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Introduction to chiral symmetry in QCD

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QCD

QCD is the theory of the strong interaction, with quarks and gluons as fundamental fields. This is a gauge theory with the local symmetry group $SU(N_c)$, acting in the internal space of color degrees of freedom, with $N_c = 3$. Leaving N_c as a free parameter allows one to study the properties of the theory in more generality. The number of generators of the group $SU(N_c)$ is $(N_c^2 - 1)$.

Quark fields belong to the defining fundamental representation of the color group, which is N_c-dimensional, antiquark fields to the complex conjugate representation of the latter (N_c-dimensional), while gluon fields belong to the adjoint representation ($(N_c^2 - 1)$-dimensional). They are distinguished with color indices i, j, I:

$$\psi^i, \quad \overline{\psi}_j, \quad A_\mu^I, \quad i, j = 1, \ldots, N_c, \quad I = 1, \ldots, N_c^2 - 1.$$

The Lagrangian density of the theory is

$$\mathcal{L} = \overline{\psi}_j (i\gamma^\mu D^j{}_{i,\mu} - m\delta^j{}_i)\psi^i - \frac{1}{4} F^I_{\mu\nu} F^{I,\mu\nu}.$$

D is the covariant derivative:

$$D^j{}_{i,\mu} = \delta^j{}_i \partial_\mu + ig(T^I)^j{}_i A^I_\mu,$$

where the Ts are $N_c \times N_c$ matrix representations of the generators of the color group acting in the defining fundamental representations. $F^I_{\mu\nu}$ is the gluon field strength. g is the coupling constant (dimensionless).

Note that the masses of all quarks are equal. Necessary for the group invariance.

Flavor

QCD is a confining theory, in the sense that quarks and gluons are not observed in nature as individual particles. Only gauge invariant objects should be observable. Hadrons are gauge invariant (color singlet) bound states of quarks and gluons and are observed in nature as individual free particles.

From the spectroscopy of hadrons one deduces that there are several types of quark (six) with different masses, having the same properties with respect to the gluon fields. We have to distinguish them with a new quantum number, whose origin is not yet known, called flavor. One thus has the quarks $u,\ d,\ s,\ c,\ b,\ t$.

In the present study, we will not deal, in general with the colored sector of quarks. We will mainly be interested by the flavor sector. To simplify notation, we omit the color indices from the quark and gluon fields, but keep the flavor indices; the gluons do not have flavor indices. The quark part of the Lagrangian density can be rewritten in the form

$$\mathcal{L}_q = \sum_{f=1}^{N_f} \overline{\psi}_f \Big(i\gamma^\mu (\partial_\mu + igA_\mu) - m_f \Big) \psi_f.$$

N_f represents the number of flavors, which actually is six, but keeping it as a free parameter allows us to study the problem in more generality.

The quarks u, d, s are much lighter than the quarks c, b, t, in the sense of comparing their free masses of the Lagrangian density (deduced from the spectroscopy of hadrons).

Flavor symmetry

We consider the idealized situation where all N_f quarks have the same mass m. We assign the quark and antiquark fields to the N_f-dimensional defining fundamental representation and to its complex conjugate one, respectively, with respect to the group $SU(N_f)$ acting in the internal space of flavors. The quark Lagrangian density becomes

$$\mathcal{L}_q = \overline{\psi}_a\Big(i\gamma^\mu(\partial_\mu + igA_\mu) - m\Big)\psi^a, \qquad a = 1, \ldots, N_f.$$

It is invariant under the continuous global transformations of the group, with spacetime independent parameters. Designating the latter by α^A ($A = 1, \ldots, N_f^2 - 1$), one has in infinitesimal form

$$\delta\psi^a = -i\delta\alpha^A (T^A)^a{}_b \psi^b, \qquad \delta\overline{\psi}_a = i\delta\alpha^A \overline{\psi}_b (T^A)^b{}_a, \qquad \delta A_\mu = 0,$$

where T^A are $N_f \times N_F$ hermitian matrices.

The matrices T are representatives of the generators of the group and hence satisfy the $SU(N_f)$ algebra

$$[T^A, T^B] = if_{ABC}T^C, \qquad A, B, C = 1, \ldots, N_f^2 - 1.$$

The fs are the structure constants of the algebra; they are real and completely antisymmetric in their indices.

Using Noether's theorem, one finds $(N_f^2 - 1)$ conserved currents

$$j_\mu^A(x) = -i\frac{\partial \mathcal{L}_q}{\partial(\partial^\mu \psi^a)}(T^A)^a{}_b\psi^b = \overline{\psi}_a\gamma_\mu(T^A)^a{}_b\psi^b,$$

$$\partial^\mu j_\mu^A = 0.$$

The generators of the group transformations (also called charges) are obtained from j_0^A by space integration:

$$Q^A = \int d^3x j_0^A(x).$$

Because of current conservation, the generators are independent of time (the fields are assumed to vanish at infinity) and therefore they commute with the Hamiltonian of the system:

$$[H, Q^A] = 0.$$

They satisfy, as operators, the $SU(N_f)$ algebra

$$[Q^A, Q^B] = if_{ABC}Q^C, \qquad A, B, C = 1, \ldots, N_f^2 - 1.$$

The system under consideration is thus characterized by the existence of N_f quark fields with equal free masses m. Since the interaction term itself is invariant under the group transformations, the latter property (equality of masses) is maintained after renormalization.

We might also transcribe the transformation properties of the fields into similar properties of states. Introducing N_f one-particle quark states $|\mathrm{p}, a>$ (spin labels are omitted), created by the fields ψ^a, and ignoring for the moment the confinement problem, we obtain

$$Q^A|\mathrm{p}, a> = -(T^A)_{ab}|\mathrm{p}, b> \ .$$

The commutation property of the generators with the Hamiltonian ensures that the various one-particle states of the fundamental representation multiplet have equal masses m.

This mode of realization of the symmetry is called the Wigner-Weyl mode.

Other types of relationship can be found between observables involving the multiplet particles.

For instance, we can consider the form factors $< \mathrm{p}', b|j_\mu^A(0)|\mathrm{p}, a >$ ..

