Constraining effective actions via scattering amplitudes

Confinement 2016, Thessaloniki

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arXiv: 1605.08697

with

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02/09/2016
Outline

1) Introduction

2) Supersymmetry constraints on the effective action of N=4 SYM in the Coulomb branch

3) Constraints from breaking conformal symmetry in the form of soft theorems

4) Scale invariance vs conformal symmetry

5) Conclusions and remarks
Introduction

Requirement of having a consistent $S$-matrix can impose highly non-trivial constraints.

We apply these ideas to effective theories.
Introduction

Requirement of having a consistent $S$-matrix can impose highly non-trivial constraints

We apply these ideas to effective theories

four-derivative terms should have positive coefficients

Allan, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi (2006)

$\mathcal{L}_{\text{eff}} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{c}{\Lambda^4} (\partial^\mu \phi \partial_\mu \phi)^2 + \ldots$

Argument

$\frac{c}{\Lambda^4} = \frac{2}{\pi} \int_{s_0}^{\infty} ds \frac{\text{Im} A_4(s,0)}{s^3} = \frac{2}{\pi} \int_{s_0}^{\infty} ds \frac{s \sigma(s)}{s^3} > 0$
Introduction

Requirement of having a consistent S-matrix can impose highly non-trivial constraints.

We apply these ideas to effective theories.

Four-derivative terms should have positive coefficients.

Allan, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi (2006)

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{e}{\Lambda^4} (\partial^\mu \phi \partial_\mu \phi)^2 + \ldots \]

Positivity for the proof of the a-theorem in 4D

Komargodski, Schwimmer (2008)

\[ c = \Delta a = a_{\text{UV}} - a_{\text{IR}} > 0 \]
Effective action of $N=4$ SYM in the Coulomb branch

The Coulomb branch

$$\langle 0 | \varphi | 0 \rangle = v$$

Spontaneous gauge symmetry breaking

$$U(N + 1) \rightarrow U(N) \times U(1)$$

We focus on the $U(1)$ part
Effective action of $N=4$ SYM in the Coulomb branch

The Coulomb branch
\[ \langle 0 | \varphi | 0 \rangle = \nu \]

Spontaneous gauge symmetry breaking
\[ U(N + 1) \rightarrow U(N) \times U(1) \]

We focus on the $U(1)$ part

Expanding in large $m_w$, the effective action becomes

\[
S_{\text{eff}} = -\frac{1}{4} F^2 + \frac{c_4^{(0)}}{m_w^4} F^4 + \frac{c_6^{(0)}}{m_w^8} F^6 + \frac{c_4^{(2)}}{m_w^8} \partial^4 F^4 + \frac{c_4^{(3)}}{m_w^{10}} \partial^6 F^4 + \ldots
\]
Effective action of \( \mathbf{N}=4 \) SYM in the Coulomb branch

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\]

Perturbation contributions

\[
A_{\text{1-loop}}^4 = F^4 \int \frac{d^4 \ell}{(2\pi)^4} \frac{g^4N}{(\ell^2 + m_w^2)((\ell + p_1)^2 + m_w^2)((\ell + p_{12})^2 + m_w^2)((\ell - p_4)^2 + m_w^2)}
\]
Effective action of $\mathcal{N}=4$ SYM in the Coulomb branch

$$S_{\text{eff}} = -\frac{1}{4} F^2 + \frac{c_4^{(0)}}{m_w^4} F^4 + \frac{c_6^{(0)}}{m_w^8} F^6 + \frac{c_4^{(2)}}{m_w^8} \partial^4 F^4 + \frac{c_4^{(3)}}{m_w^{10}} \partial^6 F^4 + \ldots$$

Perturbation contributions

Bianchi, Morales, Wen (2015)

$$\partial^k F^4 : \quad \frac{g^4 N}{(4\pi)^2 m^4} \left[ \frac{1}{2} + \frac{1}{240 m^4} (s^2 + t^2 + u^2) + \ldots + \frac{1}{192192 m^{16}} \left( (s^6 + t^6 + u^6) - \frac{7}{75} (s^2 t^2 u^2) \right) \right] + \ldots$$
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\[ \partial^k F^4 : \quad \frac{g^6 N(N+1)}{(4\pi)^4 m^4} \left[ 0 + \frac{1}{24 m^4} (s^2 + t^2 + u^2) + \ldots + \frac{7541}{59875200 m^{16}} \left( (s^6 + t^6 + u^6) - \frac{249}{7541} (s^2 t^2 u^2) \right) \right] + \ldots \]
Effective action of N=4 SYM in the Coulomb branch

