lambda_{\rm QCD}$ for weakly-coupled quarkonia (J/ψ, η_c, $\Upsilon(1S)$, η_b, ...);
$mv \sim \Lambda_{\rm QCD}$ for strongly-coupled quarkonia (excited states);

$k_\gamma \sim mv^2$ for hindered M1 transitions, most E1 transitions;
$k_\gamma \sim mv^4$ for allowed M1 transitions.

$\Rightarrow \quad k_\gamma r \ll 1$

Degrees of freedom

- Degrees of freedom at scales lower than mv:

  Q-$\bar{Q}$ states, with energy $\sim \Lambda_{\rm QCD}$, mv^2 and momentum $\lesssim mv$
  $\Rightarrow$ *(i)* singlet $\rm S$ *(ii)* octet $\rm O$ [if $mv \gg \Lambda_{\rm QCD}$]

  Gluons with energy and momentum $\sim \Lambda_{\rm QCD}$, mv^2 [if $mv \gg \Lambda_{\rm QCD}$]

  Photons of energy and momentum lower than mv.

- Power counting:

  $p \sim \frac{1}{r} \sim mv$;
  all gauge fields are multipole expanded: $A(R,r,t) = A(R,t) + \mathbf{r}\cdot\boldsymbol{\nabla} A(R,t) + \dots$
  and scale like $(\Lambda_{\rm QCD}$ or $mv^2)^{\rm dimension}$.

EFT Lagrangian

$$\mathcal{L}_{\rm pNRQCD} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} - \frac{1}{4}F^{\rm em}_{\mu\nu}F^{\mu\nu\,\rm em}$$

$$+\int d^3r\,{\rm Tr}\left\{{\rm S}^\dagger\left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s\right){\rm S}\right.$$

$$\left.+\,{\rm O}^\dagger\left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o\right){\rm O}\right\}$$

LO in r

$$+{\rm Tr}\left\{{\rm O}^\dagger\mathbf{r}\cdot g\mathbf{E}\,{\rm S} + {\rm S}^\dagger\mathbf{r}\cdot g\mathbf{E}\,{\rm O}\right\}$$

$$+\frac{1}{2}{\rm Tr}\left\{{\rm O}^\dagger\mathbf{r}\cdot g\mathbf{E}\,{\rm O} + {\rm O}^\dagger{\rm O}\mathbf{r}\cdot g\mathbf{E}\right\}$$ [if $mv \gg \Lambda_{\rm QCD}$]

$$+\cdots$$

NLO in r

$$+\mathcal{L}_\gamma$$

$\mathcal{L}_\gamma$

$$\mathcal{L}_\gamma = \mathcal{L}_\gamma^{\rm M1} + \mathcal{L}_\gamma^{\rm E1} + \dots$$

$$\begin{aligned}
\mathcal{L}_\gamma^{\rm M1} = \; & {\rm Tr} \Bigg\{ \frac{1}{2m} \, V_1^{\rm M1} \, \left\{ {\rm S}^\dagger, \boldsymbol{\sigma} \cdot ee_Q \mathbf{B}^{\rm em} \right\} {\rm S} \\
& + \frac{1}{2m} \, V_1^{\rm M1} \, \left\{ {\rm O}^\dagger, \boldsymbol{\sigma} \cdot ee_Q \mathbf{B}^{\rm em} \right\} {\rm O} \qquad [\text{if } mv \gg \Lambda_{\rm QCD}] \\
& + \frac{1}{4m^2} \, \frac{V_2^{\rm M1}}{r} \, \left\{ {\rm S}^\dagger, \boldsymbol{\sigma} \cdot \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times ee_Q \mathbf{B}^{\rm em} \right) \right] \right\} {\rm S} \\
& + \frac{1}{4m^2} \, \frac{V_3^{\rm M1}}{r} \, \left\{ {\rm S}^\dagger, \boldsymbol{\sigma} \cdot ee_Q \mathbf{B}^{\rm em} \right\} {\rm S} \\
& + \frac{1}{4m^3} \, V_4^{\rm M1} \, \left\{ {\rm S}^\dagger, \boldsymbol{\sigma} \cdot ee_Q \mathbf{B}^{\rm em} \right\} \boldsymbol{\nabla}_r^2 {\rm S} + \cdots \Bigg\}
\end{aligned}$$

- Brambilla Jia Vairo PR D73 (2006) 054005

$\mathcal{L}_\gamma$

$$
\begin{aligned}
\mathcal{L}_\gamma^{\rm E1} \quad = \quad & \mathrm{Tr}\Big\{ V_1^{\rm E1}\ \mathrm{S}^\dagger \mathbf{r}\cdot ee_Q\mathbf{E}^{\rm em}\mathrm{S} \\
& + V_1^{\rm E1}\ \mathrm{O}^\dagger \mathbf{r}\cdot ee_Q\mathbf{E}^{\rm em}\mathrm{O} \qquad [\text{if } mv \gg \Lambda_{\rm QCD}] \\
& + \frac{1}{24}\ V_2^{\rm E1}\ \mathrm{S}^\dagger \mathbf{r}\cdot[(\mathbf{r}\cdot\boldsymbol{\nabla})^2 ee_Q\mathbf{E}^{\rm em}]\mathrm{S} \\
& + \frac{i}{4m}\ V_3^{\rm E1}\ \mathrm{S}^\dagger\{\boldsymbol{\nabla}\cdot, \mathbf{r}\times ee_Q\mathbf{B}^{\rm em}\}\mathrm{S} \\
& + \frac{i}{12m}\ V_4^{\rm E1}\ \mathrm{S}^\dagger\{\boldsymbol{\nabla}_r\cdot, \mathbf{r}\times[(\mathbf{r}\cdot\boldsymbol{\nabla}) ee_Q\mathbf{B}^{\rm em}]\}\mathrm{S} \\
& + \frac{1}{4m}\ V_5^{\rm E1}\ [\mathrm{S}^\dagger, \boldsymbol{\sigma}]\cdot[(\mathbf{r}\cdot\boldsymbol{\nabla}) ee_Q\mathbf{B}^{\rm em}]\mathrm{S} \\
& - \frac{i}{4m^2}\ V_6^{\rm E1}\ [\mathrm{S}^\dagger, \boldsymbol{\sigma}]\cdot(ee_Q\mathbf{E}^{\rm em}\times\boldsymbol{\nabla}_r)\mathrm{S} + \cdots \Big\}
\end{aligned}
$$