Using the transformation properties of the states and of the current under the action of the charges (the current belongs to the adjoint representation and therefore transforms like the charges), one deduces the property

$$< \mathrm{p}', b|j_\mu^A(0)|\mathrm{p}, a >= (T^A)_{ab} F((p - p')^2),$$

which shows that the whole set of various form factors actually depend on a single form factor. This is of course a manifestation of the well-known Wigner-Eckart theorem in a more general form.

Analogous properties can also be deduced for scattering amplitudes.

Considering now the world of hadrons and taking into account the fact that hadrons are bound states of quarks (constructed from quark fields), one deduces that the previous properties should also be reflected in the hadronic world, at least in the limit of equal quark masses: hadronic states should belong to irreducubible representations of the group $SU(N_f)$, deduced from their composition of quark fields, and should form multiplets with degenerate masses. Their form factors, scattering amplitudes, coupling constants, should be expressed, through group matrix elements, in terms of a fewer of them, etc.

Explicit flavor symmetry breaking

What happens if we introduce different masses for the quark fields, which ultimately is the real situation?

$$m_a = m + (\Delta m)_a$$

$$\mathcal{L}_q = \overline{\psi}_a \Big(i\gamma^\mu (\partial_\mu + igA_\mu) - m_a \Big) \psi^a, \qquad a = 1, \ldots, N_f.$$

The mass term of the Lagrangian density is no longer invariant under the group transformations of $SU(N_f)$ and therefore the current obtained previously from Noether's theorem is not conserved. One finds

$$\partial^\mu j_\mu^A = -i \sum_{a,b=1}^{N_f} (m_a - m_b) \overline{\psi}_a (T^A)^a{}_b \psi^b \neq 0.$$

The nonconservation of the current is thus due to the mass differences within the representation multiplet.

However, we might assume that we are in a phenomenological situation where the mass differences are much smaller than the interaction mass scale, which for QCD is of the order of $\Lambda_{QCD} \sim 300$ MeV. In such a case, we are entitled to treat the effects of the mass differences as perturbations with respect to the symmetric limit where all the masses are equal. We might write the Lagrangian density in the form

$$\mathcal{L} = \mathcal{L}_0 + \Delta\mathcal{L},$$

where $\mathcal{L}_0$ is the Lagrangian density with equal masses m and $\Delta\mathcal{L}$ corresponds to mass difference terms.

The above procedure is well defined in QFT, in the sense that it does not destabilize the results of the symmetric theory after renormalization. The reason is that mass operators are soft operators, because their quantized dimension is 2 for scalar fields and 3 for fermion fields, smaller than the dimension 4 of the Lagrangian density and in particular of its interaction part.

This has the effect that mass terms introduce mild effects through renormalization and at the end their effects remain perturbative. In particular, they do not affect the current with anomalous dimensions.

The situation would be different if we had introduced the symmetry breaking through the coupling constants of the interaction terms, by assigning a different coupling constant to each flavor type quark. Those have dimension 4 and their effect on renormalization is hard. At the end, one generally does not find any trace of approximate symmetry, even if at the beginning the coupling constants had been only slightly modified.

Approximate flavor symmetry with hadrons

Since hadrons are bound states of quarks and gluons, $SU(N_f)$ flavor symmetry, and more generally its approximate realization, should be reflected in their properties. In the exact symmetry case, hadrons should be classified in $SU(N_f)$ multiplets with degenerate masses.

Considering the real world with six flavor quarks, we observe that they are divided in two groups: the light quarks $u,\ d,\ s$ and the heavy quarks $c,\ b,\ t$. The mass differences between the two categories being large (> 1 GeV), an approximate symmetry can be expected only within the space of the three light quarks. Therefore, flavor symmetry would be concerned either with $SU(2)$ (isospin symmetry), involving the quarks $u,\ d$, or with $SU(3)$, involving the quarks $u,\ d,\ s$.

Concerning isospin symmetry, one notes the approximate equalities of the masses of the proton and the neutron, of the charged and neutral pions, of the kaons and of many other groups of particles.

The nucleons and the kaons could be placed in doublet (fundamental) representations of $SU(2)$ (isospin 1/2), the pions in the triplet (adjoint) representation (isospin 1), the Δs in the quadruplet representation (isospin 3/2), etc.

Since the mass differences within each multiplet are very small, of the order of a few MeV, one deduces that the difference between the masses of the quarks u and d is also of the same order:

$$|m_d - m_u| \sim \text{a few MeV}.$$

(Precise calculations should also include the contributions of the electromagnetic interaction, which also are of the same order.)

For the realization of $SU(3)$ symmetry, the quarks u, d, s are assigned to the defining representation $\mathbf{3}$, while their antiparticles to the representation $\overline{\mathbf{3}}$.

Since mesons are made from one quark and one antiquark fields, they would be classified through the product representation

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1},$$

where $\mathbf{8}$ is the octet representation and $\mathbf{1}$ the singlet. On phenomenological grounds, one notices that for example the vector mesons ρ, K^*, ω, ϕ can be grouped in an octet plus a singlet; similarly for the pseudoscalar mesons π, K, η, η', although for them the mass differences are much larger than for the vector mesons (but the understanding of its cause comes with chiral symmetry).

Baryons, being made from three quark fields, would be classified through the product representation

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \;=\; \mathbf{10} \;\oplus\; \mathbf{8} \;\oplus \mathbf{8} \;\oplus \mathbf{1}.$$

This is the case for example of the baryons $\boldsymbol{N}$, $\boldsymbol{\Sigma}$, $\boldsymbol{\Xi}$ and $\boldsymbol{\Lambda}$, which may be classified in an octet, of the baryons $\boldsymbol{\Delta}$, $\boldsymbol{\Sigma'}$, $\boldsymbol{\Xi'}$ and $\boldsymbol{\Omega}$, which may be classified in a decuplet, etc.

The approximate $\boldsymbol{SU(3)}$ symmetry leads to many relations for the mass differences of a given multiplet (Gell-Mann–Okubo formulas), for the coupling constants of particles belonging to multiplets, for form factors, etc.

