Bose-Einstein condensation and Silver Blaze using 2-loop 2PI

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Motivation

- Functional methods at finite density are of great interest, because of the phase diagram of strongly interacting matter.
- Understanding the subtleties which lie in the renormalization of 2PI at finite μ.
- Understanding the Silver Blaze phenomenon in a simple model.
- In Andersen, PRD **75** 065011 (2007) pion condensation is discussed in LO-$1/N$ approximation of 2PI. Some general features are hidden, which are present in the 2-loop approximation.
- Therefore as a first step we chose the charged scalar model, and included chemical potential in it.

Introduction to 2PI

A bilocal source is introduced in the generating functional

$$Z[J,K] = e^{W[J,K]} = \int \mathcal{D}\varphi \, \exp\left[-S_0 - S_{\text{int}} + \varphi \cdot J + \varphi \cdot K \cdot \varphi \right]$$

The 2PI effective action defined through a double Legendre transform

$$\gamma[\phi, G] = W[J,K] - \int d^4x \underbrace{\frac{\delta W[J,K]}{\delta J(x)}}_{\phi(x)} J(x) - \int d^4x \int d^4y \underbrace{\frac{\delta W[J,K]}{\delta K(x,y)}}_{[\phi(x)\phi(y)+G(x,y)]/2} K(x,y)$$

The physical $\bar{\phi}(x)$ and $\bar{G}(x,y)$ are determined from stationarity conditions at vanishing sources ($J, K \to 0$)

$$\left.\frac{\delta\gamma[\phi,G]}{\delta\phi(x)}\right|_{\bar{\phi}(x)} = 0, \qquad \left.\frac{\delta\gamma[\phi,G]}{\delta G(x,y)}\right|_{\bar{G}(x,y)} = 0$$

$\gamma[\phi, G]$ can be written as shown in Cornwall et al., PRD **10**, 2428 (1974)

$$\gamma[\phi, G] = S_0(\phi) + \frac{1}{2}\mathrm{Tr}\log G^{-1} + \frac{1}{2}\mathrm{Tr}\left[G_0^{-1}G - 1\right] + \gamma_{\mathrm{int}}[\phi, G]$$

S_0 is the free action,

G_0 is the free propagator,

$\gamma_{\mathrm{int}}[\phi, G]$ contains all the 2PI graphs constructed with vertices from $S_{\mathrm{int}}(\phi + \varphi)$.

The Tr is to be understood in all indices and as integration over coordinates.

The 1PI effective action is recovered: $\Gamma_{\mathrm{1PI}}[\phi] = \gamma[\phi, \bar{G}]$.

The chemical potential only enters through the free action S_0 and the free propagator G_0.

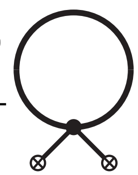
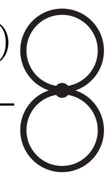
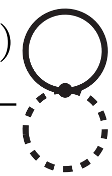
Equations

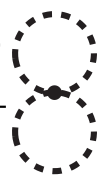
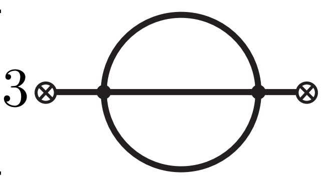
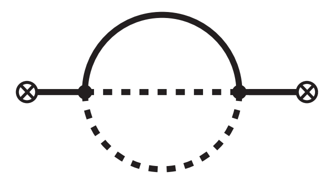
The symmetry of the theory is $SO(2)$ in the presence of μ. We represent the field as $\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$, $\varphi_a \in \mathbb{R}$, and $\langle \varphi_a \rangle = \delta_{a,1}\phi$. The free and full propagators are

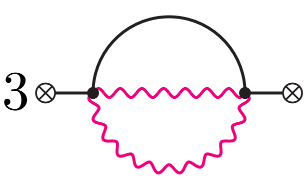
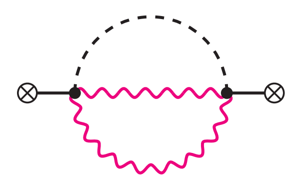
$$G_0^{-1} = \begin{pmatrix} Z_0 Q^2 + m_0^2 - Z_0\mu^2 & -2Z_0\mu\omega \\ 2Z_0\mu\omega & Z_0Q^2 + m_0^2 - Z_0\mu^2 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} G_L & G_A \\ -G_A & G_T \end{pmatrix}.$$

The 2PI potential truncated at 2-loops can written as

$$\gamma[\phi, G_L, G_T, G_A] = \frac{1}{2}\mathrm{Tr}\int_Q^T \left[\log(G^{-1}(Q)) + G_0^{-1}(Q)\cdot G(Q)\right] + \frac{1}{2}(m_2^2 - \mu^2 Z_2)\phi^2$$

$+\dfrac{\lambda_4\phi^4}{48} + \dfrac{\lambda_2^{(A+2B)}}{24}$  $+\dfrac{\lambda_2^{(A)}}{24}$  $+\dfrac{\lambda_0^{(A+2B)}}{48}$  $+\dfrac{\lambda_0^{(A)}}{24}$ 

$+\dfrac{\lambda_0^{(A+2B)}}{48}$  $-\dfrac{\lambda_\star^2}{144}\Big[3$  $+$ 

$+2\Big(3$  $-$  $\Big)\Big]$,

$\lambda_{0,2}^{(\alpha A+\beta B)} \equiv \alpha\lambda_{0,2}^{(A)} + \beta\lambda_{0,2}^{(B)}$, $G_L =$, $G_T =$, $G_A =$ , $\phi =$ .

