On a strong coupling property of QCD

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I. A Reminder on Effective Locality (6 years old)

- Discovered in a functional approach to Lagrangian QCD
- Bearing on fermionic Green's functions
- Using exact *Fradkin's representations* for $G_F(x, y|A)$ and $L(A) = Tr \ln i\not{D}(A)$,
- a standart *linearization* (Halpern's 1977-78) of the non-abelian $\mathbf{F}^2$,
- and functional *differential* identities.

Effective Locality is peculiar to the non-abelian structure of QCD

Reminder: A few basic elements

- Covariant gauge-dependent gluon propagator,

$$D^{ab(\zeta)}_{F,\mu\nu}(k) = \frac{i\delta^{ab}}{k^2 + i\varepsilon}\left[g_{\mu\nu} - \zeta\, k_\mu k_\nu / k^2\right]\,,\ \zeta = \lambda/(1-\lambda)$$

- Fermionic (quark) propagator in an external gluon field A^a_μ

$$G_F(x,y\,|A) = \langle x|[i\gamma^\mu(\partial_\mu - i\,g\,A^a_\mu\lambda_a) - m]^{-1}|y\rangle$$

- Closed-fermion loop functional,

$$L[A] = Tr\ln[1 - i\,g\,(\gamma A\lambda)\,S_F]\,,\qquad S_F = G_F[gA = 0]$$

Reminder: Not so familiar..

- Example of a remarkable functional <u>differential</u> identity being used

$$\mathcal{F}[\frac{1}{i}\frac{\delta}{\delta j}]\; e^{\frac{i}{2}\int j\cdot D_F^{(\zeta)}\cdot j} = e^{\frac{i}{2}\int j\cdot D_F^{(\zeta)}\cdot j} e^{\mathbf{D}_A^{(\zeta)}}\, \mathcal{F}[A]\big|_{A=\int D_F^{(\zeta)}\cdot j}$$

where $\mathbf{D}_A^{(\zeta)}$ is the *linkage operator*

$$\mathbf{D}_A^{(\zeta)} = -\frac{i}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \frac{\delta}{\delta A_\mu^a(x)}\, \mathbf{D}_F^{(\zeta)}\Big|_{\mu\nu}^{ab}(x-y)\, \frac{\delta}{\delta A_\nu^b(y)}$$

Reminder. What Fradkin's representations look like

$$\langle p|\mathbf{G}_F[A]|y\rangle = -\frac{1}{(2\pi)^2} e^{-ip\cdot y}\, i\int_0^\infty ds\, e^{-ism^2}\; e^{-\frac{1}{2}\,Tr\ln(2h)}$$

$$\times \int d[u]\{m - i\gamma\cdot[p - gA(y-u(s))]\}\; e^{\frac{i}{4}\int_0^s ds'\,[u'(s')]^2} e^{ip\cdot u(s)}$$

$$\times \boxed{\left(e^{g\int_0^s ds'\sigma\cdot F(y-u(s'))}\; e^{-ig\int_0^s ds'\,u'(s')\cdot A(y-u(s'))}\right)_+}$$

$h(s_1,s_2) = \int_0^s \mathrm{d}s'\Theta(s_1-s')\Theta(s_2-s')$. Auxiliary field variables, $\Omega^a(s_1)$, $\bar{\Omega}^b(s_2)$, $\alpha^a(s_1)$, $\alpha^b(s_2)$ are required to circumvent Schwinger proper-time s'-ordering (and take both $\mathbf{G}_F[A]$ and $\mathbf{L}[A]$ to *gaussian forms*)

EL is not readable on $\mathbf{Z}_{QCD}[j,\eta,\bar{\eta}]$ itself, but on its fermionic Green's functions

Reminder (Halpern'77)

$\chi^a_{\mu\nu}$: a real-valued field introduced so as to linearize the non-abelian $F^{\mu\nu}F_{\mu\nu}$ dependence of the original QCD Lagrangian density

$$e^{-\frac{i}{4}\int \mathbf{F}^a_{\mu\nu}\mathbf{F}^{\mu\nu}_a} = \mathcal{N}_\chi \int \mathrm{d}[\chi]\, e^{\frac{i}{4}\int \chi^a_{\mu\nu}\chi^a_{\mu\nu} + \frac{i}{2}\int \chi^a_{\mu\nu}\mathbf{F}^{\mu\nu}_a}$$

EL : a formal functional statement..