1-instanton effective action for Sp(2N) gauge theory

\[ S_{\text{eff}}^{1-\text{inst}} = c' \frac{g^4}{\pi^6} e^{2\pi i \tau} \int \frac{d^4 x \, d^8 \theta \sqrt{\det 4N} \, 2\Phi_{Au,Bv}}{\sqrt{\det 2N} \left( \Phi^{AB} \Phi_{AB} + \frac{1}{g} \bar{F} + \frac{1}{\sqrt{2g}} \bar{\Lambda}_A (\Phi^{-1})^{AB} \bar{\Lambda}_B \right)_{\dot{\alpha}u,\dot{\beta}v}} \]

Bianchi, Morales, Wen (2015)

Integrating out the Grassmann variables and expanding in $1/v$

\[ S_{\text{eff}}^{1-\text{inst}} = e^{2\pi i \tau} \sum_{\ell} \left( 0 \times (F_-^2 F_{2\ell}^+ + F_{2\ell}^+ F_-^2) + c_1 F_-^4 F_{2\ell}^2 + c_2 \partial^4 F_-^2 F_{2\ell}^2 + \ldots \right) \]

Suggesting a non-renormalization theorem for the operator

\[ F_-^2 F_{2\ell}^2 \]
Effective action of N=4 SYM in the Coulomb branch

This is a known non-renormalization theorem that $F^4$ is one-loop exact in N=4 SYM

Dine, Seiberg (1997)

$F^6$ was found it is not generated at one-loop, but it appears at two-loop, and it was conjectured to be two-loop exact

Buchbinder, Petrov, Tseytlin (2001)

Indeed, exploiting the N=4 Supersymmetry it has been shown that the MHV operators

$$F_-^2 F_+^{2l}$$

is l-loop exact

Chen, Hunag, Wen (2015)
Soft theorems: QFT

Leading order

vertex \times \text{propagator} = 2a_1(k_2 - k_1) \times \frac{1}{(k_1 + k_2) - m^2}
Soft theorems: QFT

Leading order

The vertex interaction cannot be as general as possible!

\[ \text{vertex} \times \text{propagator} = 2a_1(k_2 - k_1) \times \frac{1}{(k_1 + k_2) - m^2} \]

no dependence on \( k \) soft

order \( 1/k_1 \)
Gauge theories: soft theorems

Low (1954)

1) Color-ordered gluon amplitudes in quantum field theory

\( k_s = \delta \hat{k}_s, \delta \to 0 \)

\[
\frac{\langle n1 \rangle}{\langle ns \rangle \langle s2 \rangle} \left( 1 + \frac{\langle sn \rangle}{\langle 1n \rangle} \bar{u}_s \frac{\partial}{\partial \bar{u}_1} + \frac{\langle s1 \rangle}{\langle n1 \rangle} \bar{u}_s \frac{\partial}{\partial \bar{u}_n} \right)
\]

Leading order

\[ \frac{1}{k_s} \]

Sub-leading order

\( k_s^0 \)
Gravity: soft theorems
Gross-Jackiw, Weinberg (1965)

2) Graviton amplitudes in quantum field theory

\[ k_s = \delta \hat{k}_s, \delta \to 0 \]

\[
\sum_{i \neq s} \left[ \frac{k_i \cdot h_s \cdot k_i}{k_s \cdot k_i} + 2 \frac{k_i \cdot h_s \cdot J_i \cdot k_s}{k_i \cdot k_s} + \frac{k_s \cdot J_i \cdot h_s \cdot J_i \cdot k_s}{k_i \cdot k_s} \right]
\]

\[
J_i = k_i^{[\mu} \frac{\partial}{\partial k_{i\nu}]} + 2h_i^{\lambda[\mu} \frac{\partial}{\partial h_{i\nu}\lambda]} + \text{Universal Sub-sub-leading order}
\]