Equations

The field expectation value ϕ and the components of the full propagator are determined from stationarity conditions:

$$
\begin{aligned}
0 &= \left.\frac{\delta\gamma[\phi, G_L, G_T, G_A]}{\delta\phi}\right|_{\bar{\phi},\bar{G}_L,\bar{G}_T,\bar{G}_A} = \left.\frac{\delta\gamma[\phi, G_L, G_T, G_A]}{\delta G_L}\right|_{\phi,\bar{G}_L,\bar{G}_T,\bar{G}_A} \\
&= \left.\frac{\delta\gamma[\phi, G_L, G_T, G_A]}{\delta G_T}\right|_{\phi,\bar{G}_L,\bar{G}_T,\bar{G}_A} = \left.\frac{\delta\gamma[\phi, G_L, G_T, G_A]}{\delta G_A}\right|_{\phi,\bar{G}_L,\bar{G}_T,\bar{G}_A},
\end{aligned}
$$

which yield equations for the gap masses defined from the inverse propagator:

$$
\begin{aligned}
\bar{M}^2_{L,T}(Q) &= \frac{\bar{G}_{T,L}}{\bar{G}_L\bar{G}_T + \bar{G}_A^2} + Z_0(\mu^2 - Q^2), \\
\bar{M}^2_A(Q) &= -\frac{\bar{G}_A}{\bar{G}_L\bar{G}_T + \bar{G}_A^2} + Z_0 2\mu\omega,
\end{aligned}
$$

and the field equation with the structure

$$
0 = \bar{\phi}\tilde{f}(\bar{\phi}, \bar{G}_L(\phi = \bar{\phi}), \bar{G}_T(\phi = \bar{\phi}), \bar{G}_A(\phi = \bar{\phi})) = \bar{\phi}f(\bar{\phi}).
$$

Curvature masses

To study the phase transition we monitor the curvature mass tensor. It is defined using the 1PI potential

$$\gamma(\phi) \equiv \gamma[\phi, \bar{G}_L, \bar{G}_T, \bar{G}_A]$$

as

$$\hat{M}^2_{ab} = \frac{\partial^2 \gamma(\phi)}{\partial \phi_a \partial \phi_b} + \delta_{ab}\mu^2 = \hat{M}^2_L \frac{\phi_a \phi_b}{\phi^2} + \hat{M}^2_T \left(\delta_{ab} - \frac{\phi_a \phi_b}{\phi^2}\right).$$

Evaluating the derivatives yield

$$\hat{M}^2_L = 4\bar{\phi}^2 \left.\frac{df(\phi)}{d\phi}\right|_{\bar{\phi}} + 2f(\bar{\phi}) + \mu^2, \qquad \hat{M}^2_T = 2f(\bar{\phi}) + \mu^2$$

At $\bar{\phi} = 0\,:\, \hat{M}^2_L = \hat{M}^2_T$ (symmetry restoration), at $\bar{\phi} \neq 0\,:\, \hat{M}^2_T = \mu^2$ (Goldstone theorem).

Numerics

We solve the coupled field and gap equations iteratively.

We discretize the propagators on a $N_\tau \times N_s$ grid:

$$\omega_n = 2\pi n T,\ n \in [0..N_\tau - 1],\ \text{and } k = (s+1)\frac{\Lambda}{N_s},\ s \in [0..N_s - 1].$$

- Numerical method was developed in Markó et al., PRD **86** 085031 (2012).
- Rotation invariance $\Rightarrow$ only 1D in momentum space.
- Convolutions are done using FFT techniques.
- Only adjustment needed: $G_A \to \omega_n g_A$. While FFT is also applicable to odd functions, the stored frequencies would be shifted, which in the iterative process of solving the equations leads to loss of information.