The mass differences within an $SU(3)$ multiplet being of the order of 100 MeV, one deduces that the mass difference between the quark s and the quarks u and d is of the same order of magnitude and much greater than the mass difference between u and d:

$$\Big(m_s - (m_u + m_d)/2\Big) \sim 100 \text{ MeV} \gg |(m_d - m_u)| \sim \text{a few MeV}.$$

The approximate $SU(3)$ symmetry can also be used for hadrons containing heavy quarks; these, however, should stand as backgrounds for the group analysis.

For example one could apply the analysis to mesons made of $\overline{q}_h q_\ell$, or to baryons made of $q_h q_\ell q_\ell$, where $q_h = c,\ b,\ t$ and $q_\ell = u,\ d,\ s$.

Chiral symmetry

Fermion fields may undergo other types of unitary transformation than those met with the flavor symmetry. They are generated with the inclusion of the matrix γ_5 in the former transformations and are called axial flavor transformations, since they change the parity properties of fields. In infinitesimal form they are of the type

$$\delta\psi^a = -i\delta\alpha^A (T^A)^a{}_b \gamma_5 \psi^b, \quad \delta\overline{\psi}_a = -i\delta\alpha^A \overline{\psi}_b \gamma_5 (T^A)^b{}_a, \quad \delta A_\mu = 0,$$

where the indices a, b, A refer to the flavor group $SU(N_f)$ representations met before and the Ts are (hermitian) matrices representing the corresponding generators in the fundamental representation.

We consider the quark part of the QCD Lagrangian density with equal mass quarks:

$$\mathcal{L}_q = \overline{\psi}_a \Big(i\gamma^\mu (\partial_\mu + igA_\mu) - m \Big) \psi^a, \qquad a = 1, \ldots, N_f.$$

Under the above transformations, this Lagrangian density is not invariant, because of the presence of the mass terms:

$$\delta\mathcal{L} = 2im\,\delta\alpha^A\,\overline{\psi}_a(T^A)_{\ b}\,\gamma_5\psi^b.$$

Thus, invariance under axial flavor transformations requires vanishing of the quark mass terms. Contrary to the ordinary flavor symmetry transformations, equality of masses is no longer sufficient for ensuring invariance.

In the more general case of unequal masses, the mass terms can be represented in the form of a diagonal matrix $\mathcal{M}$, such that $\mathcal{M} = \mathrm{diag}(m_1, m_2, \ldots, m_{N_f})$. In that case the variation of the Lagrangian density is

$$\delta\mathcal{L} = i\,\delta\alpha^A\,\overline{\psi}_a\{\mathcal{M}, T^A\}^a_{\ b}\,\gamma_5\psi^b,$$

where $\{,\}$ is the anticommutator.

We now consider the case of massless quarks. The Lagrangian density is invariant under both the flavor and axial flavor transformations. The conserved currents are

$$j_\mu^A(x) = \overline{\psi}_a \gamma_\mu (T^A)^a{}_b \psi^b, \qquad \partial^\mu j_\mu^A = 0,$$

$$j_{5\mu}^A(x) = \overline{\psi}_a \gamma_\mu \gamma_5 (T^A)^a{}_b \psi^b, \qquad \partial^\mu j_{5\mu}^A = 0.$$

The corresponding charges are defined from space integration on the current densities:

$$Q^A = \int d^3x j_0^A(x), \qquad Q_5^A = \int d^3x j_{50}^A(x).$$

The flavor and axial flavor transformations form the set of chiral transformations. The corresponding charges satisfy the following algebra:

$$[Q^A, Q^B] = if_{ABC}Q^C, \quad [Q^A, Q_5^B] = if_{ABC}Q_5^C,$$

$$[Q_5^A, Q_5^B] = if_{ABC}Q^C, \quad A, B, C = 1, \ldots, N_f^2 - 1.$$

Note that the axial charges do not form alone an algebra. The previous algebra can, however, be simplified and become more transparent. Define

$$Q_L^A = \frac{1}{2}(Q^A - Q_5^A), \qquad Q_R^A = \frac{1}{2}(Q^A + Q_5^A).$$

(L for left-handed and R for right-handed.) One obtains

$$[Q_L^A, Q_L^B] = if_{ABC}Q_L^C, \qquad [Q_R^A, Q_R^B] = if_{ABC}Q_R^C,$$

$$[Q_L^A, Q_R^B] = 0.$$

Therefore, the left-handed and right-handed charges are decoupled and operate separately. Each of them generate an $SU(N_f)$ group of transformations. The whole chiral group is then decomposed into the direct product of two $SU(N_f)$ groups, which will be labeled with the subscripts L and R, respectively:

$$\text{Chiral group} \; = \; SU(N_f)_L \otimes SU(N_f)_R.$$

The ordinary flavor transformations form a subgroup of these, denoted $SU(N_f)_V$:

$$\text{Flavor group} \; = \; SU(N_f)_V.$$

(The subscript V refers to the vector nature of the corresponding currents.)

Explicit chiral symmetry breaking

In nature, quarks have masses. Therefore, chiral symmetry cannot be an exact symmetry of the QCD Lagrangian. The quark mass terms introduce an explicit chiral symmetry breaking.

The symmetry breaking could be treated as a perturbation only if the quark masses are much smaller than the QCD mass scale. This eliminates the heavy quarks $c,\ b,\ t$ from the domain of investigations. We are left with the sector of light quarks $u,\ d,\ s$ and the approximate chiral symmetry $SU(3)_L \otimes SU(3)_R$.

What would be the signature of this approximate symmetry in nature?

The axial charges Q_5^A are pseudoscalar objects. When acting on a massive state in the chiral symmetry limit (massless quarks), they would produce a new state with the same mass and spin, but with opposite parity:

$$Q_5^A|m, s, \mathrm{p}, +, a >= -(T^A)_{ab}|m, s, \mathrm{p}, -, b > .$$

Thus, if we adopt the Wigner-Weyl mode of realization for chiral symmetry, we should have found parity doublets for all massive states.