With

$$\mathcal{F}_{\mathrm{I}}[A] = \exp\left[\frac{i}{2}\int A\bar{\mathcal{K}}(2n)A + i\int \bar{\mathcal{Q}}(n)A\right], \quad \mathcal{F}_{\mathrm{II}}[A] = \exp\left(\mathbf{L}[A]\right)$$

The *functional* statement of EL for $2n$-points fermionic Green's functions can be read off

$$\begin{aligned} e^{\mathbf{D}_A}\mathcal{F}_{\mathrm{I}}[A]\,\mathcal{F}_{\mathrm{II}}[A] = \quad & \mathcal{N} \quad \exp\left[-\frac{i}{2}\int \bar{\mathcal{Q}}(n)\hat{\mathcal{K}}^{-1}\bar{\mathcal{Q}}(n) + \frac{1}{2}Tr\ln\hat{\mathcal{K}}\right] \\ & \times \quad \exp\left[\frac{i}{2}\int\frac{\delta}{\delta A}\hat{\mathcal{K}}^{-1}\frac{\delta}{\delta A} - \int\bar{\mathcal{Q}}(n)\hat{\mathcal{K}}^{-1}\frac{\delta}{\delta A}\right] \\ & \times \quad \exp\left(\mathbf{L}[A]\right) \end{aligned} \tag{1}$$

at

$$\bar{\mathcal{K}}(2n) = (D_F^{(\zeta)})^{-1} + \hat{\mathcal{K}}(2n), \quad \hat{\mathcal{K}}(2n)^{ab}_{\mu\nu} = (\ \mathcal{K}_{\mathcal{S}}(2n) + gf\chi\ )^{ab}_{\mu\nu}$$

EL : a formal functional statement

1. $\hat{\mathcal{K}}^{-1} = (\mathcal{K}_{\mathcal{S}} + g(f \cdot \chi))^{-1}$ is local $\langle x|\mathcal{O}|y\rangle = \mathcal{O}(x)\,\delta^{(4)}(x-y)$ as well as the extra contributions of $\mathbf{L}[A]$ to $\hat{\mathcal{K}}$ and $\bar{\mathcal{Q}}$, the contributions of (1) depend on the Fradkin variables $u_i(s_i')$ and the space-time coordinates y_i in a specific but local way

2. Nothing in (1) refers to $D_F^{(\zeta)}$ or to any other choice of a propagator. Gauge-Invariance is achieved as a matter of Gauge-Independence! hoped for by R.P. Feynman in QED (cf. 'Quantum Field Theory In A Nutshell', A. Zee)

Antecedents

- In a Field Strength formulation of the pure YM case, a surprising effective local interaction (Duality, indeed) (H. Reinhardt, K. Langfeld, L.v. Smekal, 93's)

$$\int d^4z \, \partial^\lambda \chi^a_{\lambda\mu}(z) \, \left([(gf\chi)]^{-1}\right)^{\mu\nu}_{ab}(z) \, \partial^\rho \chi^b_{\rho\nu}(z)$$

- H.M. Fried in 'Functional Methods and Eikonal Models' (Eds. Frontières, 1990)

- EL seen within a quantization of QCD by means of *functional differentiation*. Not visible within a functional integration quantization.

Intriguing connections to EL

Effective locality goes along with an enigmatic mass scale μ and leads to an almost linearly confining potential. For light/dynamical quarks (not static!)

$$V(r) = \xi\mu(r\mu)^{1-\xi}$$

If $\xi << 1$ is a sound aspect of the matter, then it connects to

- Levy-flight mode of propagation of confined quarks
- non-commutative geometry of transverse scattering planes (i.e., De Moyal planes)

.. still speculative..

II. QCD amplitudes by *Random Matrix*

Thanks to EL,

At $g >> 1$ and quenched and eikonal approximations, *Random Matrix Theory* allows one to calculate $2n$-point fermionic Green's functions/amplitudes in a generic closed form.

