On a strong coupling property of QCD

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I. A Reminder on Effective Locality (6 years old)

- Discovered in a functional approach to Lagrangian QCD
- Bearing on fermionic Green’s functions
- Using exact Fradkin’s representations for $G_F(x, y|A)$ and $L(A) = Tr \ln iD(A)$,
- a standard linearization (Halpern’s 1977-78) of the non-abelian $F^2$,
- and functional differential identities.

Effective Locality is peculiar to the non-abelian structure of QCD
Reminder: A few basic elements

- Covariant gauge-dependent gluon propagator,

\[ D_{F,\mu\nu}^{ab}(\zeta)(k) = \frac{i\delta^{ab}}{k^2 + i\epsilon} \left[ g_{\mu\nu} - \zeta k_\mu k_\nu / k^2 \right] , \quad \zeta = \lambda / (1 - \lambda) \]

- Fermionic (quark) propagator in an external gluon field \( A_\mu^a \)

\[ G_F(x, y | A) = \langle x | \left[ i\gamma^\mu (\partial_\mu - ig A_\mu^a \lambda_a) - m \right]^{-1} | y \rangle \]

- Closed-fermion loop functional,

\[ L[A] = Tr \ln[1 - ig (\gamma A \lambda) S_F] , \quad S_F = G_F[gA = 0] \]
Reminder: Not so familiar..

- Example of a remarkable functional differential identity being used

\[
\mathcal{F}\left[ \frac{1}{i} \frac{\delta}{\delta j} \right] e^{i \int j \cdot D_{\mathcal{F}}^{(\zeta)} \cdot j} = e^{i \frac{1}{2} \int j \cdot D_{\mathcal{F}}^{(\zeta)} \cdot j} e^{D_{A}^{(\zeta)}} \mathcal{F}[A] \bigg|_{A=\int D_{\mathcal{F}}^{(\zeta)} \cdot j}
\]

where \( D_{A}^{(\zeta)} \) is the linkage operator

\[
D_{A}^{(\zeta)} = -\frac{i}{2} \int d^{4}x \ d^{4}y \ \frac{\delta}{\delta A_{\mu}^{a}(x)} \ D_{\mathcal{F}}^{(\zeta)} \bigg|_{\mu \nu}^{ab} (x - y) \ \frac{\delta}{\delta A_{\nu}^{b}(y)}
\]
Reminder. What Fradkin’s representations look like

\[ \langle p | G_F[A] | y \rangle = -\frac{1}{(2\pi)^2} e^{-ip \cdot y} i \int_0^\infty ds \ e^{-ism^2} e^{-\frac{1}{2} Tr \ln(2h)} \]

\[ \times \int d[u] \{ m - i\gamma \cdot [p - gA(y - u(s))] \} \ e^{\frac{i}{4} \int_0^s ds' [u'(s')]^2} e^{ip \cdot u(s)} \]

\[ \times \left( e^{g \int_0^s ds' \sigma \cdot F(y - u(s))} e^{-ig \int_0^s ds' u'(s') \cdot A(y - u(s'))} \right) \]

\[ h(s_1, s_2) = \int_0^s ds' \Theta(s_1 - s') \Theta(s_2 - s'). \] Auxiliary field variables, \( \Omega^a(s_1), \tilde{\Omega}^b(s_2), \alpha^a(s_1), \alpha^b(s_2) \) are required to circumvent Schwinger proper-time \( s' \)-ordering (and take both \( G_F[A] \) and \( L[A] \) to gaussian forms)

\( EL \) is not readable on \( Z_{QCD}[j, \eta, \tilde{\eta}] \) itself, but on its fermionic Green’s functions
\( \chi_{\mu \nu}^a \): a real-valued field introduced so as to linearize the non-abelian \( F^\mu_\nu F_{\mu \nu} \) dependence of the original QCD Lagrangian density

\[
e^{-\frac{i}{4} \int F_{\mu \nu}^a F_{\mu \nu}^a} = \mathcal{N}_\chi \int d[\chi] \ e^{\frac{i}{4} \int \chi_{\mu \nu}^a \chi_{\mu \nu}^a + \frac{i}{2} \int \chi_{\mu \nu}^a F_{\mu \nu}^a}
\]
EL : a formal functional statement.

With

\[ F_I[A] = \exp \left[ \frac{i}{2} \int A \bar{K}(2n) A + i \int \bar{Q}(n) A \right], \quad F_{II}[A] = \exp (L[A]) \]

The *functional* statement of EL for 2\(n\)-points fermionic Green’s functions can be read off

\[
e^{D_A F_I[A]} F_{II}[A] = \mathcal{N} \exp \left[ -\frac{i}{2} \int \bar{Q}(n) \hat{K}^{-1} \bar{Q}(n) + \frac{1}{2} \text{Tr} \ln \hat{K} \right] \times \exp \left[ \frac{i}{2} \int \frac{\delta}{\delta A} \hat{K}^{-1} \frac{\delta}{\delta A} - \int \bar{Q}(n) \hat{K}^{-1} \frac{\delta}{\delta A} \right] \times \exp (L[A]) \quad (1)\]

at

\[ \bar{K}(2n) = (D_F^{(\zeta)})^{-1} + \hat{K}(2n), \quad \hat{K}(2n)_{\mu\nu} = (K_s(2n) + gf\chi)_{\mu\nu} \]
EL : a formal functional statement