- Brambilla Pietrulewicz Vairo PRD 85 (2012) 094005

Matching

The matching consists in the calculation of the coefficients V.
They get contributions from

- hard modes ($\sim m$):

$$\bar{\psi}(i\not{D} - m)\psi \to \psi^\dagger \left(iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{c_F^{\rm em}}{2m}\boldsymbol{\sigma}\cdot ee_Q\mathbf{B}^{\rm em} + \cdots \right)\psi$$

  From HQET:

$$c_F^{\rm em} \equiv 1 + \kappa^{\rm em} = 1 + \frac{2}{3}\frac{\alpha_{\rm s}}{\pi} + \ldots$$

  is the quark magnetic moment.

  ◦ `Grozin Marquard Piclum Steinhauser NP B789 (2008) 277 (3 loops)`

- soft modes ($\sim mv$).

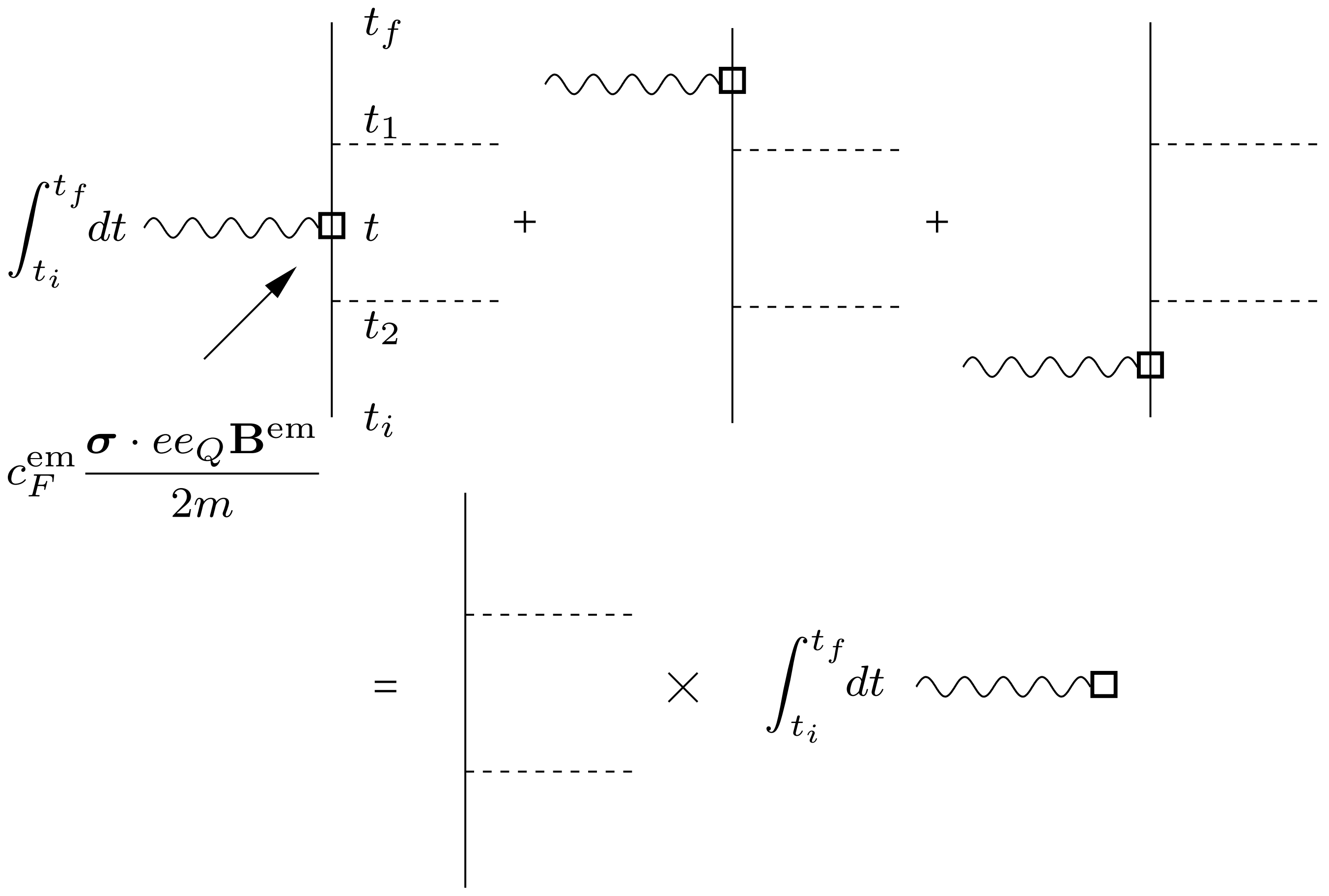
M1 operator at $\mathcal{O}(1)$

$$V_1^{\mathrm{M1}} \left\{ S^\dagger, \frac{\boldsymbol{\sigma} \cdot e\mathbf{B}^{\mathrm{em}}}{2m} \right\} S$$

$$V_1^{\mathrm{M1}} = \left(\text{hard} \right) \times \left(\text{soft} \right)$$

- $$\left(\text{hard} \right) = c_F^{\mathrm{em}} = 1 + \frac{2\alpha_{\mathrm{s}}(m)}{3\pi} + \cdots$$

- Since $\boldsymbol{\sigma} \cdot e\mathbf{B}^{\mathrm{em}}(\mathbf{R})$ behaves like the identity operator to all orders V_1^{M1} does not get soft contributions.



Diagrammatic factorization of the magnetic dipole coupling in the SU(3)$_f$ limit.