F. Cachazo, A. Strominger
arXiv:1404.4091
Soft theorems for $\mathcal{N}=4$ SYM in the Coulomb branch

We focus on the scalar sector

The 6 massless scalars coincide with the 6 Goldstone bosons

\[ \varphi, \phi^I \quad I = 1, \ldots, 5 \]

- Conformal symmetry breaking
- $R$-symmetry breaking

\[ SO(6) \rightarrow SO(5) \]
Soft theorems for N=4 SYM in the Coulomb branch

We focus on the scalar sector

The 6 massless scalars coincide with the 6 Goldstone bosons

\[ \varphi, \phi^I \quad I = 1, \ldots, 5 \]

Conformal symmetry breaking

R-symmetry breaking

\[ SO(6) \to SO(5) \]

Due to the (broken) symmetries, amplitudes must satisfy

\[ A_n(\phi_1, \ldots, \phi_{n-1}, \pi_n)|_{p_n \to 0} = \sum_{i=1}^{n-1} \langle \phi_1, \ldots, \delta \phi_i, \ldots, \phi_{n-1} \rangle |_{LSZ} \]

Infinitesimal transformation of the hard particle
Soft theorems from spontaneously breaking of conformal and R-symmetry

Conformal symmetry breaking

Amplitudes with a soft dilation

\[ \delta \phi = [\mathcal{D}, \phi] = i(d + x^\mu \partial_\mu)\phi, \]
\[ \delta_\mu \phi = [\mathcal{K}_\mu, \phi] = i \left((2x_\mu x_\nu - \eta_{\mu\nu} x^2) \partial^\nu + 2d x_\mu\right)\phi \]

These transformations lead to the soft-dilaton theorem

\[ \nu A_n \bigg|_{p_n \rightarrow \tau p_n} \rightarrow \left(\cdots + S_n^{(0)} + \tau S_n^{(1)}\right) A_{n-1} + \mathcal{O}(\tau^2) \]

Di Vecchia, Marotta, Mojaza, Noble (2015)

\[ S_n^{(0)} = \sum_{i=1}^{n-1} \left(p_i \cdot \frac{\partial}{\partial p_i} + \frac{d-2}{2}\right) - d \quad \text{From } \mathcal{D} \]
\[ S_n^{(1)} = p_n^\mu \sum_{i=1}^{n-1} \left[ \frac{1}{2} \left(2 p_i^\nu \frac{\partial^2}{\partial p_i^\nu \partial p_i^\mu} - p_{i\mu} \frac{\partial^2}{\partial p_{i\nu} \partial p_i^\nu} \right) + \frac{d-2}{2} \frac{\partial}{\partial p_i^\mu} \right] \quad \text{From } \mathcal{K}_\mu \]
Soft theorems from spontaneously breaking of conformal and R-symmetry

\[ A_n(\phi_1, \cdots, \phi_n)|_{p_n \to 0} = \sum_i A_{n-1}(\cdots, \delta_i \phi_i, \cdots) + O(p_n^1) \]

Adler’s zero is a particular case

In this case the hard particles are rotated by the broken generators leading to a non-zero result

We verified soft theorems by explicit computations of 1-loop amplitudes in the Coulomb branch up to 6-points and of instanton amplitudes
Soft theorems: explicit verification at 1-loop

one-loop: integrands from 10-D SYM
or 6D SYM

Mafra, Schlotterer (2014)
Brandhuber, Korres, Koschade, Travaglini (2010)

All soft theorems satisfied
Soft theorems: explicit verification at 1-instanton

one-instanton: effective action for $Sp(2N)$ gauge fields
Bianchi, Morales, Wen (2015)

\[
S_{\text{eff}}^{1-\text{inst}} = \frac{c}{\pi^6} \frac{g^4}{\pi} e^{2\pi i \tau} \int \frac{d^4 x d^8 \theta \sqrt{\text{det}_{4N} 2\Phi_{\alpha u, \beta v}}}{\sqrt{\text{det}_{2N} \left( \Phi^{AB} \Phi_{AB} + \frac{1}{g} \tilde{F} + \frac{1}{\sqrt{2g}} \Lambda_A (\Phi^{-1})^{AB} \Lambda_B \right)_{\dot{\alpha} u, \dot{\beta} v}}}
\]