Silver Blaze

We can formulate our theory using complex fields as well:

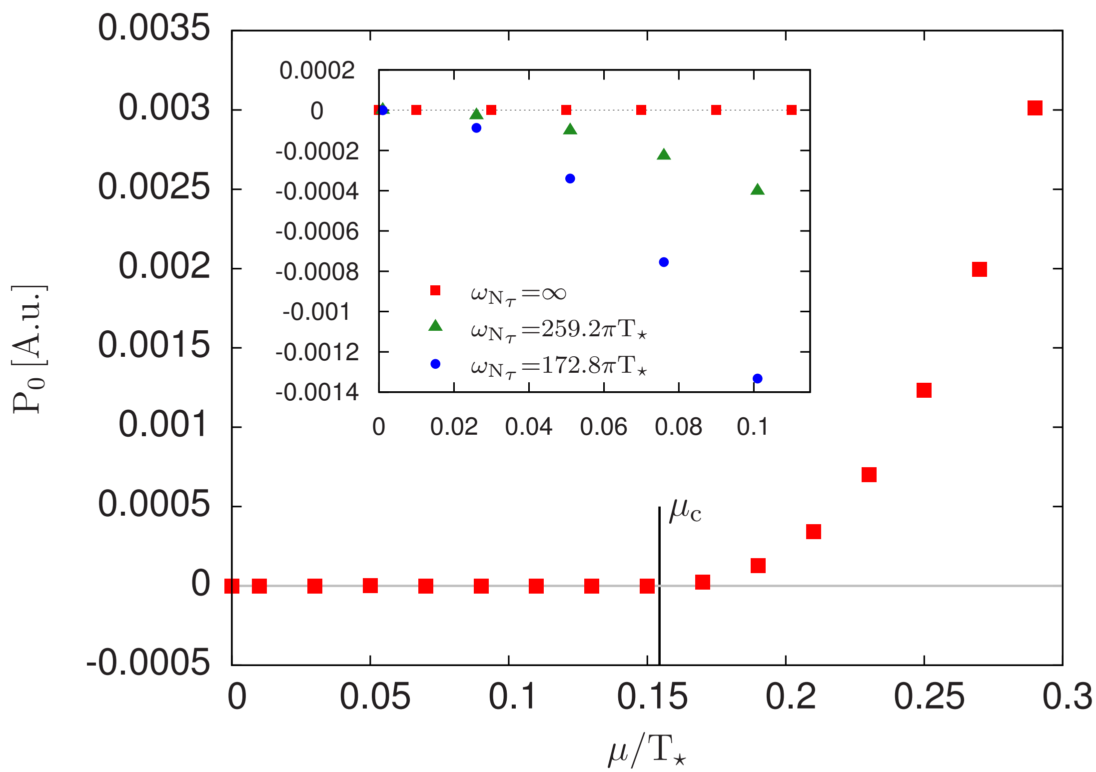
$$\Phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)\,, \text{ and } \Phi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)\,.$$

Then the Lagrangian is invariant under the gauge-transformation

$$\Phi \to e^{i\alpha\tau}\Phi\,,\ \Phi^* \to e^{-i\alpha\tau}\Phi^*\,,\ \mu \to \mu - i\alpha\,.$$

- $Z_\mu = Z_{\mu-i\alpha}$ provided that $\alpha = \omega_n$, a Matsubara-frequency in order to maintain the periodicity of the fields.
- $T \neq 0$: Periodicity in the imaginary μ direction (Roberge-Weiss periodicity).
- $T = 0$: ω_n becomes continuous $\to$ analytic continuation: Z_μ is μ-independent up to analyticity boundary μ_c. This is the Silver Blaze property Cohen, Phys. Rev. Lett. **91**, 222001 (2003)
- Generalization to n-point functions at $T = \phi = 0$: μ-dependence is just a shift of external frequencies.

Silver Blaze



- In any 2PI truncation the Silver Blaze is realized.
- Provided UV regularization and discretization keeps the gauge-transformation property.
- We use finite Matsubara-frequencies. We have to take the $T \to 0$ limit such that $2\pi N_\tau T \to \infty$.
- On the lattice μ is introduced on links, similarly to gauge fields.

Renormalization

Renormalization is based on Markó et al., PRD **87** 105001 (2013).

- Prescriptions on 2- and 4-point functions.
- Multiply defined n-point functions $\Rightarrow$ Renorm conditions AND consistency conditions in terms of two parameters: $m_\star^2$, $\lambda_\star$.
- At $T = T_\star$ (fixed renormalization scale), $\mu = \bar{\phi} = 0$.
- No new counterterms are needed compared to $\mu = 0$ case.
- Except for field renormalization, which is special in the homogeneous 2-loop approximation.
- At 2-loop order: no diagram in the gap equation has momentum dependent divergence.
  But the field equation has the setting-sun at zero external momentum (homogeneity).
- Shift of external frequencies by μ in n-point functions: need for Z_2.