- *The argument is similar to the factorization of the QCD corrections in $b \to u\, e^- \bar{\nu}_e$, which leads to* $\mathcal{L}_{\rm eff} = -4 G_F/\sqrt{2}\, V_{ub}\, \bar{e}_L \gamma_\mu \nu_L\, \bar{u}_L \gamma^\mu b_L$ *to all orders in $\alpha_{\rm s}$.*

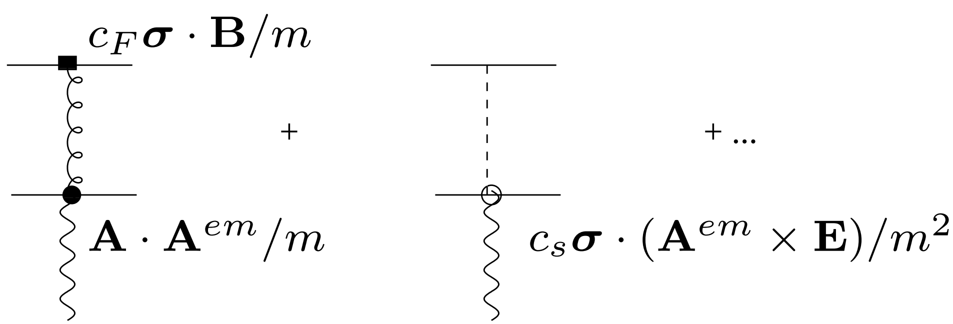
M1 operator at $\mathcal{O}(1)$

$$V_1^{\mathrm{M1}} \left\{ S^\dagger, \frac{\boldsymbol{\sigma} \cdot e\mathbf{B}^{\mathrm{em}}}{2m} \right\} S$$

- $$V_1^{\mathrm{M1}} = 1 + \frac{2\alpha_{\mathrm{s}}(m)}{3\pi} + \cdots$$
- No large quarkonium anomalous magnetic moment!
  - Dudek Edwards Richards PR D73 (2006) 074507 (lattice)

M1 operators at $\mathcal{O}(v^2)$

$$\frac{1}{4m^2}\,\frac{V_2^{\rm M1}}{r}\,\left\{\mathrm{S}^\dagger, \boldsymbol{\sigma}\cdot\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times ee_Q\mathbf{B}^{\rm em}\right)\right]\right\}\mathrm{S} \quad \text{and} \quad \frac{1}{4m^2}\,\frac{V_3^{\rm M1}}{r}\,\left\{\mathrm{S}^\dagger, \boldsymbol{\sigma}\cdot ee_Q\mathbf{B}^{\rm em}\right\}\mathrm{S}$$



$$= \left(\text{hard}\right)\times\left(\text{soft}\right)$$

- to all orders $\left(\text{hard}\right) = 2c_F - c_s = 1$; $\left(\text{soft}\right) = r^2 V_s'/2$
  - ○ Brambilla Gromes Vairo PLB 576 (2003) 314 (Poincaré invariance)
    Luke Manohar PLB 286 (1992) 348 (reparameterization invariance)

- $V_2^{\rm M1} = r^2 V_s'/2$ and $V_3^{\rm M1} = 0$

- No (effective) scalar interaction!

M1 operators at $\mathcal{O}(v^2)$

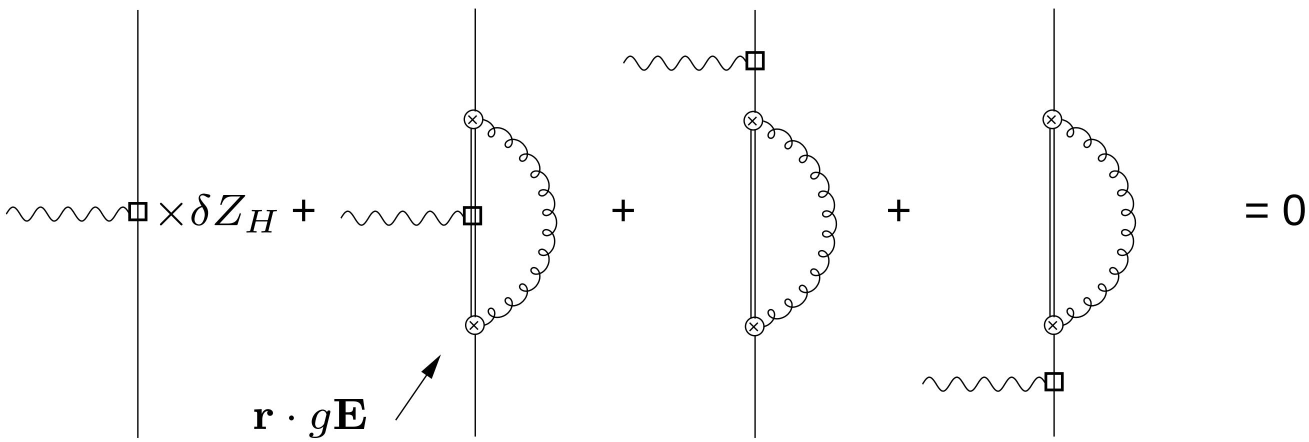
$$V_4^{\rm M1}\left\{S^\dagger, \frac{\boldsymbol{\sigma}\cdot e\mathbf{B}^{em}}{4m^3}\right\}\boldsymbol{\nabla}_r^2 S$$

$$V_4^{\rm M1} = \left(\text{hard}\right)\times\left(\text{soft}\right)$$

- $\left(\text{hard}\right) = 1$
  - Manohar PR D56 (1997) 230 (reparameterization invariance)
- $\left(\text{soft}\right) = 1$ to all orders
  - Brambilla Pietrulewicz Vairo PRD 85 (2012) 094005
- $V_4^{\rm M1} = 1$

$\mathcal{O}(v^2)$ corrections to weakly-coupled quarkonia

Coupling of photons with octets: $V_1^{\rm M1}\left\{O^\dagger, \frac{\boldsymbol{\sigma}\cdot e\mathbf{B}^{em}}{2m}\right\}O$ [if $mv \gg \Lambda_{\rm QCD}$]



- If $mv^2 \sim \Lambda_{\rm QCD}$ the above graphs are potentially of order $\Lambda^2_{\rm QCD}/(mv)^2 \sim v^2$.
- The contribution vanishes, for $\boldsymbol{\sigma}\cdot e\mathbf{B}^{em}(\mathbf{R})$ behaves like the identity operator.
- There are no non-perturbative contributions at $\mathcal{O}(v^2)$!