Turning on just the scalars, the super fields become

\[
\begin{align*}
\Phi_{AB} &= \Phi_{AB}, \\
\Lambda_{A\dot{\alpha}} &= i \theta^B \partial_{\dot{\alpha}} \Phi_{AB}, \\
\tilde{F}_{\dot{\alpha} \dot{\beta}} &= \frac{1}{2} \theta^{A \alpha} \theta^{B \beta} \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \Phi_{AB}
\end{align*}
\]
Soft theorems: explicit verification at 1-instanton

One-instanton: effective action for $Sp(2N)$ gauge fields

Bianchi, Morales, Wen (2015)

The final effective action produces interactions with at most 8-derivatives (dimension $s^4$)
Soft theorems: explicit verification at 1-instanton

**One-instanton**: effective action for Sp(2N) gauge fields

Bianchi, Morales, Wen (2015)

\[
S_{\text{eff}}^{1-\text{inst}} = c' \frac{g^4}{\pi^6} e^{2\pi i \tau} \int \frac{d^4 x \ d^8 \theta \sqrt{\det_{4N} 2 \bar{\Phi}_{\alpha u, \beta v}}}{\sqrt{\det_{2N} \left( \Phi^{AB} \bar{\Phi}_{AB} + \frac{1}{g} \bar{\mathcal{F}} + \frac{1}{\sqrt{2g}} \bar{\Lambda}_A (\Phi^{-1})^{AB} \Lambda_B \right)_{\dot{\alpha u}, \dot{\beta v}}}}
\]

The final effective action produces interactions with at most 8-derivatives (dimension $s^4$)

\[
S_{\text{dilaton}} = \int d^4 x \left[ (S_{\mu \nu} S^{\mu \nu})^2 - S_{\mu \nu} S^{\nu \rho} S_{\rho \sigma} S^{\sigma \mu} \right]
\]

\[
S_{\mu \nu} = \frac{\partial_\mu \partial_\nu \varphi}{\varphi^2} - 2 \frac{\partial_\mu \varphi \partial_\nu \varphi}{\varphi^3}.
\]
Soft theorems: explicit verification at 1-instanton

one-instanton: effective action for Sp(2N) gauge fields


\[ \nu^8 A_4^{\text{inst}} = \frac{1}{32} \left( s_{12}^2 + P_4 \right)^2, \quad \nu^9 A_5^{\text{inst}} = -\frac{1}{36} \left( s_{12}^2 + P_5 \right)^2, \]
\[ \nu^{10} A_6^{\text{inst}} = -\frac{2}{3} s_{12}^4 - 6 s_{12}^2 s_{23}^2 + \frac{17}{18} s_{123}^4 + \frac{15}{2} s_{123}^2 s_{124}^2 + P_6, \]
\[ \nu^{11} A_7^{\text{inst}} = 4 s_{12}^4 + 40 s_{12}^2 s_{23}^2 - \frac{5}{3} s_{123}^4 - 25 s_{123}^2 s_{124}^2 + P_7, \]
\[ \nu^{12} A_8^{\text{inst}} = -\frac{809}{144} s_{12}^4 - \frac{395}{8} s_{12}^2 s_{13}^2 + \frac{1339}{576} s_{123}^4 + \frac{595}{32} s_{123}^2 s_{124}^2 + \frac{535}{32} s_{123} s_{145}^2 + P_8, \]
\[ \nu^{13} A_9^{\text{inst}} = \frac{3935}{294} s_{12}^4 + \frac{846}{7} s_{12}^2 s_{13}^2 - \frac{475}{126} s_{123}^4 - \frac{491}{14} s_{123}^2 s_{124}^2 - \frac{535}{14} s_{123} s_{145}^2 + P_9. \]

All soft theorems satisfied up to the 9-point amplitude
Systematics of soft theorems: recursion relations

How to utilize soft-theorems systematically?

Recursion relations
Systematics of soft theorems: recursion relations

How to utilize soft-theorems systematically?