These forms comply with a general conjecture, that all QFT's Green's functions are expandable in terms of G_{pq}^{mn}-Meijer's special functions.

(D.D. Ferrante, G.S. and Z. Guralnik, C. Pehlevan. S. Gukov and E. Witten, *etc.*, 2008-2011)

By the *measure image theorem* (Wiener space), EL enables the passage ..

.. from infinite dimensional functional integrations (over $\chi^a_{\mu\nu}$-fields),

$$\int \mathrm{d}[\chi] = \prod_{x \in \mathcal{M}} \prod_{a=1}^{N^2-1} \prod_{0=\mu<\nu}^{3} \int \mathrm{d}[\chi^a_{\mu\nu}](x)$$

.. to ordinary Lebesgue integrations over finite-dimensional spaces (that of real symmetric $N \times N$ traceless random matrices at $N \equiv D \times (N_c^2 - 1) = 32$).

.. that is, EL enables the passage to *Random Matrix* integration

With,

$$\sum_{a=1}^{N_c^2-1} \chi^a_{\mu\nu} \otimes T^a = i\,M\,, \qquad M_{ij} = M_{ji} \in \mathbb{R}\,,\;\; 1 \leq i,j \leq N\,, \qquad Tr\,M = 0.$$

One has,

$$\begin{aligned}
& \mathrm{d}(\sum_{a=1}^{N_c^2-1} \chi^a{}_{\mu\nu} \otimes T^a) \\
= \;& i\mathrm{d}M = i\mathrm{d}M_{11}\,\mathrm{d}M_{12}\cdots\mathrm{d}M_{NN} \\
= \;& i\left|\frac{\partial(M_{11},\cdots,M_{NN})}{\partial(\xi_1,\cdots,\xi_N;\,p_1,\cdots,p_{N(N-1)/2})}\right| \mathrm{d}\xi_1\cdots\mathrm{d}\xi_N\,\mathrm{d}p_1\cdots\mathrm{d}p_{N(N-1)/2} \\
= \;& i\prod_{i=1}^{N}\mathrm{d}\xi_i \prod_{i<j}|\xi_i-\xi_j|^{\kappa=1}\,\mathrm{d}p_1\;..\;\mathrm{d}p_{N(N-1)/2}\,F(p)\,,
\end{aligned}$$

.. *Random Matrix* integration

dM splits into

- an integration over a spectrum of pairwise opposite real eigenvalues $-\infty \leq \xi_i \leq +\infty$,

$$\mathrm{Sp}\, M = \left\{ (\xi_i, \xi_{N-i+1} = -\xi_i) \right\} \,, \forall i = 1, 2, \ldots, N/2\,.$$

- an integration over the orthogonal group $O_N(\mathbb{R})$ endowed with *some Haar measure* $\mathrm{d}p_1 \,..\, \mathrm{d}p_{N(N-1)/2}\, F(p)$

..*Random Matrix* integration

- Integrating on the spectrum yields the *Meijer's functions*

$$\int_0^\infty \mathrm{d}\xi_i \, \xi_i^p \, e^{-\xi_i^2 - \frac{b_i}{\xi_i}} = \frac{1}{2\sqrt{\pi}} G^{30}_{03}\left(\frac{b_i^2}{4} \middle| \frac{p+1}{2}, \frac{1}{2}, 0 \right) .$$

analytic in both their arguments and parameters!

- Integrating on the orthogonal group $O_N(\mathbb{R})$ displays the full color algebraic dependences of Green's functions and related amplitudes

III. Color algebraic dependences (4-pt Green's function; one specie of quarks of mass m)

In the $g >> 1$ limit, at quenched and eikonal approximations, a result proportional to

$$\sum_{\text{monomials}} \Bigg\langle \prod_{1\leq i\leq N}^{\sum q_i = N(N-1)/2} \left[1 - i(-1)^{q_i}\right] \times \left[\frac{\sqrt{2iN_c}\sqrt{\hat{s}(\hat{s}-4m^2)}}{m^2}\right] \frac{[(\mathcal{O}\mathcal{T})_i]^{-2}}{g\varphi(b)} \times G^{30}_{03}\left(\left[\frac{g\varphi(b)}{\sqrt{32iN_c}}\frac{m^2}{\sqrt{\hat{s}(\hat{s}-4m^2)}}\right]^2 \left[(\mathcal{O}\mathcal{T})_i\right]^4 \Bigg| \frac{1}{2}, \frac{3+2q_i}{4}, 1\right) \Bigg\rangle_{O_N(\mathbb{R})}$$

thanks to exact integrations on the ξ_i-spectrum, Fradkin's variables and auxiliary fields.