When the light quarks obtain masses, the degeneracy of masses within the parity doublets would be removed, but the masses would still remain close to each other.

However, no parity doublets of massive particles with approximately equal masses are found in the hadronic world, neither for mesons, nor for baryons.

This observation forces us to abandon the Wigner-Weyl mode of realization for chiral symmetry.

The other alternative that remains is the phenomenon of spontaneous symmetry breaking, also called the Nambu-Goldstone mode of realization of chiral symmetry.

Spontaneous chiral symmetry breaking

An inherent assumption within the Wigner-Weyl mode is that the ground state of the theory (the vacuum state in QFT) is invariant under the symmetry group of transformations. $\Longrightarrow$ the generators of the transformations annihilate the vacuum state:

$$Q^A|0>= 0.$$

Since one-particle states are constructed from the action of the fields on the vacuum state, the above property guarantees that one-particle states do transform as elements of irreducible representations of the symmetry group. Then, the action of a generator on a one-particle state gives again another one-particle state of the same representation.

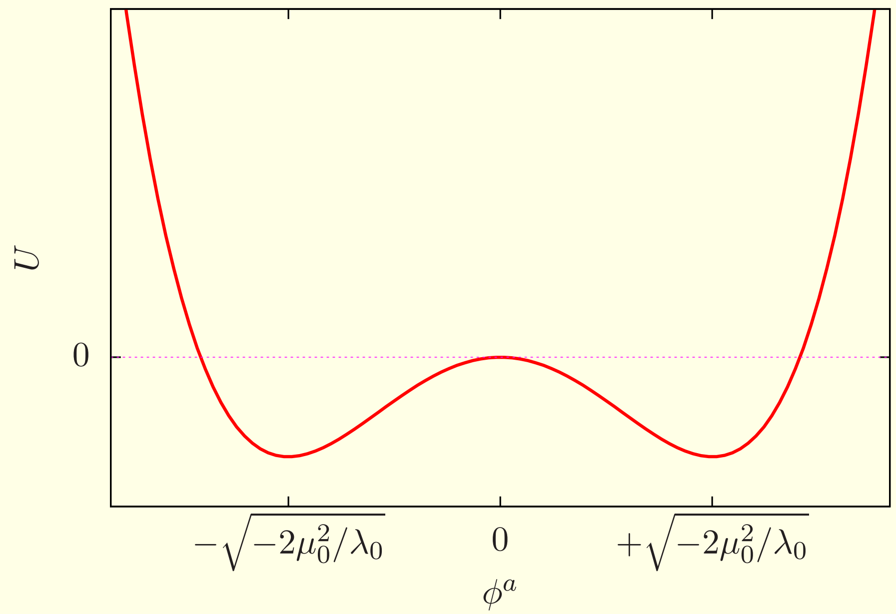
However, the vacuum state (the ground state) may not be invariant under the symmetry group of transformations, even if the Lagrangian is. In that case, it is not a symmetric state and the generators of the symmetry group do not annihilate it:

$$Q_5^A|0> \neq 0.$$

It is said that the symmetry is spontaneously broken.

A similar well-known situation is found in the $O(N)$ model or the sigma-model of spin-0 fields. Here, one studies the properties of the potential energy at the classical level. According to the values of the parameters of theory, the potential energy may have a non-symmetric ground state.

Shape of the potential energy $\boldsymbol{U}$.



The symmetric state is not the ground state, while the ground state is not symmetric.

Absence of parity doublets for massive hadronic states forces us to explore the possibility of spontaneous breaking of chiral symmetry. The axial charges, when applied on the vacuum state, would give new states:

$$Q_5^A|0>=|\chi^A,->, \qquad A=1,...,8.$$

These states have the same quantum properties as their generating axial charges. In particular, they are pseudoscalar states. Since in the symmetric limit the charges commute with the Hamiltonian, their energy is zero. This is possible only if there exist massless pseudoscalar particles, which might also generate by superposition other many-particle zero energy states. This is the content of the Goldstone theorem.

Note: the above states $|\chi^A,->$ are not one-particle states, but rather a superposition of many-massless-particle states; their norm is infinite.

Spontaneous chiral symmetry breaking is thus manifested by the existence of eight pseudoscalar massless particles (mesons), called Nambu-Goldstone bosons.

Spontaneous breaking concerns, however, only the axial sector of the charges. The ordinary flavor symmetry is still realized with the Wigner-Weyl mode. Therefore the symmetry group $SU(3)_L \otimes SU(3)_R$ is spontaneously broken down to the flavor group $SU(3)_V$:

$$SU(3)_L \otimes SU(3)_R \longrightarrow SU(3)_V.$$

In the real world, where quark have masses, chiral symmetry will undergo an additional explicit symmetry breaking. Under this effect, the eight Nambu-Goldstone bosons will acquire small masses, as compared to the masses of massive hadrons.

Considering the spectroscopy of mesons, one indeed notices the existence of eight light pseudoscalar mesons, $\pi,\ K,\ \eta$. They can be identified, in the chiral limit, with the eight Nambu-Goldstone bosons.

One would expect that the masses-squared of these particles are proportional to the masses of the quarks making their content, which would explain in turn the mass differences and hierarchies between them.

$$M_P^2 = O(\mathcal{M}).$$

For the other hadronic massive states, one should have the expansion

$$M_h^2 \ = \ M_{h0}^2 \ + \ O(\mathcal{M}),$$

M_{h0}^2 being the same for all members of a flavor multiplet and different from zero in the chiral limit. This explains why the lightest pseudoscalar meson masses are more sensitive to the quark masses than those of the massive hadrons.

Properties of Goldstone bosons

Consider, in the chiral limit, the coupling of the Goldstone bosons to the axial-vector current:

$$< 0|j_{5\mu}^{A}|P^{B}, p >= \ i\, \delta^{A}_{\ B}\, p_{\mu}\, F_{P}.$$

$F_P \neq 0$, otherwise, the axial charges, which are constructed from j_{50}^{A}, could not create from the vacuum massless states.