Recursion relations

Standard BCFW cannot apply here!

Soft-BCFW

Cheung, Kampf, Novotny, Sheng, Trnka (2015)

\[ p_i^\flat = (1 - a_i z) p_i \]

\[ \sum_{i=1}^{n} a_i p_i = 0 \]

Soft limit for the particle i for \( z \to \frac{1}{a_i} \)
Systematics of soft theorems: recursion relations

How to utilize soft-theorems systematically?

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Standard BCFW cannot apply here!

Soft-BCFW
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\[ \sum_{i=1}^{n} a_i p_i = 0 \]

Soft limit for the particle \( i \) for \( z \to \frac{1}{a_i} \)

The amplitude can be expressed as a contour integral

\[ A_n = A_n(0) = \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z} \frac{A_n(z)}{F_n^{(\sigma)}(z)} \]
Systematics of soft theorems: recursion relations

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\[
A_n = A_n(0) = \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z} A_n(z) F_n^{(\sigma)}(z)
\]

An additional function is introduced such that

\[
F_n^{(\sigma)}(z) = \prod_{i=1}^{n} (1 - a_i z)^{\sigma}
\]

for large \( z \)

\[
\frac{A_n(z)}{F_n^{(\sigma)}(z)} \sim 0
\]
Systematics of soft theorems: recursion relations

The amplitude can be expressed as a contour integral

\[ A_n = A_n(0) = \frac{1}{2\pi i} \oint_{z=0} dz \frac{A_n(z)}{z} F_n^{(\sigma)}(z) \]

\[ F_n^{(\sigma)}(z) = \prod_{i=1}^{n} (1 - a_i z)^{\sigma} \]

An additional function is introduced such that for large \( z \)

\[ \frac{A_n(z)}{F_n^{(\sigma)}(z)} \sim 0 \]

This construction useful only if we know the residues of the additional poles

That’s the input of the soft-theorem
Systematics of soft theorems: recursion relations

\[ F_n^{(\sigma)}(z) = \prod_{i=1}^{n} (1 - a_i z)^\sigma \]

Sigma is determined by the order that amplitudes have universal soft behavior

For broken conformal symmetry sigma=2

\[ F_n^{(2)}(z) = \prod_{i=1}^{n} (1 - a_i z)^2 \sim z^{2n} \]

Moreover

\[ A_n(z) \sim z^m, \quad z \to \infty \quad \frac{A_n(z)}{F_n^{(\sigma)}(z)} \to z^{m-n\sigma} \]

\[ m - n\sigma < 0 \]

Relation between the degree of the amplitude and the number of points
Systematics of soft theorems: recursion relations

For order $s^k$ amplitudes with $n$-dilatons the recursion is valid if

$$n > k$$

In other words, knowing the $k$-point amplitude at order $s^k$ for $k \leq n$ allows us to construct all the amplitudes up to order $s^n$.

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<th>6</th>
<th>7</th>
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Computed by other means

Single Soft → Soft-BCFW
Systematics of soft theorems: recursion relations

Example: If each scalar mostly carries one-derivative, then amplitudes at $2n$, $(2n+1)$ points mostly go as $s^n$

<table>
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Everything determined by the coefficient of the 4-point function at order $s^2$

For $N=4$ SYM, uniquely determined to be the coefficient from conformal DBI
Scale invariance vs Conformal symmetry

Scale vs Conformal in QFT
Scale invariance vs Conformal symmetry

Scale vs Conformal in QFT

Amplitudes language: to what extent does the sub-leading soft theorem due to conformal boosts follow from the leading behavior stemming from dilatation symmetry?
Any 5-point amplitude constrained by the leading soft theorem automatically satisfy the sub-leading soft theorem (verified up to $s^{11}$)
Scale invariance vs Conformal symmetry

A seven-point example
Using both leading and sub-leading soft theorems

\[ A^{(3)}_6 = -c_5^{(3)}(s_{12}^3 + \mathcal{P}_6) - \left( \frac{c_5^{(3)}}{2} + (c_4^{(2)})^2 \right) (s_{123}^3 + \mathcal{P}_6) \]
\[ + \quad (c_4^{(2)})^2 \left( (s_{12}^2 + s_{13}^2 + s_{23}^2) \frac{1}{s_{123}} (s_{45}^2 + s_{46}^2 + s_{56}^2) + \mathcal{P}_6 \right) \]
Scale invariance vs Conformal symmetry