1. \( \hat{\mathcal{K}}^{-1} = (\mathcal{K}_s + g(f \cdot \chi))^{-1} \) is local
   \( \langle x|O|y \rangle = O(x) \delta^{(4)}(x - y) \) as well as the extra contributions of \( L[A] \) to \( \hat{\mathcal{K}} \) and \( \bar{Q} \), the contributions of (1) depend on the Fradkin variables \( u_i(s_i') \) and the space-time coordinates \( y_i \) in a specific but local way

2. Nothing in (1) refers to \( D_F^{(\zeta)} \) or to any other choice of a propagator. **Gauge-Invariance** is achieved as a matter of **Gauge-Independence!** hoped for by R.P. Feynman in QED (cf. ‘Quantum Field Theory In A Nutshell’, A. Zee)
Antecedents

- In a Field Strength formulation of the pure YM case, a surprising effective local interaction (Duality, indeed) (H. Reinhardt, K. Langfeld, L.v. Smekal, 93’s)

\[ \int d^4 z \, \partial^\lambda \chi_{\lambda \mu}^a (z) \, \left( \left[ (gf\chi) \right]^{-1} \right)_{ab}^{\mu \nu} (z) \, \partial^\rho \chi_{\rho \nu}^b (z) \]


- EL seen within a quantization of QCD by means of functional differentiation. Not visible within a functional integration quantization.
Intriguing connections to EL

Effective locality goes along with an enigmatic mass scale $\mu$ and leads to an almost linearly confining potential. For light/dynamical quarks (not static!)

$$V(r) = \xi \mu (r\mu)^{1-\xi}$$

If $\xi \ll 1$ is a sound aspect of the matter, then it connects to

- Levy-flight mode of propagation of confined quarks
- non-commutative geometry of transverse scattering planes (i.e., De Moyal planes)

.. still speculative..
II. QCD amplitudes by *Random Matrix*

Thanks to EL,

At \( g \gg 1 \) and quenched and eikonal approximations, *Random Matrix Theory* allows one to calculate \( 2n \)-point fermionic Green’s functions/amplitudes in a generic closed form.

These forms comply with a general conjecture, that all QFT’s Green’s functions are expandable in terms of \( G_{pq}^{mn} \)-Meijer’s special functions.

By the *measure image theorem* (Wiener space), EL enables the passage..

.. from *infinite dimensional* functional integrations (over $\chi_{\mu\nu}^a$-fields),

\[
\int d[\chi] = \prod_{x \in \mathcal{M}} \prod_{a=1}^{N^2-1} \prod_{0=\mu<\nu}^{3} \int d[\chi_{\mu\nu}^a](x)
\]

.. to ordinary Lebesgue integrations over *finite-dimensional* spaces (that of real symmetric $N \times N$ traceless random matrices at $N \equiv D \times (N_c^2 - 1) = 32$).
.. that is, EL enables the passage to Random Matrix integration 

With,

$$\sum_{a=1}^{N_c^2-1} \chi_{\mu \nu}^a \otimes T^a = iM, \quad M_{ij} = M_{ji} \in \mathbb{R}, \quad 1 \leq i, j \leq N, \quad \text{Tr} M = 0.$$ 

One has,

$$d\left( \sum_{a=1}^{N_c^2-1} \chi_{\mu \nu}^a \otimes T^a \right) = idM = idM_{11} dM_{12} \cdots dM_{NN}$$

$$= i \begin{vmatrix} \frac{\partial (M_{11}, \cdots, M_{NN})}{\partial (\xi_1, \cdots, \xi_N; p_1, \cdots, p_{N(N-1)/2})} \end{vmatrix} d\xi_1 \cdots d\xi_N d\rho_1 \cdots d\rho_{N(N-1)/2}$$

$$= i \prod_{i=1}^{N} d\xi_i \prod_{i<j} |\xi_i - \xi_j|^{\kappa=1} d\rho_1 \cdots d\rho_{N(N-1)/2} F(p),$$
Random Matrix integration

dM splits into

- an integration over a spectrum of pairwise opposite real eigenvalues $-\infty \leq \xi_i \leq +\infty$,

$$Sp M = \{ (\xi_i, \xi_{N-i+1} = -\xi_i) \} , \forall i = 1, 2, \ldots, N/2 .$$

- an integration over the orthogonal group $O_N(\mathbb{R})$ endowed with some Haar measure $d\rho_1 \ldots d\rho_{N(N-1)/2} F(\rho)$
- Integrating on the spectrum yields the **Meijer’s functions**

\[
\int_0^\infty d\xi \xi^p e^{-\xi^2 - \frac{b_i}{\xi}} = \frac{1}{2\sqrt{\pi}} G_{03}^{30} \left( \frac{b_i^2}{4} \middle| \frac{p+1}{2}, \frac{1}{2}, 0 \right).
\]

analytic in both their arguments and parameters!