- This is not the case for strongly-coupled quarkonia:

  non-perturbative corrections affect the operator $\frac{1}{m^3}\,\frac{V_5^{\rm M1}}{r^2}\,\left\{{\rm S}^\dagger, \boldsymbol{\sigma}\cdot ee_Q\mathbf{B}^{\rm em}\right\}{\rm S}$.

M1 hindered transitions

- One new operator contributes:

$$-\frac{1}{16m^2}\, c_S^{\rm em}\, \left[{\rm S}^\dagger, \boldsymbol{\sigma}\cdot\left[-i\boldsymbol{\nabla}_r\times, {\bf r}^i(\boldsymbol{\nabla}^i ee_Q{\bf E}^{\rm em})\right]\right]{\rm S}$$

- Two new wave-function corrections contribute:

  (1) induced by the spin-spin potential $V^{\rm ss}$;

  (2) recoil correction induced by the spin-orbit potential;

  Due to the recoil, the final state develops a nonzero P-wave component suppressed by a factor $v\,k_\gamma/m$ (through the spin-orbit operator $-\frac{1}{4\,m^2}\frac{V_S^{(0)\,\prime}}{2}\,{\rm Tr}\left\{\{{\rm S}^\dagger,\boldsymbol{\sigma}\}\cdot[\hat{\bf r}\times(-i\boldsymbol{\nabla})]\,{\rm S}\right\}$*), which, in a $n^3S_1\to n'\,^1S_0\,\gamma$ transition, can be reached from the initial $^3S_1$ state through a $1/v$ enhanced E1 transition.*

M1 transitions

$$\Gamma_{n^3S_1 \to n^1S_0\,\gamma} = \frac{4}{3}\alpha e_Q^2 \frac{k_\gamma^3}{m^2}\left[1 + \frac{4\alpha_s(m)}{3\pi} - \frac{5}{3}\langle nS|\frac{\mathbf{p}^2}{m^2}|nS\rangle\right]$$

$$\Gamma_{n^3S_1 \to n'^1S_0\,\gamma} = \frac{4}{3}\alpha e_Q^2 \frac{k_\gamma^3}{m^2}\left[\langle n'S|\left(-\frac{k_\gamma^2\mathbf{r}^2}{24} - \frac{5}{6}\frac{\mathbf{p}^2}{m^2}\right)|nS\rangle + \frac{1}{m^2}\frac{\langle n'S|V^{\rm ss}(\mathbf{r})|nS\rangle}{E_n^{(0)} - E_{n'}^{(0)}}\right]^2 \quad \text{for} \quad n \neq n'$$

$$\Gamma_{n^3P_J \to n^1P_1\,\gamma} = \frac{4}{3}\alpha e_Q^2 \frac{k_\gamma^3}{m^2}\left[1 + \frac{4\alpha_s(m)}{3\pi} - d_J\,\langle nP|\frac{\mathbf{p}^2}{m^2}|nP\rangle\right]$$

$$\Gamma_{n^1P_1 \to n^3P_J\,\gamma} = (2J+1)\frac{4}{9}\alpha e_Q^2 \frac{k_\gamma^3}{m^2}\left[1 + \frac{4\alpha_s(m)}{3\pi} - d_J\,\langle nP|\frac{\mathbf{p}^2}{m^2}|nP\rangle\right]$$

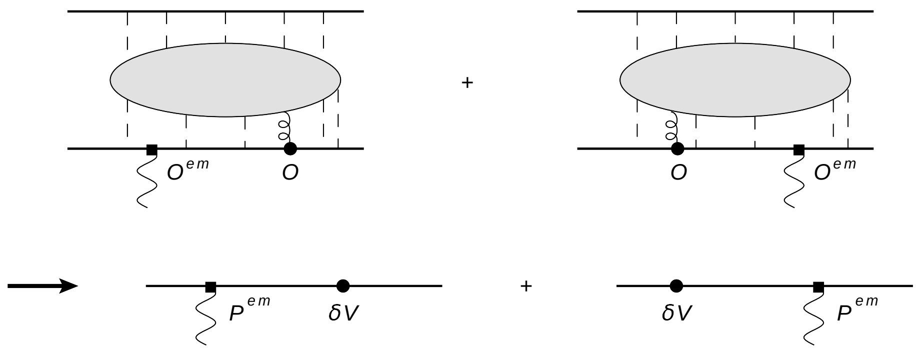
where $d_0 = 1$, $d_1 = 2$ and $d_2 = 8/5$.

- Brambilla Jia Vairo PR D73 (2006) 054005

E1 transitions

E1 transitions always involve excited states. These are likely strongly coupled.

- Operators contributing at relative order v^2 to E1 transitions are not affected by non-perturbative soft corrections.



$$V_1^{\rm E1} = V_2^{\rm E1} = V_3^{\rm E1} = V_4^{\rm E1} = 1$$

$$V_5^{\rm E1} = c_F^{\rm em} = 1 + \frac{2\alpha_{\rm s}(m)}{3\pi} + \cdots\,, \qquad V_6^{\rm E1} = 2c_F^{\rm em} - 1 = 1 + \frac{4\alpha_{\rm s}(m)}{3\pi} + \cdots$$

○ Brambilla Pietrulewicz Vairo PRD 85 (2012) 094005

E1 transitions

E1 transitions always involve excited states. These are likely strongly coupled.