A seven-point example
Using both leading and sub-leading soft theorems

\[ A_6^{(3)} = -c_5^{(3)}(s_{12}^3 + P_6) - \left( \frac{c_5^{(3)}}{2} + (c_4^{(2)})^2 \right) (s_{123}^3 + P_6) \]
\[ + \quad (c_4^{(2)})^2 \left( (s_{12}^2 + s_{13}^2 + s_{23}^2) \frac{1}{s_{123}} (s_{45}^2 + s_{46}^2 + s_{56}^2) + P_6 \right) \]

Leading soft alone allows us to determine the seven-point amplitude

\[ A_7^{(3)} = c_5^{(3)}(s_{12}^3 + P_7) + \left( c_5^{(3)} + 3(c_4^{(2)})^2 \right) (s_{123}^3 + P_7) - (c_4^{(2)})^2 A_7^{\text{fac}} \]

Satisfy the sub-leading soft theorem automatically
Scale invariance vs Conformal symmetry

The $s^4$, $s^5$ case without factorization diagrams

Number of points
$s^4$: 9-points amplitude determined by just the leading soft
$s^5$: 11-points amplitude determined by just the leading soft

Conjecture: amplitudes with $n>k$ determined only by the leading soft
Conclusions and remarks

The requirement of having a consistent S-matrix can impose highly non-trivial constraints on the theory.

Both SUSY and soft theorems can strongly constrain the effective action of $N=4$ SYM in the Coulomb branch, and lead to new non-renormalization theorems.

We observe amplitudes determined by leading soft automatically satisfy the sub-leading soft theorem.

Explore other possible constraints such as SUSY at higher points as well as $SL(2,\mathbb{Z})$ symmetry.
Spinor helicity formalism

Scattering amplitudes computed in QFT involve momenta and polarizations

\[ \{ k_1^{\mu_1}, k_2^{\mu_2}, \ldots, k_n^{\mu_n} \} \]
\[ \{ \epsilon_1^\mu, u_2, \bar{v}_3, \ldots, \epsilon_n^{\rho\sigma} \} \]

For massless particles

\[ k^2 = 0 \]

\[ k_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^{\mu} k_\mu = \bar{u}_\alpha \bar{u}_{\dot{\alpha}} \]

Left/Right handed spinors
Spinor helicity formalism

For massless particles

\[ k^2 = 0 \]

\[ k \alpha \dot{\alpha} = \sigma_{\alpha \dot{\alpha}}^\mu k_\mu = u_\alpha \bar{u} \dot{\alpha} \]

Example:

\[ 2k_1 \cdot k_2 = u_1^\alpha \varepsilon_{\alpha \beta} v_2^\beta \bar{u} \dot{\alpha}_1 \varepsilon_{\dot{\alpha} \dot{\beta}} \bar{v} \dot{\beta}_2 \]

\[ |i\rangle = u_i \quad \langle ij \rangle = u_i^\alpha \varepsilon_{\alpha \beta} u_j^\beta \]

\[ |i\rangle = \bar{u}_i \quad [ji] = \bar{u}_i^{\dot{\alpha}} \varepsilon_{\dot{\alpha} \dot{\beta}} \bar{u}_j^{\dot{\beta}} \]

\[ 2k_1 k_2 = \langle 12 \rangle [21] \]
Spinor helicity formalism

For polarizations

\begin{align*}
a^- & \rightarrow \frac{|k\rangle[q]}{[kq]} \\
a^+ & \rightarrow \frac{|q\rangle[k]}{\langle kq \rangle} \\
h^{--} & \rightarrow \frac{|k\rangle[q\rceil k\rangle[q]}{[kq]^2}
\end{align*}

q is a reference momentum reminding the gauge dependence

\[ a^\mu \rightarrow a^\mu + \alpha k^\mu \]

Every massless amplitude can be expressed using just helicity spinors
Helicity amplitudes