- Integrating on the orthogonal group $O_N(\mathbb{R})$ displays the full color algebraic dependences of Green’s functions and related amplitudes
III. Color algebraic dependences (4-pt Green’s function; one specie of quarks of mass $m$)

In the $g \gg 1$ limit, at quenched and eikonal approximations, a result proportional to

$$\sum_{\text{monomials}} \left( \sum_{q_i=N(N-1)/2} \prod_{1 \leq i \leq N} [1 - i(-1)^{q_i}] \right)$$

$$\times \left[ \frac{\sqrt{2iN_c} \sqrt{\hat{s}(\hat{s} - 4m^2)}}{m^2} \right] \left[ \frac{((OT)_i)^{-2}}{g\varphi(b)} \right]$$

$$\times G_{03}^{30} \left( \left[ \frac{g\varphi(b)}{\sqrt{32iN_c} \sqrt{\hat{s}(\hat{s} - 4m^2)}} \right] \frac{m^2}{g\varphi(b)} \right) \left[ (OT)_i \right] \frac{1}{2}, \frac{3 + 2q_i}{4}, 1 \right) \left| O_N(R) \right>$$

thanks to exact integrations on the $\xi_j$-spectrum, Fradkin’s variables and auxiliary fields.

Reads as an $O_N(R)$-averaged finite sum of finite products of Meijer functions.
Then, defining..

\[ z_i \equiv \lambda [(OT)_i]^4, \quad \lambda \equiv \left( \frac{g\varphi(b)}{\sqrt{32iN_c}} \frac{m^2}{\sqrt{s(s - 4m^2)}} \right)^2, \]

.. the matrix-valued argument of the \( G_{03}^{30} \) Meijer function, one has \( |z_i| << 1 \) at \( g >> 1 \). Expanding, one finds for all \( i \), \( 1 \leq i \leq N \), a leading order algebraic dependence of,

\[ < \sqrt{Z_i} >_{O_N(R)} = \frac{\sqrt{\lambda}}{N} D_{C_{2f} 1_{3 \times 3}}, \]

where \( C_{2f} \) is the quadratic Casimir operator eigenvalue on the fundamental representation, \( C_{2f} = C_F = 4/3 \).

\[ C_2(R) = \sum_{a=1}^{8} T_a^2(R), \]

the \( T_a(R) \) denote the \( SU_{c}(3) \) Lie algebra generators in a given representation \( R \).
.. while at next to leading order..

\[
< z_i >_{O_N(\mathbb{R})} = \left( \frac{\sqrt{\lambda}}{N} \right)^2 \left( (DC_{2f})^2 + (DC_{3f}) \right) 1_{3 \times 3},
\]

with,

\[
\sum_{a,b,c=1}^{N_c^2-1} d_{abc} t^a t^b t^c \equiv C_{3f} 1_{3 \times 3} \quad C_{3f} = 10/9,
\]

the second Casimir operator of the rank-2 Lie algebra of \( SU(3)_c \). Likewise,

\[
< z_i \sqrt{z_i} >_{O_N(\mathbb{R})} = \left( \frac{\sqrt{\lambda}}{N} \right)^3 \left( 2 + \left( \frac{5}{6} \right)^2 \right) (DC_{2f})^2 + (DC_{2f})(DC_{3f}) + 3(DC_{3f}) 1_{3 \times 3}.
\]
and though relative to dynamical rather than static quarks..

.. at $g = 15$, the relative deviations to a pure $C_{2f}$ behaviour seem to agree with the allowed $\sim 5\%$ of the Lattice predictions,

$$
\delta_C = \frac{\sqrt{\lambda}}{N} \left( \frac{C_{3f}}{C_{2f}} + DC_{2f} \right) \frac{\sum_1^N b_i + \sum_2^N a_i \sum_1^{i-1} a_j}{\sum_1^N a_i} + O\left( \frac{\sqrt{\lambda}}{N} \right)^2,
$$

$$
a_i \equiv -2\frac{\Gamma\left(\frac{2q_i - 1}{4}\right)}{\Gamma\left(\frac{2q_i + 1}{4}\right)}, \quad b_i \equiv \frac{8}{(2q_i - 3)},
$$

one finds,

$$
\delta_C(N - 1, N - 2, \cdots, 1, 0) \approx -1.0\%,
$$

and for the full sum of monomials (but at $N = 4$!),

$$
\frac{-1798,211444}{147,543465} \frac{\sqrt{\lambda}}{4} \frac{DC_{3f} + (DC_{2f})^2}{DC_{2f}} = -7.5\%.
$$
- these results extend to any $2n$-point fermionic Green’s functions

- are preserved by a restoration of unitarity (i.e., the unquenched case)

Collaboration

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Papers