- However, non-perturbative corrections affect the quarkonium wave-functions: at large distances the quarkonium potentials are non-perturbative.
- For weakly-coupled quarkonia, non-perturbative corrections to the quarkonium wave-functions also involve octet fields and are of relative order v^2: unlike M1 dipoles, E1 dipoles do not commute with the octet Hamiltonian.

E1 transitions

$$\Gamma_{n^3P_J\to n'^3S_1\gamma} = \Gamma^{\rm E1}_{n^3P_J\to n'^3S_1\gamma}\left[1+R^{S=1}_{nn'}(J)-\frac{k_\gamma^2}{60}\frac{I_5(n1\to n'0)}{I_3(n1\to n'0)}\right.$$
$$\left.-\frac{k_\gamma}{6m}+\kappa^{\rm em}\frac{k_\gamma}{2m}\left(\frac{J(J+1)}{2}-2\right)\right]$$

$$\Gamma_{n^1P_1\to n'^1S_0\gamma} = \Gamma^{\rm E1}_{n^1P_1\to n'^1S_0\gamma}\left[1+R^{S=0}_{nn'}-\frac{k_\gamma}{6m}-\frac{k_\gamma^2}{60}\frac{I_5(n1\to n'0)}{I_3(n1\to n'0)}\right]$$

$$\Gamma_{n^3S_1\to n'^3P_J\gamma} = \frac{2J+1}{3}\Gamma^{\rm E1}_{n^3S_1\to n'^3P_J\gamma}\left[1+R^{S=1}_{nn'}(J)-\frac{k_\gamma^2}{60}\frac{I_5(n'1\to n0)}{I_3(n'1\to n0)}\right.$$
$$\left.+\frac{k_\gamma}{6m}-\kappa^{\rm em}\frac{k_\gamma}{2m}\left(\frac{J(J+1)}{2}-2\right)\right]$$

where $R^{S=1}_{nn'}(J)$ and $R^{S=0}_{nn'}$ are the (non-perturbative) initial and final state corrections.

- Brambilla Pietrulewicz Vairo PRD 85 (2012) 094005

$J/\psi \to \eta_c \gamma$

$$\Gamma_{J/\psi \to \eta_c \gamma} = \int \frac{d^3 k}{(2\pi)^3} (2\pi)\delta(E_p^{J/\psi} - k - E_k^{\eta_c}) \, |\langle \gamma(k)\eta_c | \mathcal{L}_\gamma | J/\psi \rangle|^2$$

$J/\psi \to \eta_c \gamma$

Up to order v^2 the transition $J/\psi \to \eta_c \gamma$ is completely accessible by perturbation theory.

$$\Gamma_{J/\psi\to\eta_c\gamma} = \frac{16}{3}\alpha e_c^2 \frac{k_\gamma^3}{M_{J/\psi}^2}\left[1 + 4\frac{\alpha_s(M_{J/\psi}/2)}{3\,\pi} - \frac{32}{27}\alpha_s(p_{J/\psi})^2\right]$$

The normalization scale for the α_s inherited from κ^{em} is the charm mass ($\alpha_s(M_{J/\psi}/2) \approx 0.35 \sim v^2$), and for the α_s, which comes from the Coulomb potential, is the typical momentum transfer $p_{J/\psi} \approx 2m\alpha_s(p_{J/\psi})/3 \approx 0.8$ GeV $\sim mv$.

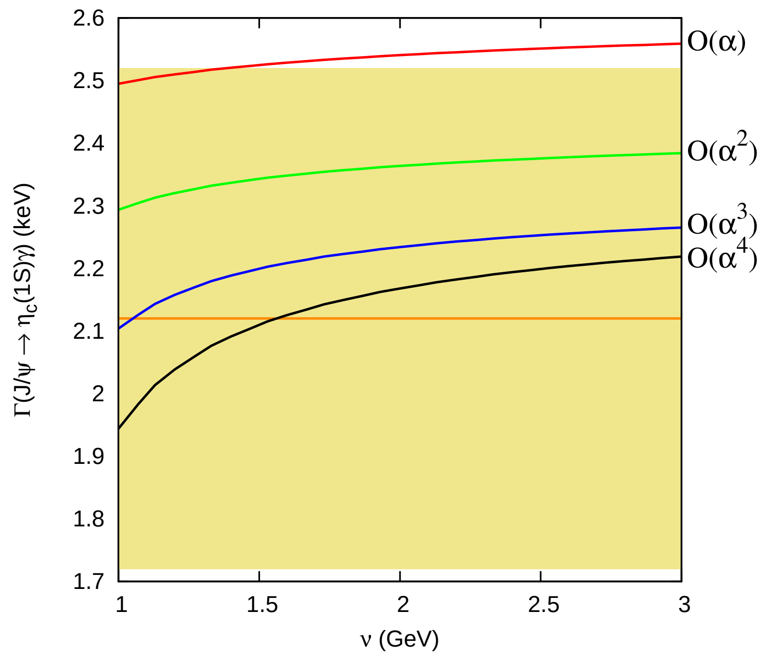
$$\Gamma_{J/\psi\to\eta_c\gamma} = (1.5 \pm 1.0)\ \text{keV}$$

to be compared with the non-relativistic result ≈ 2.83 keV.

