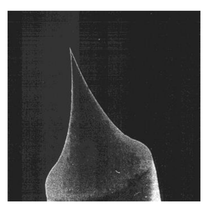

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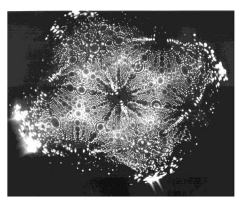
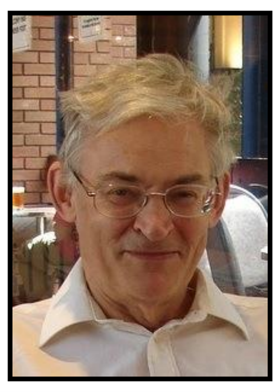


RECENT IMPROVEMENTS IN FOWLER-NORDHEIM THEORY [A SUMMARY FOR NON-EXPERTS]

Richard G Forbes



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SUMMARY

This poster provides an overview of attempts made by the present author and collaborators, from about 2006 onwards, to improve "mainstream" Fowler-Nordheim theory and theory relating to the interpretation of Fowler-Nordheim plots.

1. INTRODUCTION

Fowler-Nordheim (FN) tunnelling is electron tunnelling through an exact or rounded triangular barrier. *Deep tunnelling* is tunnelling well below the top of the barrier, at a level where simplified tunnelling theory applies. *Cold field electron emission (CFE)* is a statistical emission regime where most electrons escape by deep tunnelling from states near the emitter Fermi level.

Fowler-Nordheim-type (FN-type) equations are a large family of approximate equations used to describe CFE. The original FN-type equation described CFE from the conduction band of a bulk free-electron metal with a smooth, flat planar surface. Slightly more general situations can be adequately described by slightly more sophisticated FN-type equations. In practice, FN-type equations are also often used to describe CFE from non-metals, particularly when analyzing FN plots (see below), but this is not strictly valid.

The theory of how to develop and use FN-type equations to describe CFE and analyze related experimental results is often called *Fowler-Nordheim theory*. This poster provides an overview of recent (since about 2006) attempts to improve FN theory, made by the author and collaborators, including attempts to put FN theory onto an improved scientific basis. The theory given here covers both *single-tip field emitters (STFEs)* and *large-area field emitters (LAFEs)* comprising many individual emission sites.

2. FIELD EMISSION THROUGH A SCHOTTKY-NORDHEIM BARRIER

For a planar or quasi-planar field emitter, the best tunnelling-barrier model is the *Schottky-Nordheim (SN) barrier.* A SN barrier with zero-field height equal to the local work function ϕ is described by the *motive energy*

$$M^{SN}(z) = \phi - eF_L z - e^2/16\pi\varepsilon_0 z \; , \qquad (1)$$

where e is the elementary positive charge, ε_0 the electric constant, F_L the *local barrier field*, and z distance measured from the emitter's electrical surface.

The *reference field* F_R needed to reduce this barrier to zero is

$$F_R = c_S^{-2}\phi^2 = (4\pi\varepsilon_0/e^3)\,\phi^2 \; ,(2)$$

where c_S is the *Schottky constant* [1]. The *scaled barrier field* f is defined by

$$f \equiv F_L/F_R = c_S^2 F_L \phi^{-2} \approx (1.439965 \text{ eV V nm}^{-1})\, F_L\phi^{-2} \; . \qquad (3)$$

This parameter f, introduced in 2006 [2,3], plays an important role in modern FN theory.

In the simple-JWKB approach, the *probability* D *of tunnelling* through this SN barrier is written physically in the form

$$D \approx \exp[-\nu_F^{SN} b\phi^{3/2}/F_L] \; , \qquad (4)$$

where b is the *second FN constant* [≈ 6.830890 eV$^{-3/2}$ V nm^{-1}] [1], and ν_F^{SN} is the physical *barrier form correction factor* for this SN barrier.

When deriving tunnelling theory for SN barriers, it is found that

$$\nu_F^{SN} = v(f) \; , \qquad (5)$$

where $v(x)$ is a defined mathematical function, sometimes called the *principal SN barrier function*, and x is a purely mathematical variable. In the course of the derivation, x is set equal to the SN-barrier modelling parameter f.