Using conservation of the axial current and $\partial^{\mu} j_{5\mu}^{A} = i[P_{\mu}, j_{5\mu}^{A}]$:

$$< 0|\partial^{\mu} j_{5\mu}^{A}|P^{B}, p >= \ 0 \ = \ \delta^{A}_{\ B}\, p^{2}\, F_{P} \ = \ \delta^{A}_{\ B}\, M_{P}^{2}\, F_{P} \ = \ 0 \quad (M_{P} = 0).$$

$\Longleftrightarrow$ necessity of $M_P = 0$.

Consider the coupling of a massive pseudoscalar state (for example a radial excitation of the Goldstone boson) to the axial-vector current:

$$< 0|j_{5\mu}^{A}|P'^{B}, p > = \ i\,\delta^{A}_{\ B}\,p_{\mu}\,F_{P'}.$$

Using conservation of the axial current:

$$< 0|\partial^{\mu}j_{5\mu}^{A}|P'^{B}, p > = \ 0 \ = \ \delta^{A}_{\ B}\,p^{2}\,F_{P'} \ = \ \delta^{A}_{\ B}\,M_{P'}^{2}\,F_{P'}.$$

$\Longrightarrow \quad F_{P'} = 0 \quad (M_{P'} \neq 0)$.

$\Longrightarrow$ the massive pseudoscalar states decouple from the axial-vector current.

Goldstone bosons: $M_P = 0$, $F_P \neq 0$.

Massive pseudoscalar mesons: $M_{P'} \neq 0$, $F_{P'} = 0$.

When quarks obtain masses, the above properties are modified by terms proportional to the quark masses.

Goldstone bosons:

$$M_P^2 = O(\mathcal{M}), \qquad F_P = F_{P0} + O(\mathcal{M}),$$

Massive pseudoscalar mesons:

$$M_{P'}^2 = M_{P'0}^2 + O(\mathcal{M}), \qquad F_{P'} = O(\mathcal{M}),$$

with $M_P^2 \ll M_{P'}^2$, $F_{P'} \ll F_P$.

On experimental grounds, from the leptonic decays of π and K:

$$F_\pi \simeq 92\,\mathrm{MeV}, \qquad F_K \simeq 110\,\mathrm{MeV}.$$

The quantity $(F_K/F_\pi - 1) \simeq 0.2$ measures the order of magnitude of flavor $SU(3)$ breaking.

Remark: In QCD, the quark masses appear as free parameters. Therefore, one expects that all hadronic physical quantities – masses, decay couplings, coupling constants, form factors, scattering amplitudes – possess analyticity properties in them, up to the existence of cuts or branching points.

These objects appear in general as residues of Green's functions at physical particle poles. Therefore, they define on-mass shell quantities. They should not be considered as functions of the mass-shell variables $p^2,\ p'^2$, etc., but only of the quark mass parameters and of the momentum transfers or of the Mandelstam variables $s,\ t,\ u$, etc., which, eventually may take unphysical values by analytic continuation.

Green's functions, on the other hand, may be functions of the mass-shell variables $p^2,\ p'^2$, etc.

Low-energy theorems

The decoupling of the massive pseudoscalar mesons from the axial-vector currents in the chiral limit (massless quarks) allows one to derive low-energy theorems concerning processes where enters at least one Goldstone boson. Most of these relations are obtained with the aid of the Ward-Takahashi identities.

Contrary to to the ordinary flavor symmetry, chiral symmetry does not yield linear relations between matrix elements of multiplets, but rather leads to relations between processes involving absorption and/or emission of Goldstone bosons at low momenta:

$$\alpha \to \beta \qquad \longleftrightarrow \qquad \alpha + n_1 P \to \beta + n_2 P.$$

Goldberger-Treiman relation

Matrix element of the axial-vector current between a proton and a neutron state in the isospin limit:

$$< p(p')|\, j_{5\mu}^{1+i2}\, |n(p) >= \overline{u}_p(p') \left[\gamma_\mu\gamma_5 g_A(q^2) + q_\mu\gamma_5 h_A(q^2) \right] u_n(p),$$

where $q = (p - p')$. g_A and h_A are the axial-vector form factors of the nucleons.

Take the divergence of the current:

$$< p(p')|\, \partial^\mu j_{5\mu}^{1+i2}\, |n(p) >= -i\Big(2M_N g_A(q^2) + q^2 h_A(q^2)\Big)\, \overline{u}_p(p')\gamma_5 u_n(p).$$

The left-hand side has singularities through the contribution of pseudoscalar intermediate states. For simplicity and illustrative purposes, let us assume that the latter can be saturated by a series of narrow-width particles (the pion and its radiative excitations). It can be shown that multipion states do not contribute to the final result.

$$< p(p')|\, \partial^\mu j_{5\mu}^{1+i2}\, |n(p)> = -2i\Big\{ \frac{M_\pi^2 F_\pi}{M_\pi^2 - q^2} g_{\pi NN} + \sum_{n=1}^{\infty} \frac{M_{\pi^n}^2 F_{\pi^n}}{M_{\pi^n}^2 - q^2} g_{\pi^n NN} \Big\}$$

$$\times \overline{u}_p(p')\gamma_5 u_n(p),$$

where $g_{\pi NN}$ and $g_{\pi^n NN}$ are the coupling constants of the pseudoscalar mesons with the nucleons. $\Longrightarrow$

$$2\frac{M_\pi^2 F_\pi}{M_\pi^2 - q^2} g_{\pi NN} + 2\sum_{n=1}^{\infty} \frac{M_{\pi^n}^2 F_{\pi^n}}{M_{\pi^n}^2 - q^2} g_{\pi^n NN} \ = \ 2M_N g_A(q^2) + q^2 h_A(q^2).$$

Take the limit $q^2 = 0$. In the right-hand side, $h_A(q^2)$ does not have a pole at this value (no massless pseudoscalars in the real world).

$$F_\pi g_{\pi NN} + \sum_{n=1}^{\infty} F_{\pi^n} g_{\pi^n NN} \ = \ M_N g_A(0).$$

Consider now the $SU(2) \otimes SU(2)$ chiral limit (massless u and d quarks). All massive pseudoscalar mesons decouple from the axial-vector current ($F_{\pi^n} = 0$ for $\mathcal{M} = 0$).

$$g_A(0) = \frac{F_\pi g_{\pi NN}}{M_N} \qquad (\mathcal{M} = 0).$$

This is an exact result of QCD in the $SU(2) \otimes SU(2)$ chiral limit (a low-energy theorem).