∘ Brambilla Jia Vairo PR D73 (2006) 054005

Improved determination of M1 transitions

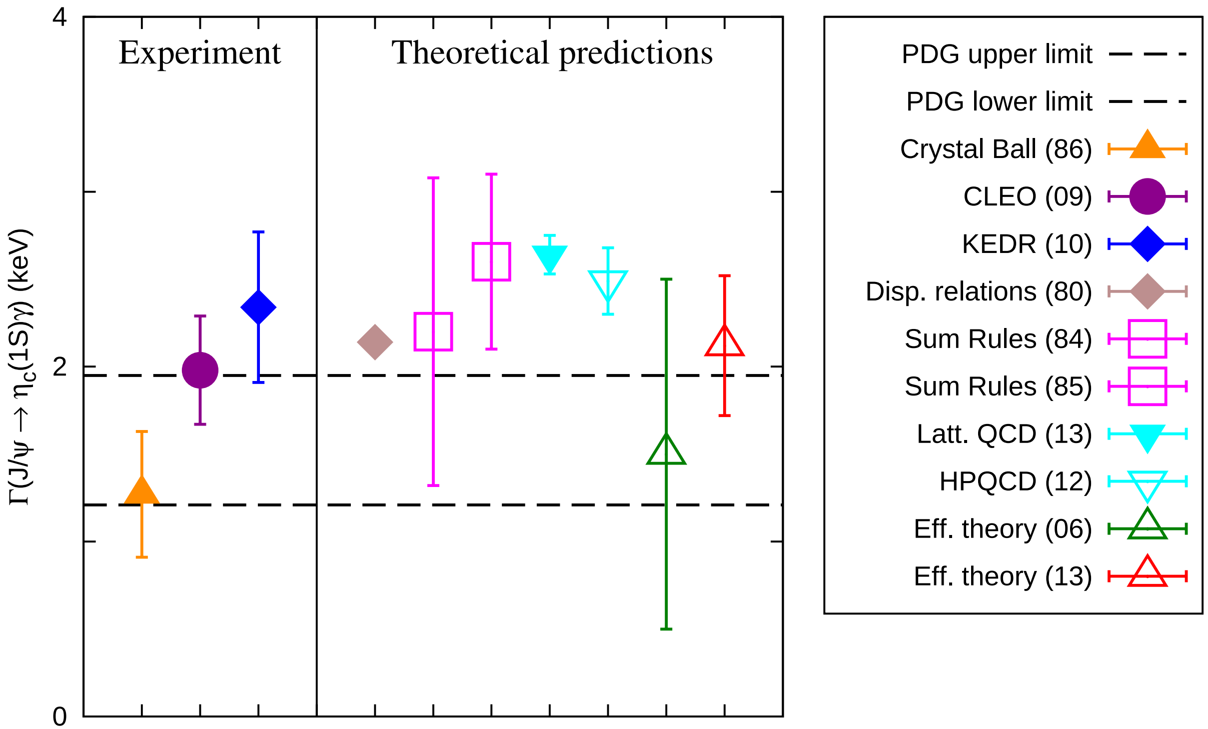
- Exact incorporation of the static potential.



$$\Gamma_{J/\psi \to \eta_c \gamma} = 2.12 \pm 0.40\ \mathrm{keV}$$

- Pineda Segovia PRD 87 (2013) 074024

$J/\psi \to \eta_c \gamma$ (experimental & theoretical status)



- Pineda Segovia PRD 87 (2013) 074024

$\Gamma_{J/\psi\to\eta_c\gamma}$ as a probe of the J/ψ potential

$$\Gamma_{J/\psi\to\eta_c\gamma} = \frac{16}{3}\alpha e_c^2 \frac{k_\gamma^3}{M_{J/\Psi}^2}\left(1 + \frac{4}{3}\frac{\alpha_{\rm s}(M_{J/\Psi}/2)}{\pi} - \frac{2}{3}\frac{\langle 1|rV_s'|1\rangle}{M_{J/\Psi}} + 2\frac{\langle 1|V_s|1\rangle}{M_{J/\Psi}}\right)$$

- If $V_s = -\frac{4}{3}\frac{\alpha_{\rm s}(\mu)}{r}$: $-\frac{2}{3}\frac{\langle 1|rV_s'|1\rangle}{M_{J/\Psi}} + 2\frac{\langle 1|V_s|1\rangle}{M_{J/\Psi}} = -\frac{32}{27}\alpha_{\rm s}(\mu)^2 < 0$
- *If* $V_s = \sigma r$: $-\frac{2}{3}\frac{\langle 1|rV_s'|1\rangle}{M_{J/\Psi}} + 2\frac{\langle 1|V_s|1\rangle}{M_{J/\Psi}} = \frac{4}{3}\frac{\sigma}{M_{J/\Psi}}\langle 1|r|1\rangle > 0$

  A scalar interaction would add a negative contribution: $-2\langle 1|V^{\rm scalar}|1\rangle/M_{J/\Psi}$.

$J/\psi \to X\,\gamma$ for $k_\gamma \lesssim$ 500 MeV

Scales:

- $\langle p \rangle \sim 1/\langle r \rangle \sim m_c v \sim$ 700 MeV - 1 GeV $\gg \Lambda_{\rm QCD}$
- $E_{J/\psi} \equiv M_{J/\psi} - 2m_c \sim m_c v^2 \sim$ 400 MeV - 600 MeV $\ll 1/\langle r \rangle$
- 0 MeV $\leq k_\gamma \lesssim$ 400 MeV - 500 MeV $\ll 1/\langle r \rangle$

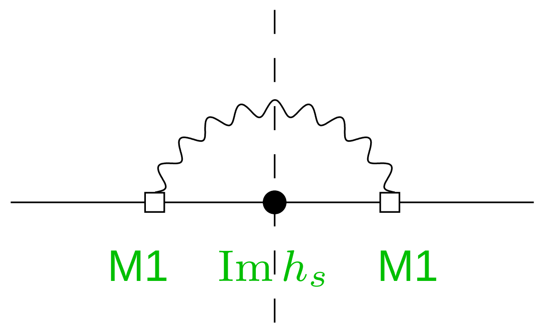
It follows that the system is

(i) non-relativistic,
(ii) weakly-coupled at the scale $1/\langle r \rangle$: $v \sim \alpha_{\rm s}$,
(iii) that we may mutipole expand in the external photon energy.