Several alternative mathematical definitions of $v(x)$ exist; probably the most convenient is the integral definition

$$v(x) = (3 \times 2^{-3/2}) \int_{b'}^{a'} (a'^2-\eta^2)^{1/2}(\eta^2-b'^2)^{1/2}\,d\eta \; , \qquad (6)$$

where a' and b' are given by

$$a' = [1+(1-x)^{1/2}]^{1/2}; \quad b' = [1-(1-x)^{1/2}]^{1/2} \; . \qquad (7)$$

3. MATHEMATICS OF THE PRINCIPAL SN-BARRIER FUNCTION v(*x*)

Improvements in understanding over the last ten years include the following.

(1) In older work a different mathematical argument was used, namely the Nordheim parameter y, given by $y=x^{1/2}$. Using x is better, – mathematically because the exact series expansion for $v(x)$ (below) contains no half-integral terms in x, physically because the relationship between f and F is linear.

(2) A good simple approximation has been found for $v(x)$, namely [2]:

$$v(x) = 1 - x + (1/6)x\ln x \; . \qquad (8)$$

This formula has an accuracy of better than 0.33% over the range $0 \le x \le 1$, which is better than other approximations of equivalent complexity, and is more than good enough for most technological purposes.

3. (cont.) SN-BARRIER MATHEMATICS (cont.)

(3) It has been shown [4] that there is a defining equation for $v(x)$, namely

$$x(1-x)d^2W/dx^2 = (3/16)W \; . \qquad (9)$$

This equation is a special case of the Gauss hypergeometric differential equation; $v(x)$ is the particular solution that meets the boundary conditions:

$$v(0) = 1; \quad \lim(x\to 0)\{dv/dx-(3/16)\ln x\} = -(9/8)\ln 2 \; . \qquad (10)$$

(4) An exact series expansion has been found [4] for $v(x)$. The lowest few terms are:

$$v(x)=1-\left(\frac{9}{8}\ln 2+\frac{3}{16}\right)x-\left(\frac{27}{256}\ln 2+\frac{3}{16}\right)x^2-O(x^3)+x\ln x\left\{\frac{3}{16}+\frac{9}{512}x+O(x^2)\right\} \qquad (11)$$

(5) Numerical expressions have been found [3] that give both $v(x)$ (using 9 terms) and dv/dx (using 11 terms) to an absolute error $|\varepsilon| \le 8\times 10^{-10}$.

4. FORMAL STRUCTURING OF FOWLER-NORDHEIM EQUATION SYSTEM

Formal structuring of the system of FN-type equations has been introduced, in order to: (a) describe the relationships between the many different FN-type equations found in the literature; (b) allow high-level formulae relating to FN plots (see below); and (c) clarify issues relating to what parameters can be predicted and/or measured reliably.

The *general (or "universal") form* for a FN-type equation is

$$Y = C_{YX}X^2 \exp[-B_X/X] \; , \qquad (12)$$

where X is any CFE *independent variable* (usually a field or voltage), Y is any CFE *dependent variable* (usually a current or current density), and B_X and C_{YX} are related parameters.

Both B_X and C_{YX} depend on the choices of X and Y and on barrier form, and C_{YX} also depends on various other physical factors. Both often exhibit weak to moderate functional dependences on the chosen variables.

The choices of X and Y determine the *form* $Y(X)$ of a FN-type equation. The *core theoretical form* (the form initially derived from theory) is the $J_L(F_L)$ form, which gives the *local emission current density (ECD)* J_L in terms of the local work function ϕ and the local barrier field F_L.

The *characteristic local barrier field* F_C is the value of F_L at some point "C" (in the emitter's electrical surface) that is considered characteristic of the emitter. In modelling, "C" is often taken at the emitter apex (or, in the case of a large-area field emitter, at the apex of the most strongly emitting individual emitter).

For a *general barrier (GB)*, the $Y(F_C)$-form FN-type equation can be written formally as the *linked equations* (13a) and (13b), and then expanded into a $Y(X)$-form equation via eq. (13c).

$$Y = c_Y \cdot J_{kC}^{GB} \; , \qquad (13a)$$

$$J_{kC}^{GB} \equiv a\phi^{-1}F_C^2 \exp[-\nu_F^{GB} b\phi^{3/2}/F_C] \; , \qquad (13b)$$

$$= a\phi^{-1}(c_X X)^2 \exp[-\nu_F^{GB} b\phi^{3/2}/(c_X X)] \; . \qquad (13c)$$

Eq. (13a) is an *auxiliary equation*, wherein c_Y is the *auxiliary parameter* linking Y to the *characteristic kernel current density* J_{kC}^{GB} for the general barrier. J_{kC}^{GB} is defined by eq. (12b), in which a is the *first FN constant* [1], and ν_F^{GB} the barrier-form correction factor for the general barrier.