Renormalization

In line with the other prescriptions, we require:

$$\begin{aligned}\left.\frac{d}{d\mu^2}\hat{M}^2_{\phi=0}\right|_{T_\star,\mu=0} &= 1-\alpha\,,\\ \left.\frac{d}{d\mu^2}\hat{M}^2_{\phi=0}\right|_{T_\star,\mu=0} &= \left.\frac{d}{d\mu^2}\bar{M}^2_{\phi=0}\right|_{T_\star,\mu=0}\,.\end{aligned}$$

Which lead to the following expressions for the field normalizations:

$$\begin{aligned}Z_2 &= Z_0+\frac{\lambda_\star^2}{6}\mathcal{B}_\star[G_\star](0)\left.\left(\frac{\partial\mathcal{T}[\bar{\mathcal{D}}]}{\partial\mu^2}\right)\right|_{T_\star,\mu=0}-\frac{\lambda_\star^2}{18}\left.\left(\frac{\partial\mathcal{S}[\bar{\mathcal{D}},\bar{\mathcal{D}}^*,\bar{\mathcal{D}}]}{\partial\mu^2}\right)\right|_{T_\star,\mu=0}\,,\\ Z_0 &= \alpha+\left.\frac{\lambda_\star}{3}\frac{\partial\mathcal{T}[\bar{\mathcal{D}}]}{\partial\mu^2}\right|_{T_\star,\mu=0}\,,\ \text{with } \bar{\mathcal{D}}^{-1}(Q)=(\omega_n+i\mu)^2+q^2+\bar{M}^2_{\phi=0}\,.\end{aligned}$$

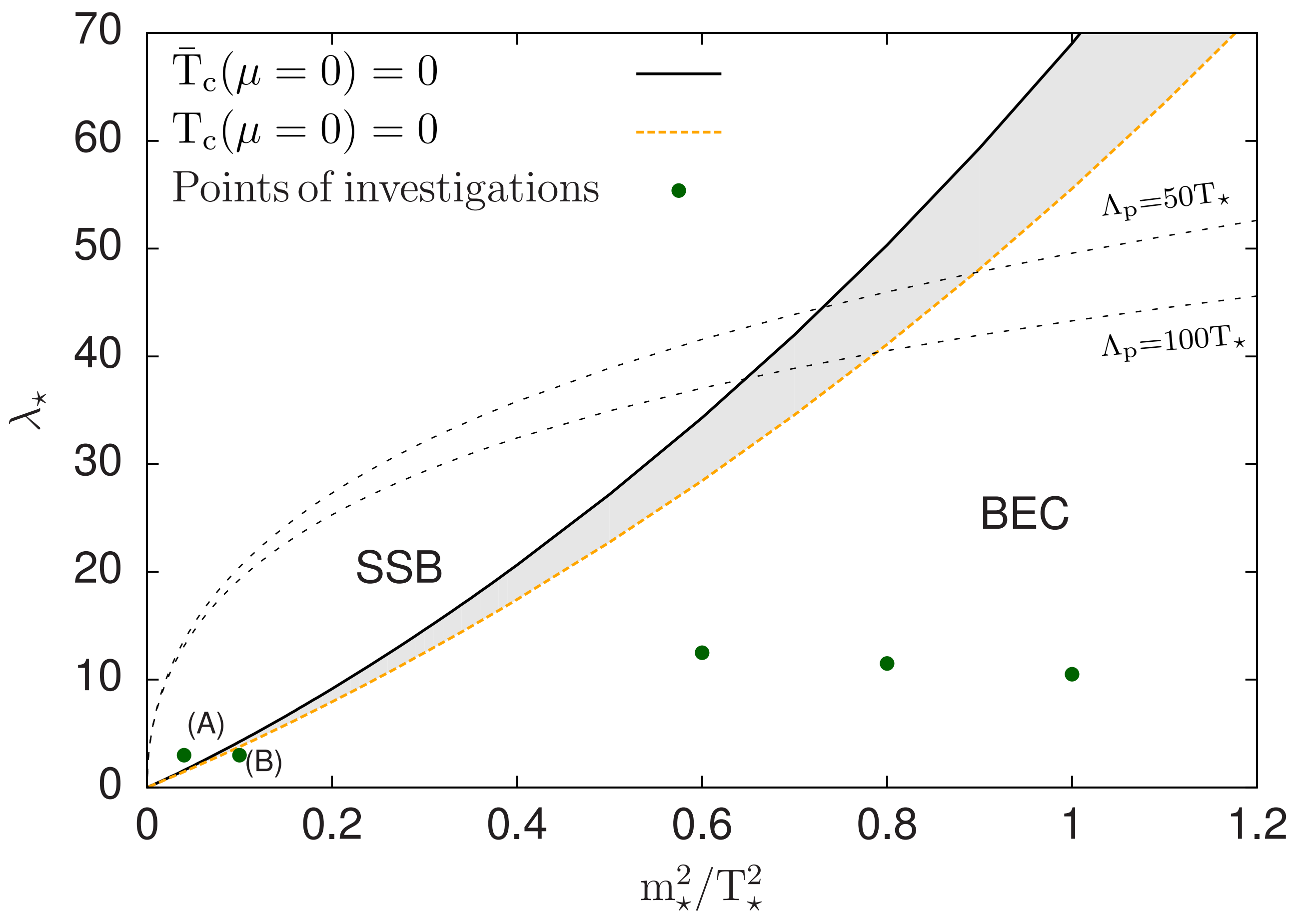
- Z_0 is finite, as the tadpole has no μ dependent divergence.
- α dependence only through Z_0. We choose Z_0, no new parameter.