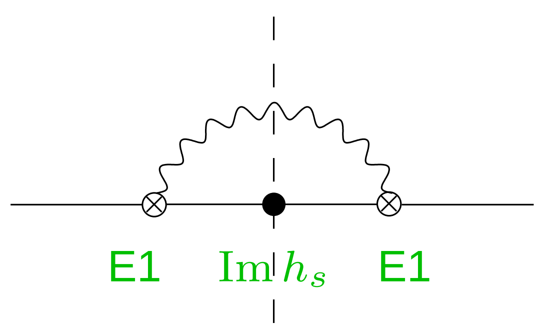
$J/\psi \to X\,\gamma$ for $k_\gamma \lesssim$ 500 MeV

Three main processes contribute to $J/\psi \to X\,\gamma$ for $k_\gamma \lesssim$ 500 MeV:

- $J/\psi \to \eta_c\,\gamma \to X\,\gamma$ [magnetic dipole interactions]



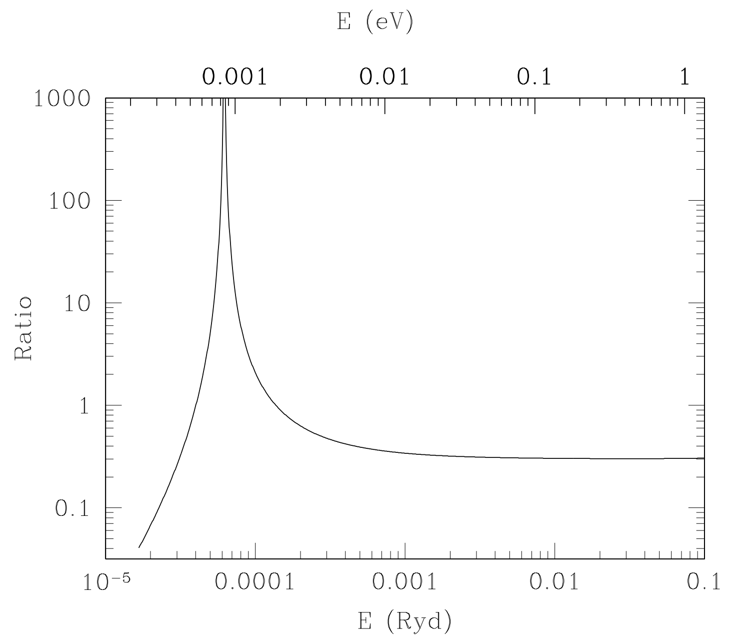
- $J/\psi \to \chi_{c0,2}(1P)\,\gamma \to X\,\gamma$ [electric dipole interactions]



- fragmentation and other background processes, included in the background functions.

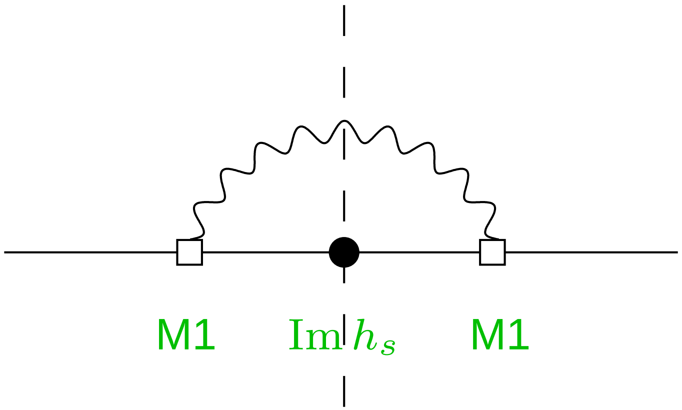
The orthopositronium decay spectrum

The situation is analogous to the photon spectrum in orthopositronium $\to 3\gamma$



- Manohar Ruiz-Femenia PRD 69 (2004) 053003
  Ruiz-Femenia NPB 788 (2008) 21, PoS EFT09 (2009) 005

$J/\psi \to \eta_c\,\gamma \to X\,\gamma$

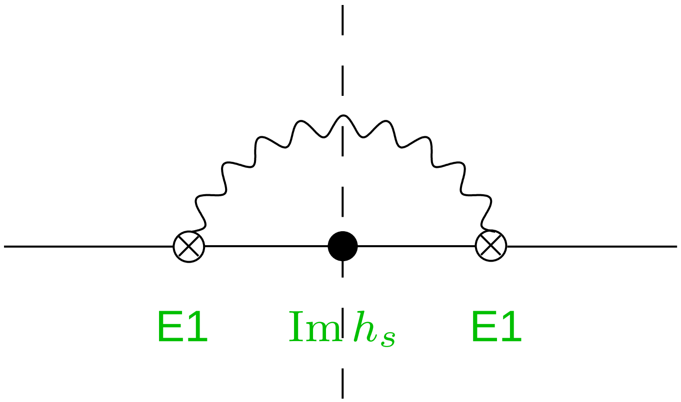


$$\frac{d\Gamma}{dk_\gamma} = \frac{64}{27}\,\frac{\alpha}{M_{J/\psi}^2}\,\frac{k_\gamma}{\pi}\,\frac{\Gamma_{\eta_c}}{2}\,\frac{k_\gamma^2}{(M_{J/\psi} - M_{\eta_c} - k_\gamma)^2 + \Gamma_{\eta_c}^2/4}$$

- For $\Gamma_{\eta_c} \to 0$ one recovers $\Gamma(J/\psi \to \eta_c\,\gamma) = \dfrac{64}{27}\alpha\dfrac{k_\gamma^3}{M_{J/\psi}^2}$
- The non-relativistic Breit–Wigner distribution goes like:

$$\frac{k_\gamma^2}{(M_{J/\psi} - M_{\eta_c} - k_\gamma)^2 + \Gamma_{\eta_c}^2/4} = \begin{cases} 1 & \text{for } k_\gamma \gg m_c\alpha_s^4 \sim M_{J/\psi} - M_{\eta_c} \\ \dfrac{k_\gamma^2}{(M_{J/\psi} - M_{\eta_c})^2} & \text{for } k_\gamma \ll m_c\alpha_s^4 \sim M_{J/\psi} - M_{\eta_c} \end{cases}$$