In eq. (13c), the auxiliary equation $F_C = c_X X$ has been used to obtain a FN-type equation containing the independent variable X of interest (often a voltage).

The merit of these linked forms is that, in suitable emission situations (including orthodox emission situations), for any given choices of barrier form and related barrier-defining parameters (often □ and F_C), the kernel current density J_{kC}^{GB} can be calculated ***exactly***. Thus, in suitable emission situations, all uncertainty in theoretical ***prediction*** is accumulated into c_Y .

On the other hand, in suitable emission situations, when Y is an accurately observable quantity, and when relevant barrier-parameter values can be deduced for insertion into (13b) or (13c), then a reasonably precise value for c_Y can be deduced **"by experiment"**. However, the value obtained will depend on what choice has been made for barrier form and hence $\square_F^{GB}$.

The "universal" equation (13) contains three selectable parameters, shown in pink, that determine the detailed equation type. The options for these parameters are considered below.

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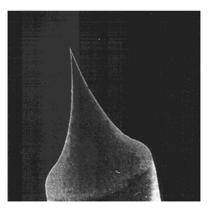

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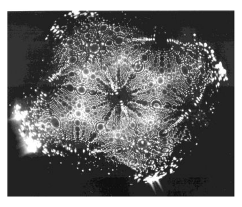

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RECENT IMPROVEMENTS IN FOWLER-NORDHEIM THEORY [A SUMMARY FOR NON-EXPERTS]: Sheet 2

5. THE SERIES-RESISTANCE PROBLEM

When resistance is present in the measuring circuit, then careful distinction is needed between *emission variables* and *measured variables*. This need has grown increasingly apparent in recent years. Figure 1 is a schematic circuit diagram for CFE measurements. Although the parallel resistance R_p can usually be eliminated by careful system design, series resistance often cannot.

In this case, the measured voltage V_m is not equal to the emission voltage V_e, and a (current-dependent) *voltage ratio* Θ can be defined by

$$\Theta = V_e/V_m = R_e/(R_e+R_s) \,, \qquad (14)$$

where R_e is the *emission resistance* [$\equiv V_e/i_e$], and R_s [$\equiv R_{s1}+R_{s2}$] is the *total series resistance*.

The $i_m(V_m)$-form FN-type equation (giving measured current as a function of measured voltage, for a general barrier) thus becomes

$$i_m \equiv A_f^{GB}\, a\phi^{-1}(\zeta_C^{\square 1}\Theta V_m)^2 \exp[-\nu_F^{GB} b\phi^{3/2}\zeta_C/\Theta V_m] \,, \qquad (15)$$

where ζ_C [$\equiv V_e/F_C$] is the *characteristic local conversion length (LCL)* that relates F_C to V_e (see below). The presence of Θ is a recently introduced feature.

6. AUXILIARY PARAMETERS & EQNS FOR INDEPENDENT VARIABLES

Table 1 shows the "independent" variables (X) currently used in FN theory, and the main related auxiliary parameters and equations. Some points arising are:

(1) It is convenient to classify independent variables as "theoretical variables", "emission variables", or "measured variables".

(2) To avoid present ambiguities over the meaning of the symbol β, it is suggested that use of *local conversion lengths (LCLs)* (ζ_C) should replace use of voltage-to-barrier-field conversion factors (VCFs).

(3) Usually, theory is clearer if the voltage ratio Θ is shown explicitly.

(4) Auxiliary equations of the form $F_C=c_X X$ (shown in blue in Table 1) have a special role in FN theory, as indicated earlier.

(5) For LAFEs, *field enhancement factors (FEFs)* can be derived from LCLs by the formula $\gamma_C = \zeta_M/\zeta_C$, where ζ_M is the *macroscopic conversion length.*

7. AUXILIARY PARAMETERS AND EQNS FOR DEPENDENT VARIABLES

In this section, the superscript "GB" is omitted for notational simplicity; however, the equations apply to a barrier of any specific form.

Characteristic local ECD J_C relates to characteristic kernel current density J_{kC} by

$$J_C = \lambda_C J_{kC} \,, \qquad (16)$$

where λ_C is the *characteristic local pre-exponential correction factor*. λ_C takes formal account of factors not considered elsewhere, including improved tunnelling theory, temperature, the use of atomic-level wave-functions, and electronic band-structure. For the SN barrier, our current best guess (in 2015) is that λ_C lies in the range $0.005<\lambda_C<11$, but this could be an underestimate of the range of uncertainty.