Experimental values are: $g_A \simeq 1.27$, $g_{\pi NN} \simeq 13.40$, $F_\pi \simeq 92.2$ MeV, $M_N = 938.92$ MeV. The right-hand side is $\simeq 1.32$, to be compared with the left-hand side 1.27. The discrepancy is about 4%, which is typical of the corrections coming from explicit breaking of $SU(2) \otimes SU(2)$ symmetry.

Ward-Takahashi identities

Ward-Takahashi identities (WTI) are obtained by considering Green's functions of axial-vector currents with local operators $O(x)$.

$$\int dx e^{iq.x} < \beta|T\, j_{5\mu}^{A}(x)\, O(0)|\alpha >,$$

where $|\alpha >$ and $|\beta >$ are hadronic states.

Consider the divergence of the axial-vector current:

$$-iq^{\mu} \int dx e^{iq.x} < \beta|T\, j_{5\mu}^{A}(x)\, O(0)|\alpha >= \int dx e^{iq.x} < \beta|T\, \partial^{\mu} j_{5\mu}^{A}(x)\, O(0)|\alpha >$$

$$+ \int dx e^{iq.x} \delta(x^0) < \beta|[j_{50}^{A}(x), O(0)]|\alpha > .$$

One notices the presence of the equal-time commutator, which should be evaluated in the theory. Usually, this is again a local operator.

One then takes the limit of low or zero values for q and proceeds with similar methods as in the Goldberger-Treiman case.

The method can also be generalized by considering multilocal operators, such as $O_1(x_1)O_2(x_2)\dots O_n(x_n)$.

Callan-Treiman relation

Choose $|\alpha >= |K^+ >$, $|\beta >= |0 >$, $O = j_\nu^{4-i5}$ and $j_{5\mu}^A = j_{5\mu}^3$. The equal-time commutator of the WTI yields the axial-vector current $j_{5\nu}^{4-i5}$ plus Schwinger terms that do not contribute. The matrix element involving the divergence of the axial-vector current is again saturated by the pion and its radial excitations. Now the corresponding residues are proportional to the $K_{\ell 3}$ form factors with respect to the pion and to its radial excitations.

One has the definition

$$< \pi^0(p')|j_\nu^{4-i5}|K^+(p) >= (p+p')_\nu f_+(t) + (p-p')_\nu f_-(t), \quad t = (p-p')^2,$$

with similar definitions with the radial excitations of the pion.

Take the limit $q \to 0$ in the WTI, in which case the left-hand side vanishes (no poles at $q = 0$). Then the $SU(2) \otimes SU(2)$ chiral limit is taken (massless u and d quarks). The massive pseudoscalar states decouple and one ends up with the relation

$$f_+(M_K^2) + f_-(M_K^2) = \frac{F_K}{F_\pi}.$$

The physical domain of the $K_{\ell 3}$ form factors (corresponding to the decay $K \to \pi \ell \nu$) being limited by the inequalities $m_\ell^2 \leq t \leq (M_K - M_\pi)^2$, the form factors appearing in the above relation are evaluated at the unphysical point $t = M_K^2$. Extrapolations (usually linear) are used from the physical domain to reach that point. The relation is well satisfied, with a few percent discrepancy, on experimental grounds.

The Callan-Treiman formula establishes a relation between the form factors of the process $K \to \pi \ell \nu$ and the decay coupling of the process $K \to \ell \nu$.

Pion scattering lengths (Weinberg)

Choose in the WTI for $|\alpha >$ and $|\beta >$ target particles, like N, K or π, and for O and $j_{5\mu}^A$ axial vector currents with the pion quantum numbers.

Then calculate the divergences of the two currents. The corresponding two momenta squared q^2 and k^2 are taken to zero, but q and k are maintained nonzero, at close values of M_π. One ends up with formulas for the S-wave scattering lengths of the processes $\pi + \alpha \to \pi + \alpha$, where α is the target particle, much heavier than the pion:

$$a_0^I = -\frac{M_\pi}{8\pi F_\pi^2}(1 + \frac{M_\pi}{M_\alpha})^{-1}\,[I(I+1) - I_\alpha(I_\alpha + 1) - 2],$$

where I is the total isospin of the state $|\pi\alpha >$ and I_α the isospin of the target.

The above formula is applied for the scattering processes $\pi + N \to \pi + N$ and $\pi + K \to \pi + K$.

When the target is the pion itself, the analysis should be completed by retaining higher-order terms in the kinematic variables. Crossing symmetry is also used. One then obtains the $\pi - \pi$ scattering amplitude at low energies:

$$\mathcal{M}_{ac,bd} = \frac{1}{F_\pi^2}\{\delta_{ac}\delta_{bd}(s - M_\pi^2) + \delta_{ab}\delta_{cd}(t - M_\pi^2) + \delta_{ad}\delta_{bc}(u - M_\pi^2)\},$$

where a, b, c, d are the pion isospin indices and s, t, u the Mandelstam variables.

The S-wave scattering lengths are

$$a_0^0 = \frac{7M_\pi}{32\pi F_\pi^2} \simeq 0.16\ M_\pi^{-1}, \qquad a_0^2 = -\frac{2M_\pi}{32\pi F_\pi^2} \simeq -0.046\ M_\pi^{-1}.$$

The above predictions are well satisfied experimentally within $10 - 25\%$ of discrepancy. Direct measurements of the scattering lengths are, however, not possible because of the instability of the pion under weak or electromagnetic interactions. Elaborate extrapolation procedures are used for the extraction of the scattering lengths from high-energy data.