Ex: 3-gluons vertex

\[ a_1 a_2 a_3 (k_1 - k_2) + a_2 a_3 a_1 (k_3 - k_1) + a_3 a_1 a_2 (k_3 - k_1) \]
Helicity amplitudes

Ex: 3-gluons vertex

\[ A_3(1^-2^-3^-) = 0 \]

\[ A_3(1^-2^-3^+) \propto \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \]
Helicity amplitudes

Helicity amplitudes show properties of scattering amplitudes hidden in the QFT formalism

Ex: 4-gluons amplitude

\[ A(1^+ 2^+ 3^+ 4^+) = 0 \]
\[ A(1^- 2^+ 3^+ 4^+) = 0 \]
\[ A(1^- 2^- 3^+ 4^+) \neq 0 \]
Parke-Taylor: $gg \rightarrow gg$ scattering

color ordered amplitude

$$A_s(1234) = f^{c_1c_2} x f^{c_3c_4} x \tilde{A}_s(1234)$$
Parke-Taylor: $gg ightarrow gg$ scattering

Color ordered amplitude

$$
A_s(1234) = f^{c_1 c_2 x} f^{c_3 c_4 x} \tilde{A}_s(1234)
$$

$$
A(1234) = \sum_{\sigma \in S_4 \mod \mathbb{Z}_4} \text{tr}(\sigma(1)\sigma(2)\sigma(3)\sigma(4)) \tilde{A}(\sigma(1)\sigma(2)\sigma(3)\sigma(4))
$$

$$
= 2 \times (\text{tr}(1234) \tilde{A}(1234) + \text{tr}(1324) \tilde{A}(1324) + \text{tr}(1243) \tilde{A}(1243))
$$
Parke-Taylor: $gg \rightarrow gg$ scattering

Color ordered amplitude

\[
\mathcal{A}_s(1234) = f^{c_1 c_2 x} f^{c_3 c_4 x} \tilde{\mathcal{A}}_s(1234)
\]

\[
\mathcal{A}(1234) = \sum_{\sigma \in S^4 / \mathbb{Z}_4} \text{tr}(\sigma(1)\sigma(2)\sigma(3)\sigma(4))\tilde{\mathcal{A}}(\sigma(1)\sigma(2)\sigma(3)\sigma(4))
\]

\[
= 2 \times (\text{tr}(1234)\tilde{\mathcal{A}}(1234) + \text{tr}(1324)\tilde{\mathcal{A}}(1324) + \text{tr}(1243)\tilde{\mathcal{A}}(1243))
\]

Parke-Taylor

\[
\tilde{\mathcal{A}}(1^-2^-3^+4^+) = \frac{\langle12\rangle^4}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle}
\]
Parke-Taylor: $gg \rightarrow gg$ scattering

Color ordered amplitude

$$
\mathcal{A}(1234) = \sum_{\sigma \in S^4 / \mathbb{Z}_4} \text{tr}(\sigma(1)\sigma(2)\sigma(3)\sigma(4)) \tilde{\mathcal{A}}(\sigma(1)\sigma(2)\sigma(3)\sigma(4))
$$

$$
= 2 \times (\text{tr}(1234) \tilde{\mathcal{A}}(1234) + \text{tr}(1324) \tilde{\mathcal{A}}(1324) + \text{tr}(1243) \tilde{\mathcal{A}}(1243))
$$

Parke-Taylor for MHV amps

$$
\tilde{\mathcal{A}}_n(1^-2^-3^+ \ldots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle (n-1)n \rangle \langle n1 \rangle}
$$
BCJ relations: gravity=$(YM)^2$

3 gravitons vertex

\[ M_3(1^-2^-3^+) = (A_3(1^-2^-3^+))^2 = \frac{\langle 12 \rangle^8}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2} \]

BCJ double copy relations

\[ A_n = \sum_{i \in \text{cubic}} \frac{n_i c_i}{\Pi \alpha_i p_{\alpha_i}^2} \quad \rightarrow \quad M_n = \sum_{i \in \text{cubic}} \frac{n_i^2}{\Pi \alpha_i p_{\alpha_i}^2} \]

Similar relations for the integrands at loop level!