$J/\psi \to \chi_{c0,2}(1P)\,\gamma \to X\,\gamma$



$$\frac{d\Gamma}{dk_\gamma} = \frac{32}{81}\,\frac{\alpha}{M_{J/\psi}^2}\,\frac{k_\gamma}{\pi}\left[\frac{21\,\alpha_s^2}{2\,\pi\,\alpha^2}\right]|a(k_\gamma)|^2$$

- $a(k_\gamma) = \dfrac{(1-\nu)(3+5\nu)}{3(1+\nu)^2} + \dfrac{8\nu^2(1-\nu)}{3(2-\nu)(1+\nu)^3}\,{}_2F_1(2-\nu,1;3-\nu;-(1-\nu)/(1+\nu))$

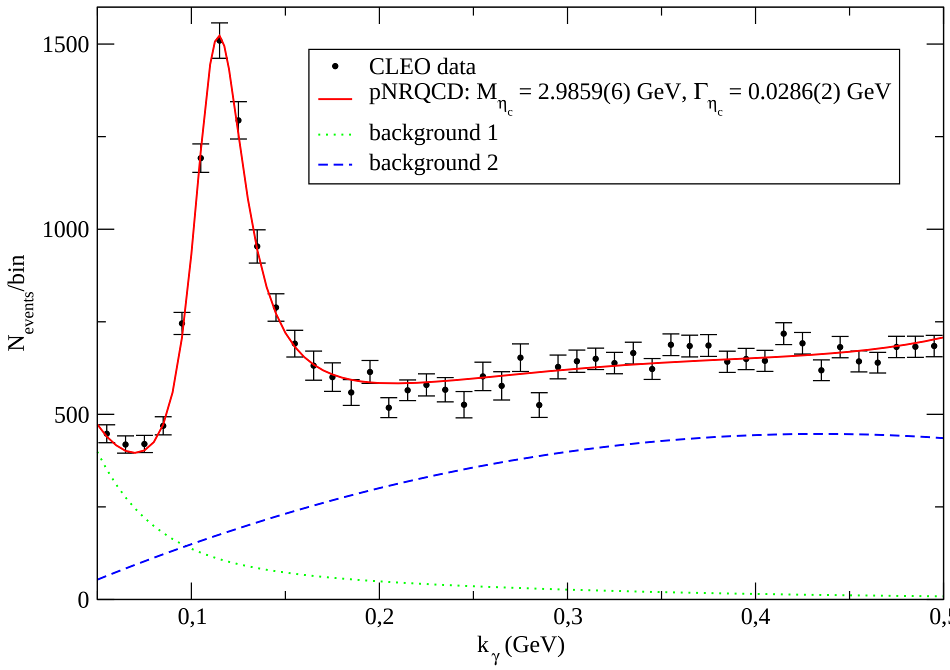
  $\nu = \sqrt{-E_{J/\psi}/(k_\gamma - E_{J/\psi})}$

  ○ Voloshin MPLA 19 (2004) 181

- $|a(k_\gamma)|^2 = \begin{cases} 1 & \text{for } k_\gamma \gg m_c\alpha_s^2 \sim E_{J/\psi} \\ k_\gamma^2/(2E_{J/\psi})^2 & \text{for } k_\gamma \ll m_c\alpha_s^2 \sim E_{J/\psi} \end{cases}$

- The two contributions are of equal order for $m_c\alpha_s \gg k_\gamma \gg m_c\alpha_s^2 \sim -E_{J/\psi}$;
- the magnetic contribution dominates for $-E_{J/\psi} \sim m_c\alpha_s^2 \gg k_\gamma \gg m_c\alpha_s^4 \sim M_{J/\psi} - M_{\eta_c}$;
- it also dominates by a factor $E_{J/\psi}^2/(M_{J/\psi} - M_{\eta_c})^2 \sim 1/\alpha_s^4$ for $k_\gamma \ll m_c\alpha_s^4 \sim M_{J/\psi} - M_{\eta_c}$.

Fit to the CLEO data



$$M_{\eta_c} \quad = \quad 2985.9 \pm 0.6\,\text{(fit) MeV} \qquad\qquad \Gamma_{\eta_c} = 28.6 \pm 0.2\,\text{(fit) MeV}$$

- Besides M_{η_c} and Γ_{η_c} the fitting parameters are the overall normalization, the signal normalization, and (three) background parameters.
- M_{η_c} is larger than CLEO's value (≈ 2982 MeV) because of the use of a non-relativistic BW (50% difference) and no damping function (50% difference).

◦ Brambilla Roig Vairo AIP Conf.Proc. 1343 (2011) 418

Further applications

- Weak coupling:
  $t\bar{t}$ threshold production, $\Upsilon(1S)$, $\eta_b(1S)$, J/ψ, $\eta_c(1S)$, B_c observables (masses, hence m_b, m_c, hfs, widhts, electromagnetic widths, radiative transitions, radiative decays, ...), also relevant for $\Upsilon(2S)$, $\eta_b(2S)$, $\chi_b(1P)$ observables.
- Strong coupling:
  lattice NRQCD, pNRQCD, exotic quarkonium states (hybrids, tetraquarks, ...), quarkonium production (factorization), exclusive hadronic decays, hadronic transitions, ...
- Quarkonium at finite temperature:
  quarkonium thermal production, quarkonium dissociation and regeneration in a fireball, spectral functions, free energies, ...

Bibliography: reviews

- N. Brambilla, A. Pineda, J. Soto and A. Vairo
  Effective field theories for heavy quarkonium
  Rev. Mod. Phys. 77 (2005) 1423
- A. Pineda
  Review of Heavy Quarkonium at weak coupling
  Prog. Part. Nucl. Phys. 67 (2012) 735
- Quarkonium Working Group [QWG]
  Heavy quarkonium physics
  CERN Yellow Report CERN-2005-005, Geneva, CERN (2005)
- Quarkonium Working Group [QWG]
  Heavy quarkonium: progress, puzzles, and opportunities
  Eur. Phys. J. C71 (2011) 1534
- Brambilla et al.
  QCD and Strongly Coupled Gauge Theories: Challenges and Perspectives
  Eur. Phys. J. C74 (2014) 2981