Adler-Weisberger relation

The starting point is the same as for the calculation of the scattering lengths. One chooses in the WTI for $|\alpha>$ and $|\beta>$ nucleon states. For O and $j_{5\mu}^A$ one chooses axial vector currents with the pion quantum numbers. The corresponding two momenta q and k are taken to zero. In this limit, there is in addition a nucleon pole that contributes, yielding as a residue the axial-vector form factor at zero momentum transfer. At the end of the operations, one obtains the isospin antisymmetric part of the pion-nucleon scattering amplitude at an unphysical point. The latter is reexpressed by means of a dispersion relation in terms of an integral over physical pion-nucleon cross sections. The final formula is

$$g_A^2 = 1 - \frac{2F_\pi^2}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}\left[\sigma^{\pi^- p}(\nu) - \sigma^{\pi^+ p}(\nu)\right], \qquad \nu = q.p\ .$$

The right-hand side yields for g_A the value 1.24, to be compared to its experimental value 1.27.

Gell-Mann–Oakes–Renner formulas

Choose in the WTI $|\alpha> = |\beta> = |0>$ and for O the divergence of the axial-vector current. One has

$$\partial^\nu j^B_{5\nu} = i\overline{\psi}_a\{\mathcal{M}, T^B\}^a_{\ b}\,\gamma_5\psi^b.$$

The operators $v^B = -i\overline{\psi}_a(T^B)^a_{\ b}\gamma_5\psi^b$ define the pseudoscalar densities. They transform, in the chiral limit, under the action of the axial charges as

$$[Q^A_5, v^B] = id_{ABC}u^C + i\frac{2}{3}\delta_{AB}u^0,$$

where the us are the scalar densities

$$u^C = \overline{\psi}_a(T^C)^a_{\ b}\psi^b, \qquad C = 1, \ldots, 8, \qquad u^0 = \overline{\psi}_a\psi^a,$$

and the coefficients d are fully symmetric in their indices; they result from the anticommutators of the matrices T: $\{T^A, T^B\} = d_{ABC}T^C + \frac{1}{3}\delta_{AB}\mathbf{1}$.

The WTI takes the form

$$-iq^{\mu}\int dx e^{iq.x} < 0|T\, j_{5\mu}^{A}(x)\, \partial^{\nu} j_{5\nu}^{B}|0 >$$

$$= \int dx e^{iq.x} < 0|T\, \partial^{\mu} j_{5\mu}^{A}(x)\, \partial^{\nu} j_{5\nu}^{B}|0 >$$

$$+ \int dx e^{iq.x}\delta(x^0) < 0|[j_{50}^{A}(x), \partial^{\nu} j_{5\nu}^{B}]|0 > .$$

Intermediate states are only pseudoscalar mesons. In the limit $q = 0$, no poles in the first term.

$$\delta_{AB}\left\{M_{PA}^2 F_{PA}^2 + \sum_{n=1}^{\infty} M_{PnA}^2 F_{PnA}^2\right\} = -\,\mathrm{tr}\{T^A, \{\mathcal{M}, T^B\}\} < 0|\frac{1}{3}u^0|0 > .$$

We have seen that

$$M^2_{PA} = O(\mathcal{M}), \qquad F_{PA} = F_{P0} + O(\mathcal{M}),$$

$$M^2_{PAn} = M^2_{Pn_0} + O(\mathcal{M}), \qquad F_{PAn} = O(\mathcal{M}),$$

$\Longrightarrow$

$$F^2_{PAn} = O(\mathcal{M}^2) \ll O(\mathcal{M}).$$

Keep only $O(\mathcal{M})$ terms:

$$\delta_{AB} M^2_{PA} F^2_{P0} = -\,\mathrm{tr}\{T^A, \{\mathcal{M}, T^B\}\} < 0|\frac{1}{3}u^0|0 > .$$

$\mathcal{M} = (m_u - m_d)T^3 - \frac{1}{\sqrt{3}}(2m_s - m_u - m_d)T^8 + \frac{1}{3}(m_u + m_d + m_s)1$;
tr$(T^A T^B) = \frac{1}{2}\delta_{AB}$.

Defining

$$B = -\frac{1}{3F_{P0}^2} < 0|u^0|0 >, \qquad \hat{m} = \frac{1}{2}(m_u + m_d),$$

one finds, neglecting electromagnetism and $\pi^0 - \eta - \eta'$ mixings,

$$M_{\pi^+}^2 = M_{\pi^0}^2 = 2\hat{m}B,$$

$$M_{K^+}^2 = (m_s + m_u)B, \qquad M_{K^0}^2 = (m_s + m_d)B,$$
$$M_\eta^2 = \frac{2}{3}(2m_s + \hat{m})B.$$

Verification of the $SU(3)_V$ Gell-Mann–Okubo formula (in the isospin limit):

$$4M_K^2 - 3M_\eta^2 - M_\pi^2 = 0.$$

The masses M_P and decay couplings F_P are physical quantities; threfore they do not depend on renormalization mass scales. However, the quark masses and the scalar densities are renormalized under interaction and depend on the renormalization mass scale, although their product does not. One must specify, when providing values for the quark masses, at which scale they have been evaluated. (Usually, they are chosen at a mass scale $\mu = 2$ GeV.) Also the ratios of quark masses are renormalization group invariant.

Numerically, one finds from the above formulas

$$\frac{m_s}{\hat{m}} = 26.0, \quad \frac{m_u}{m_d} = 0.65, \quad \frac{m_s}{m_d} = 21.5.$$

We come back to the vacuum expectation value of the scalar density $< 0|u^0|0 >$. With respect to flavor $SU(3)$, u^0 is a singlet operator. However, with respect to the chiral group $SU(3)_L \otimes SU(3)_R$, it has a more complicated structure.
Introduce left-handed and right-handed quark and antiquark fields,

$$\psi_L^a = \frac{1}{2}(1-\gamma_5)\psi^a, \ \ \psi_R^a = \frac{1}{2}(1+\gamma_5)\psi^a, \ \ \overline{\psi}_{La} = \frac{1}{2}\overline{\psi}_a(1+\gamma_5), \ \ \overline{\psi}_{Ra} = \frac{1}{2}\overline{\psi}_a(1-\gamma_5).$$

$$u^0 = \overline{\psi}_a\psi^a = \overline{\psi}_{Ra}\psi_L^a + \overline{\psi}_{La}\psi_R^a.$$

The left-handed and right-handed fields belong to representations of different groups, $SU(3)_L$ and $SU(3)_R$. One finds that u^0 belongs to the $(\overline{3}_L, 3_R) + (3_L, \overline{3}_R)$ representation of the chiral group. This is not the singlet representation.
If the vacuum were invariant under chiral transformations, then $< 0|u^0|0 >$ would be zero, according to the Wigner-Eckart theorem. Its nonvanishing is a sign that the vacuum is not invariant under chiral transformations and therefore chiral symmetry is spontaneously broken. $< 0|u^0|0 >$ represents an order parameter of spontaneous chiral symmetry breaking. Analogy with ferromagnetism.