Reads as an $O_N(\mathbb{R})$-averaged finite sum of finite products of Meijer functions

Then, defining..

$$z_i \equiv \lambda[(\mathcal{O}\mathcal{T})_i]^4 \,, \qquad \lambda \equiv \left(\frac{g\varphi(b)}{\sqrt{32iN_c}} \frac{m^2}{\sqrt{\widehat{s}(\widehat{s}-4m^2)}} \right)^2 \,,$$

.. the matrix-valued argument of the G^{30}_{03}- Meijer function, one has $|z_i| << 1$ even @ $g >> 1$. Expanding, one finds for all i, $1 \leq i \leq N$, a leading order algebraic depence of,

$$< \sqrt{z_i} >_{O_N(\mathbb{R})} = \frac{\sqrt{\lambda}}{N} D\, C_{2f}\, \mathbf{1}_{3\times 3} \,,$$

where C_{2f} is the quadratic Casimir operator eigenvalue on the fundamental representation, $C_{2f} = C_F = 4/3$.

$$C_2(\mathcal{R}) = \sum_{a=1}^{8} T_a^2(\mathcal{R}) \,,$$

the $T_a(\mathcal{R})$ denote the $SU_c(3)$ Lie algebra generators in a given representation $\mathcal{R}$.

.. while at next to leading order..

$$< z_i >_{O_N(\mathbb{R})} = (\frac{\sqrt{\lambda}}{N})^2 \left((DC_{2f})^2 + (DC_{3f}) \right) \mathbf{1}_{3\times 3} \,,$$

with,

$$\sum_{a,b,c=1}^{N_c^2-1} d_{abc}\, t^a t^b t^c \equiv C_{3f} \mathbf{1}_{3\times 3} \qquad C_{3f} = 10/9 \,,$$

the second Casimir operator of the rank-2 Lie algebra of $SU(3)_c$. Likewise,

$$< z_i \sqrt{z_i} >_{O_N(\mathbb{R})} =$$

$$(\frac{\sqrt{\lambda}}{N})^3 \left([2 + (\frac{5}{6})^2](DC_{2f})^2 + (DC_{2f})(DC_{3f}) + 3(DC_{3f}) \right) \mathbf{1}_{3\times 3} \,.$$

and though relative to *dynamical* rather than *static* quarks..

.. at $g = 15$, the relative deviations to a pure C_{2f} behaviour seem to agree with the allowed $\sim 5\%$ of the Lattice predictions,

$$\delta_C = \frac{\sqrt{\lambda}}{N} \left(\frac{C_{3f}}{C_{2f}} + DC_{2f}\right) \frac{\sum_1^N b_i + \sum_2^N a_i \sum_1^{i-1} a_j}{\sum_1^N a_i} + \mathcal{O}\left(\left(\frac{\sqrt{\lambda}}{N}\right)^2\right),$$

$$a_i \equiv -2\Gamma\left(\frac{2q_i - 1}{4}\right)/\Gamma\left(\frac{2q_i + 1}{4}\right), \qquad b_i \equiv 8/(2q_i - 3),$$

one finds,

$$\delta_C(N-1, N-2, \cdots, 1, 0) \simeq -1.0\%,$$

and for the full sum of monomials (but at $N = 4!$),

$$-\frac{1798,211444}{147,543465} \frac{\sqrt{\lambda}}{4} \frac{DC_{3f} + (DC_{2f})^2}{DC_{2f}} = -7.5\%.$$

Further ....

- these results extend to any $2n$-point fermionic Green's functions

- are preserved by a restoration of unitarity (*i.e.,*, the unquenched case)

Collaboration

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Papers

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