The *emission current* i_e is obtained by integrating the local ECD J_L over the whole emitter surface. The result can be written in the following ways:

$$i_e = \int J_L \mathrm{d}A \equiv A_n J_C = A_n\lambda_C J_{kC} \equiv A_f J_{kC} \,, \qquad (17)$$

where the *notional emission area* A_n and *formal emission area* A_f [$\equiv i_e/J_{kC}$] are defined via eq. (17).

Both parameters are needed because, for orthodox emission, it ought to be possible to extract reasonably accurate values of A_f from experiment; but A_n (which cannot be extracted accurately) appears in some existing theory (e.g. [6]) and might in principle be closer to geometrical area estimates.

For LAFEs, the *macroscopic current density* J_M is the average ECD taken over the whole LAFE *macroscopic area* (or "footprint") A_M, and can be written

$$J_M \equiv i_e/A_M = (A_n/A_M)J_C \equiv \alpha_n J_C = \alpha_n\lambda_C J_{kC} \equiv \alpha_f J_{kC} \,, \qquad (18)$$

where the *notional area efficiency* α_n and *formal area efficiency* α_f [$\equiv J_M/J_{kC}$] are defined via eq. (18).

8. EQUATION COMPLEXITY LEVELS

In the literature, many choices have been made for the barrier form (and hence $\square_F^{GB}$), and for what physical effects to include when modelling the parameter λ_C^{GB} defined above. These choices determine the *complexity level* of FN-type equations.

For bulk emitters with planar surfaces, Table 2 shows the main complexity levels historically and currently used. A given complexity level applies to all related $Y(X)$-forms of equation at the given complexity level.

TABLE 2. Complexity levels of FN-type equations

Level name	*Date*	λ_C^{GB} →	*Barrier*	ν_F^{GB} →
Elementary	1999?	1	ET	1
Original	1928	P_{FN}	ET	1
Fowler-1936	1936	4	ET	1
Extended elementary	2015	λ_C^{ET}	ET	1
Dyke-Dolan	1956	1	SN	v_F
Murphy-Good	1956	t_F^{-2}	SN	v_F
Orthodox	2013	λ_C^{SN*}	SN	v_F
New-standard	2015	λ_C^{SN}	SN	v_F
"Barrier-changes-only"	2013	λ_C^{GB*}	GB	ν_F^{GB}
General	1999	λ_C^{GB}	GB	ν_F^{GB}

For citations behind dates of introduction, see [7].
ET=Exactly triangular; SN=Schottky-Nordheim;
GB= General barrier. v_F & t_F are SN-barrier functions.
P_{FN} is FN pre-exponential (see [1]).
* denotes that parameter is to be treated as constant.

Figure 1.

Schematic circuit for measurement of field emission current-voltage characteristics, showing resistances in parallel and in series with the emission resistance R_e [$=V_e/i_e$]. Due to the resistance in series with the emission resistance, the emission voltage V_e is less than the measured voltage V_m .

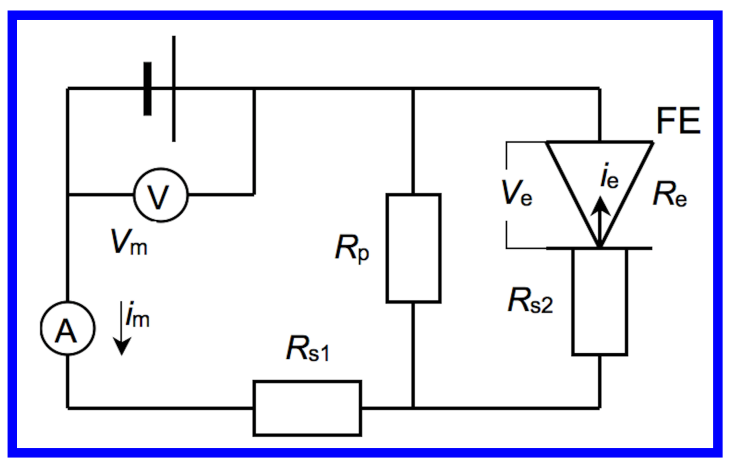

TABLE 1: Independent variables, and main related auxiliary parameters and equations