Chiral perturbation theory

After the successes of the predictions of low energy theorems, obtained in the chiral limit or in leading order of explicit chiral symmetry breaking, one naturally is interested by the calculation of corrective terms to the leading-order quantities.

Essentially, two types of correction arise.

1) Quark mass terms.

2) Many-Goldstone-boson state contributions. These do not contribute at leading orders because of damping factors coming from phase space coefficients. At nonleading orders, they are no longer negligible. They produce unitarity cuts and corresponding logarithmic functions of the momenta and the masses. Early calculations were done by Li and Pagels (1971).

Calculation of the nonleading chiral symmetry corrections through a systematic perturbative approach was proposed by Glashow and Weinberg (1967) and by Dashen (1969).

Initial calculations were done with the use of the Ward-Takahashi identities. But at higher-orders, this method becomes rapidly very complicated.

Weinberg has made several observations.

1) In spite of the fact that we are in the domain of strong interactions, the couplings of the Goldstone bosons with other particles and with themselves turned out to be relatively weak.

2) These couplings were reminiscent of derivative coupling types.

3) All results of low-energy theorems could be obtained (often more easily) by using phenomenological Lagrangians involving only the hadrons entering into the concerned processes and satisfying chiral invariance; it was sufficient to do the calculations at the tree level.

Chiral effective field theories

How to construct chiral invariant hadronic Lagrangians?

Basic ingredient: the chiral transformation properties of the Goldstone boson fields.

The latter are obtained by studying the action of the chiral charge commutators on them. Equations are obtained that can be solved. The solutions indicate, as expected, that the Goldstone boson fields transform nonlinearly under chiral transformations.

Similarly, one obtains the chiral transformation properties of massive hadronic states. The latter transform linearly, but the transformation coefficients are nonlinear functions of the Goldstone boson fields.

The construction of chiral invariant Lagrangians necessitate the introduction of chiral covariant derivatives which also allow the introduction of the interaction terms.

Once chiral invariant Lagrangians are constructed, one can go further and implement nonleading-order calculations.

One introduces explicit chiral symmetry breaking through mass terms of Goldstone bosons.

Next, one calculates loop corrections. Since Goldstone boson couplings are of the derivative type, each loop introduces, through its vertices, additional powers of the Goldstone boson momenta. At low energies, the latter are small, of the same order of magnitude as the Goldstone boson masses. Increasing the number of loops increases in turn the powers of the momenta and hence decreases the magnitude of the terms in comparison with the leading ones.

Therefore, one is led to a systematic power counting rule. The Goldstone boson mass squared M_P^2 and momenta squared, $q^2,\ k^2$, etc., are considered as perturbation theory parameters. Their higher powers will correspond to higher-order terms. Thus, the first-order corrections will involve at most one-loop diagrams.

A crucial point that remains to be dealt with is renormalization.

In order for the phenomenological hadronic Lagrangian to describe correctly the underlying theory (QCD), it is necessary to consider the most general chiral invariant Lagrangian. Therefore, the latter will involve an infinite series of terms with an increasing number of derivatives and powers of the masses. It will involve, from the start an infinite number of unknown constants. However, the Lagrangian will be ordered according to the number of derivatives or powers of masses it involves.
For instance, for the purely Goldstone boson Lagrangian one has the expansion

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots + \mathcal{L}_n + \cdots \ ,$$

where the subscript n designates the number of derivatives or the power of the masses. At each finite order of the expansion, there corresponds a finite number of unknown coupling constants. The leading-order contributions come from $\mathcal{L}_2$, those of the next-to-leading order from $\mathcal{L}_2 + \mathcal{L}_4$, etc.

Calculation of loops introduces divergences. These are then absorbed by the renormalization of the coupling constants of the higher-order parts of the Lagrangian. These coupling constants are called low energy constants (LEC). They also contain information about the contributions of the high-mass particles which do not appear explicitly in the Lagrangian.

Thus in the Lagrangian describing $\pi - \pi$ interaction, the ρ-meson is absent, but since the Lagrangian is describing only low-energy regions, the ρ-meson field is in some sense integrated out from the total Lagrangian of the theory and its effects are enclosed in the resulting coefficients.

At each finite order of the perturbation theory, there are only a finite number of unknown constants, which should be determined from experiment. Then the theory becomes predictive at that order.

The phenomenological chiral hadronic Lagrangians that are constructed from the previous principles define the chiral effective field theories designed to describe definite sectors of hadrons.

The chiral effective field theory for Goldstone bosons was first adapted to QCD by Gasser and Leutwyler (1984-1985).

Applications concerned processes involving π, K, η mesons, leading to improved precision with respect to the low-energy theorem results.

The method was generalized including $\pi - N$ and $N - N$ interactions.

Currently, nuclear physics forces are being calculated from the same type of approach.

In conclusion, the effective field theory approach enables one to convert, at low energies and in the hadronic world, the highly complicated nonperturbative structure of QCD into a more transparent and familiar framework, with the sole aid of the symmetry properties of the theory.

Other topics

Anomalies.

Theoretical (indirect) proof of spontaneous chiral symmetry breaking in QCD. ’t Hooft’s anomaly matching conditions.

QCD sum rules.

Dynamical mass generation.