Transition line

The transition temperature at chemical potential μ, is determined by

$$\hat{M}^2_{\phi=0;T=T_c(\mu),\mu} = \mu^2 \quad \text{or} \quad \bar{M}^2_{\phi=0;T=\bar{T}_c(\mu),\mu} = \mu^2 \, .$$

The $\mu = 0$ existence of T_c $(\bar{T}_c)$ splits the $m_\star^2 - \lambda_\star$ parameter plane in two

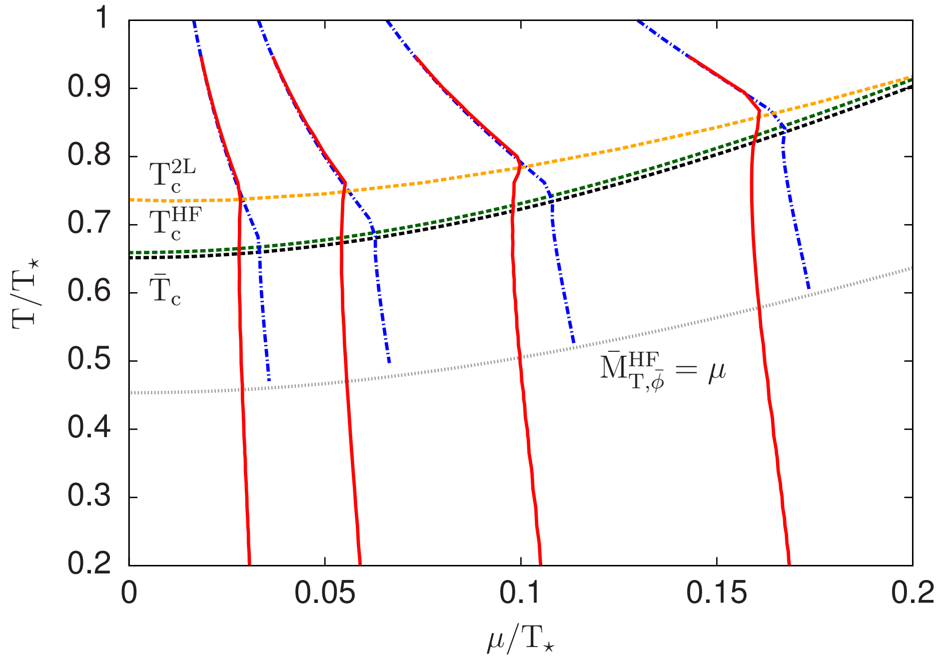


Comparison: 2-loop vs. Hartree-Fock

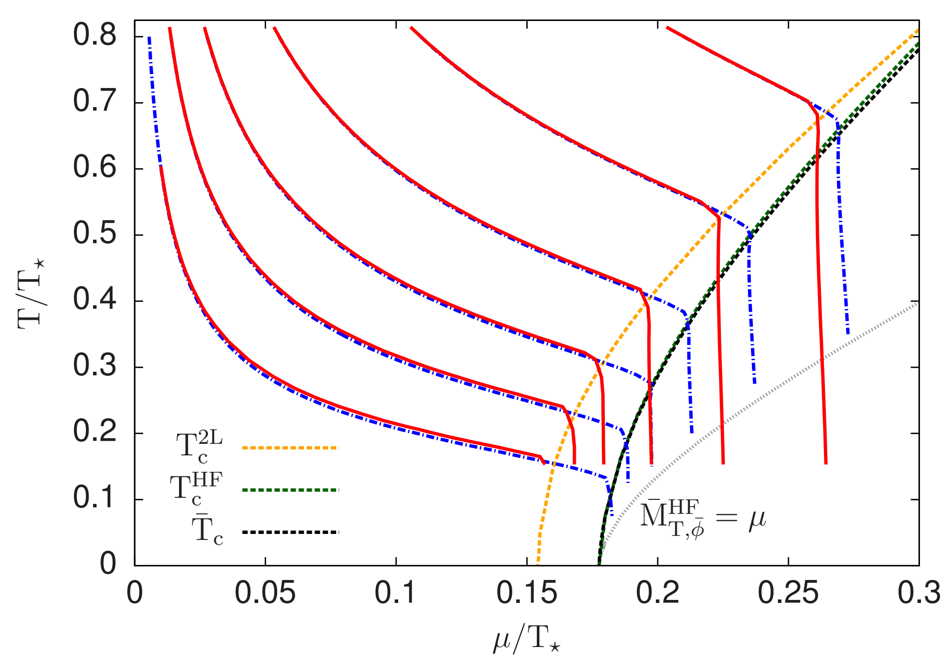
The density is defined as

$$\rho = \frac{1}{\beta V}\frac{\partial \ln Z}{\partial \mu} = \mu\bar{\phi}^2 + \mu\int_Q(\bar{G}_L(Q)+\bar{G}_T(Q)) - 2\int_Q \omega_n \bar{G}_A(Q)\,.$$

We compare the iso-density lines at given parameters in the H-F and the 2-loop approximations:



(A)



(B)