Independent variable name and symbol		**links to** (symbol)	**via auxiliary parameter** name and symbol		**Formulae**
Theoretical variables					
Characteristic local barrier field	F_C	-	-	-	-
Scaled barrier field	f	F_C	Reference field	F_R	$F_C = f F_R$
Emission variables					
Emission voltage	V_e	F_C	(True) local voltage-to-barrier-field conversion factor (VCF)[a]	$\beta_{V,C}$	$F_C = \beta_{V,C} V_e$
Emission voltage	V_e	F_C	(True) local conversion length (LCL)	ζ_C	$F_C = V_e/\zeta_C$
True macroscopic field	F_M	V_e	(True) macroscopic conversion length[b]	ζ_M	$F_M = V_e/\zeta_M$
True macroscopic field	F_M	F_C	(True) (electrostatic) macroscopic field enhancement factor (FEF)	γ_C	$F_C = \gamma_C F_M$ $\gamma_C = \zeta_M/\zeta_C$
Measured variables					
Measured voltage	V_m	V_e	Voltage ratio	Θ	$V_e = \Theta V_m$
Measured voltage	V_m	F_C	Measured-voltage-defined LCL	(ζ_C/Θ)	$F_C = \zeta_C^{-1}\Theta V_m$
Apparent macroscopic field	F_A	V_m	Macroscopic conversion length	ζ_M	$F_A = V_m/\zeta_M$
Apparent macroscopic field	F_A	F_M	Voltage ratio	Θ	$F_M = \Theta F_A$
Apparent macroscopic field	F_A	F_C	Apparent-field-defined FEF[c]	$\gamma_C^{afd} = \gamma_C\Theta$	$F_C = \gamma_C^{afd} F_A$ $F_C = \gamma_C \Theta F_A$

[a]Future use of the parameter $\square_{V,C}$ is discouraged: use $\square_C$ and related formulae instead.
[b]In planar-parallel-plate geometry, $\square_M$ is normally taken as equal to the plate separation d_{sep}.
[c]Use of the parameter $\square_C^{afd}$ is discouraged: use the combination $\square_C\square$ instead.

9. INTERPRETATION OF FOWLER-NORDHEIM PLOTS

When re-written in form (19), eq. (12) is said to be *written in FN coordinates*, and the *slope* S of the resulting *FN plot* can be written in form (20):

$$L(X^{-1}) \equiv \ln\{Y/X^2\} = \ln\{C_{YX}\} - B_X/X = \ln\{C_{YX}\} - \nu_F^{GB} b\phi^{3/2}/c_X X, \qquad (19)$$

$$S \equiv \mathrm{d}L/\mathrm{d}(X^{-1}) \equiv -\sigma b\phi^{3/2}/c_X \,, \qquad (20)$$

where σ is a *slope correction factor* (taken as 1 in elementary FN plot analysis), defined by eq. (20). If σ can be reliably predicted, i.e., if $\mathrm{d}L/\mathrm{d}(X^{-1})$ can be reliably evaluated, measurement of S allows extraction of c_X-values and related emitter characterization parameters, such as a field enhancement factor (FEF).

The presence of Θ in eq. (15) and related equations massively complicates practical FN plot analysis. When $\Theta \neq 1$, then σ can be significantly less than 1 but cannot (at present) be reliably predicted. More generally, the usual methods of FN-plot analysis work correctly **only if** the emission situation is *orthodox* [5] (which requires that $\Theta=1$, that c_X, c_Y and ϕ be constant, and that emission can be treated as taking place through a SN barrier).

Many real emitters are not orthodox; thus, it is likely [5] that many published FEF-values are **spuriously large**. Steps taken to deal with this problem and develop FN-plot theory include the following.

(1) A careful definition of emission orthodoxy has been given [5].

(2) A robust test for emission orthodoxy has been given, which can be applied to any form of FN plot [5].

(3) For emitters that fail the orthodoxy test, a process called *phenomenological adjustment* has been developed that allows c_X to be roughly estimated [7].

(4) In SN-barrier theory, new forms of *intercept correction factor* (r_{2012}) [7] and *area extraction parameter* (Λ^{SN}) [7] have been defined.

(5) For emitters that pass the orthodoxy test, better methods of extracting formal emission area A_f and formal area efficiency α_f have been developed [8].

(6) Attempts have been made [7] to determine σ by simulating the circuit in Fig. 1, using a constant total series resistance, but have proved unsuccessful.

(7) Using the "barrier effects only" approximation [7], values have been determined for σ and for the generalized intercept correction factor ρ, for the barriers relating to spherical and sphere-on-cone (SOC) emitter models.

Unfortunately, it has recently been concluded [9] that, for non-planar emitters, the usual methods of finding tunnelling probabilities may not be strictly valid quantum mechanics, and that *transformation of the motive energy* $M(z)$ may sometimes be necessary. This is an active topic of research.

Work continues on these and other attempts to improve mainstream FN theory.

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