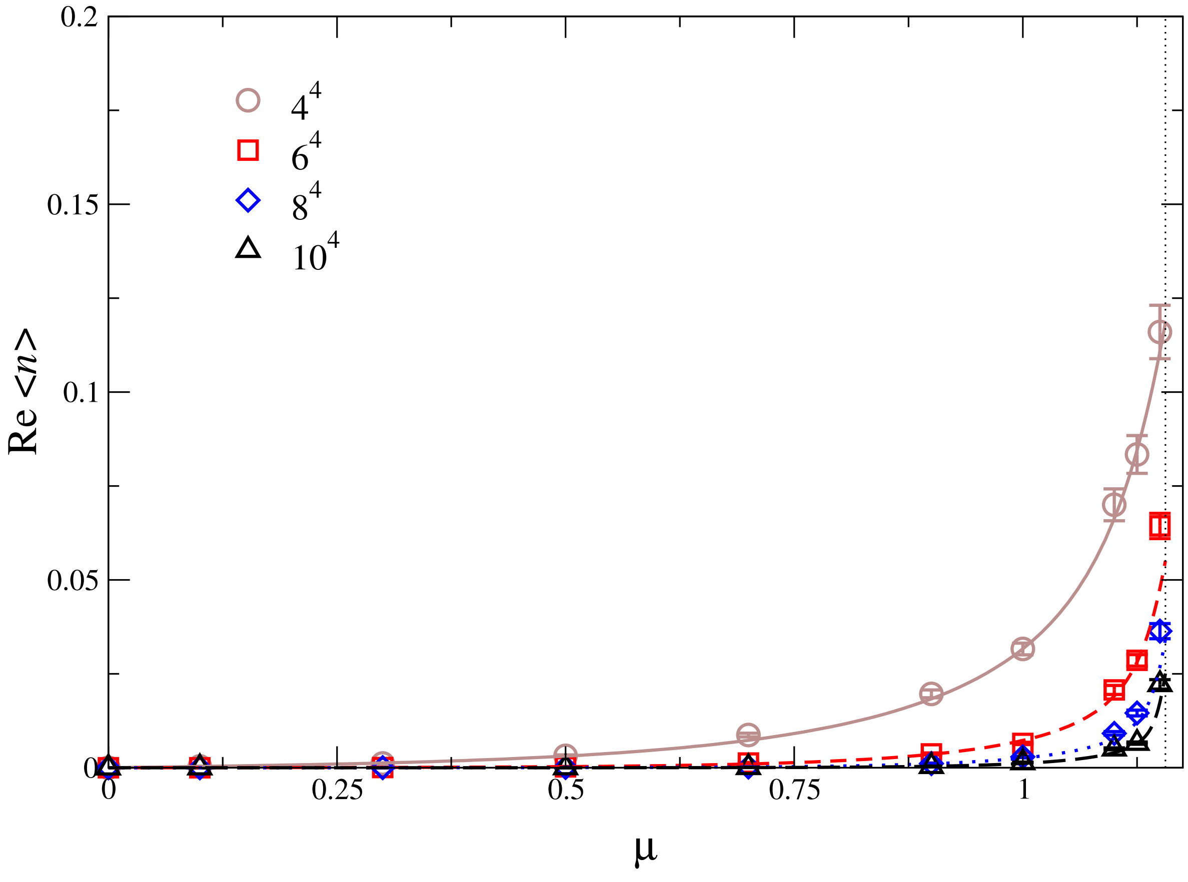
Comparison: Hartree-Fock vs. Lattice

This comparison was shown in G. Aarts, JHEP 0905 (2009) 052.

The H-F approximation, using the

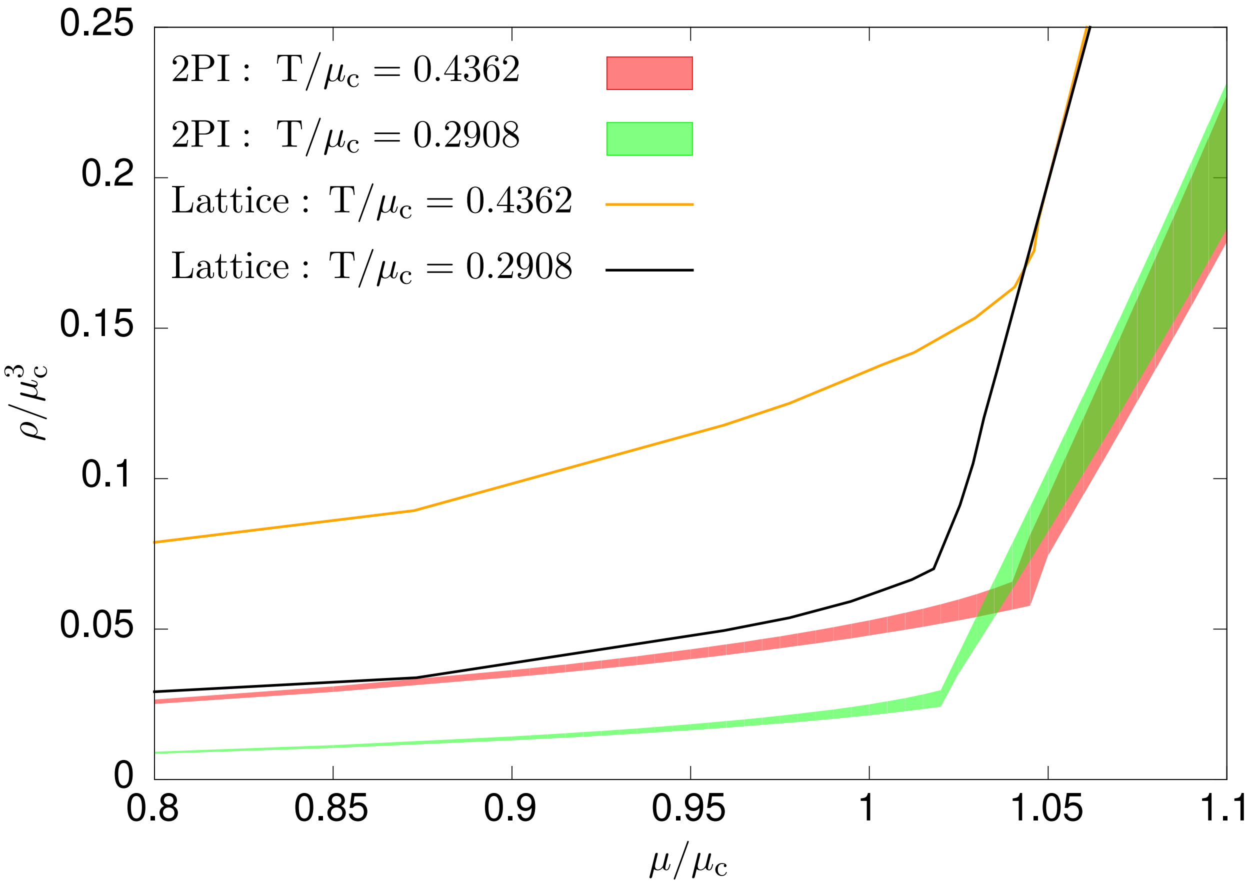
- bare lattice action,
- at fixed lattice spacing,
- in the symmetric phase,

reproduces the lattice results:



Comparison: 2-loop vs. Lattice

We compare ρ/μ_c^3 as a function of μ/μ_c, and see a qualitative, but not quantitative agreement. Lattice results are from Gattringer et al., Nucl. Phys. B **869**, 56 (2013).



Why?

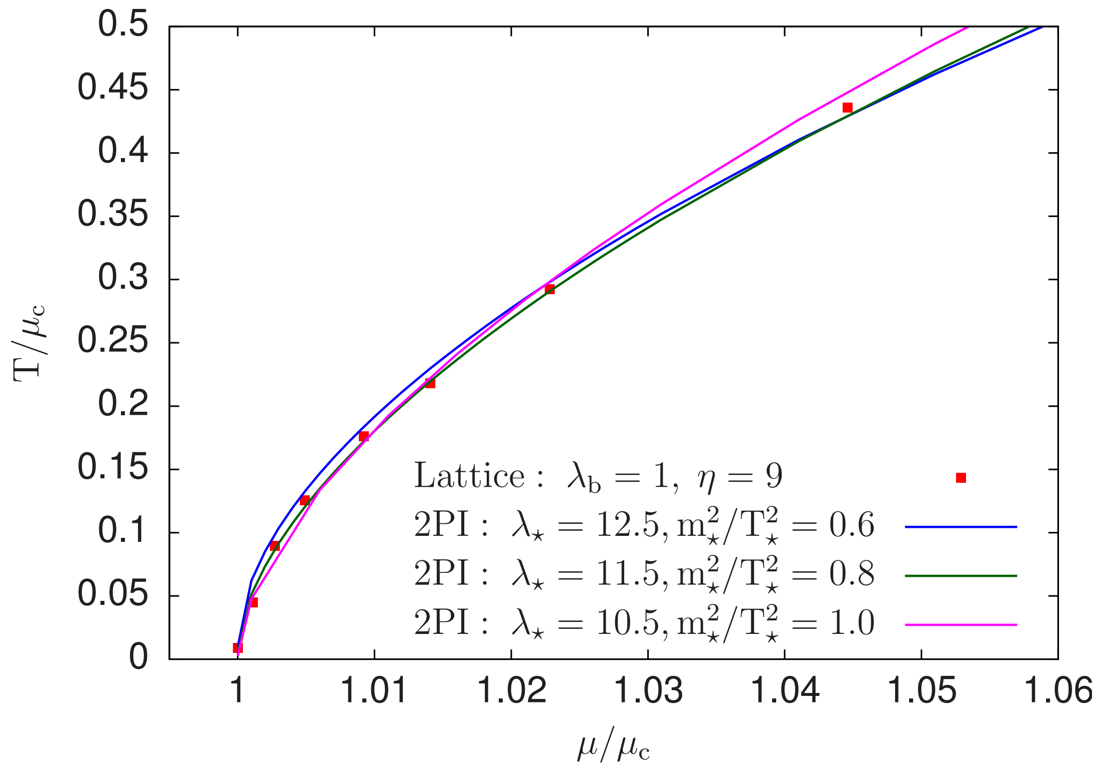
Comparison: 2-loop vs. Lattice

On the lattice: bare theory, at fixed lattice spacing.

Our 2PI: renormalized theory, in the ”continuum” limit.

- Cut-off effects are not small, as the inverse lattice spacing is comparable to physical quantities (e.g. $a\mu_c \approx 1.15$).
- We did not use the lattice action $\rightarrow$ even the bare theories differ.

Choice of parameters based on the reproduction of the $T_c(\mu)$ curve.



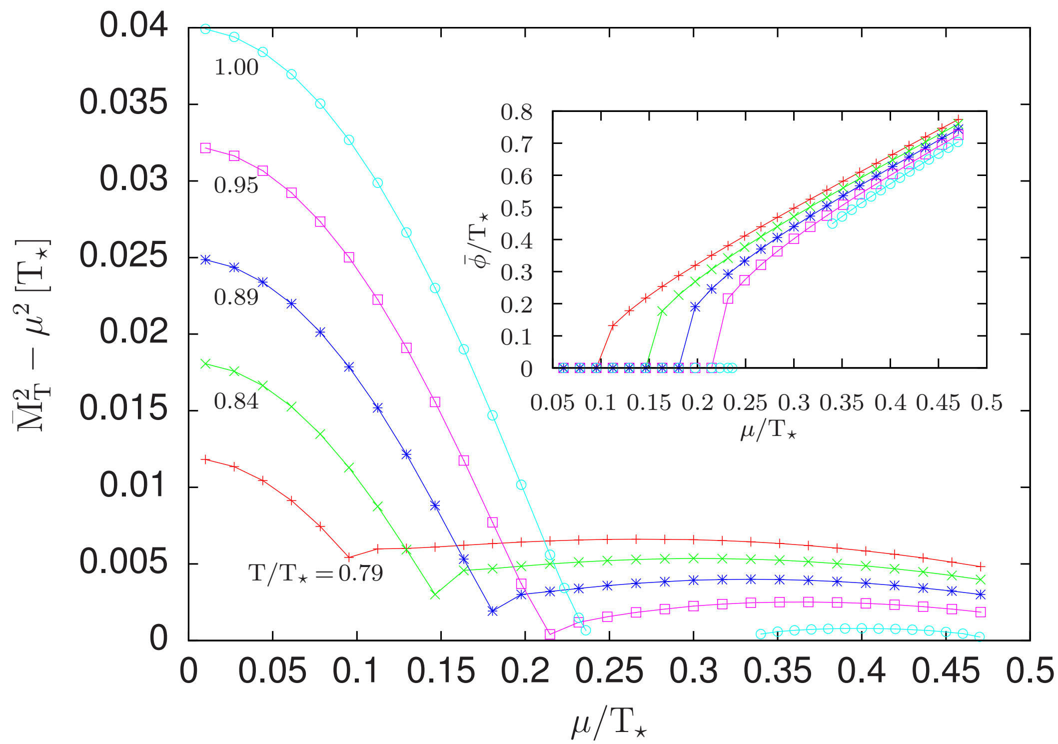
Loss of solution

We define $\bar{\mu}_c(T)$ as

$$\bar{M}^2_{\phi=0,T,\mu=\bar{\mu}_c(T)} = \bar{\mu}_c^2\,,$$

which is the inverse of $\bar{T}_c(\mu)$.

- $\mu > \bar{\mu}_c(T) \rightarrow$ no solution for gap eq at $\phi = 0$.
- $\phi_c(\mu, T)$: the smallest ϕ for which a solution of the gap equations exists.
- Solution of the coupled gap and field equations is lost when: $\bar{\phi}(\mu, T) < \phi_c(\mu, T)$.



Loss of solution

A more general look: two problems tightly connected.

- $\bar{M}_T^2 - \mu^2 < 0$
- Renders integrals meaningless.
- If Goldstone's theorem is not obeyed, this may always happen.
- Symmetry improvements ensuring GS theorem [see Pilaftsis et al., Nucl.Phys. B **874** (2013)]

- $\bar{M}_T^2 - \mu^2$ becoming small, or zero.
- May lead to infrared divergences.
- Further resummations are needed to tame them.
- Vertex resummations are good candidates, e.g. NLO-$1/N$.

Conclusions

- Renormalization program and numerical method successfully extended to $\mu \neq 0$.
- Parameter space divided into SSB and BEC regions.
- We understood and generalized the Silver Blaze property using symmetry properties.
- The used truncation preserves the Silver Blaze property.
- We found qualitative agreement with lattice results.
- At high T and μ the solution is lost, could be a general problem.
- Merge the project with the $O(4)$ investigations to study pion condensation.