Model Independent Features of Bose-Einstein Correlations

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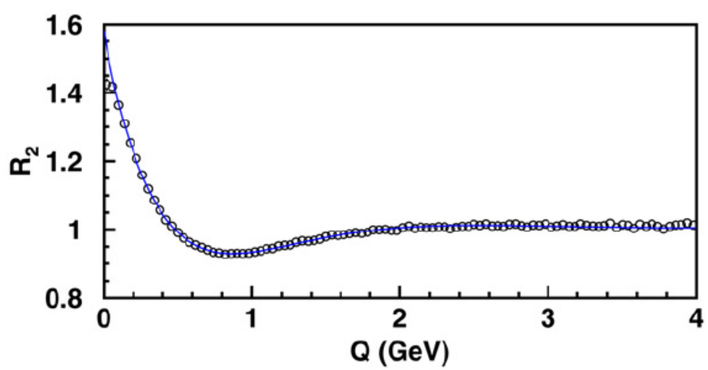

Fig. 1. The Bose–Einstein correlation function R_2 for events generated by PYTHIA. The curve corresponds to a fit of the one-sided Lévy parametrization, Eq. (13).

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Model-independent shape analysis:

- General introduction
- Edgeworth,
- Laguerre,
- Levy expansions

Summary

MODEL - INDEPENDENT SHAPE ANALYIS I.

experimental properties:

i) The correlation function tends to a constant for large values of the relative momentum Q.

ii) The correlation function has a non-trivial structure at a certain value of its argument.

The location of the non-trivial structure in the correlation function is assumed for simplicity to be close to $Q = 0$.

Model-independent but experimentally testable:

- $w(t)$ measure in an abstract H-space
- approximate form of the correlations
- t: dimensionless scale variable

$$\int dt w(t) h_n(t) h_m(t) = \delta_{n,m},$$

$$f(t) = \sum_{n=0}^{\infty} f_n \, h_n(t),$$

$$f_n = \int dt w(t) \, f(t) \, h_n(t).$$

e.g. $t = Q_I R_I$

MODEL - INDEPENDENT SHAPE ANALYIS II.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)},$$

$$R_2(\mathbf{k}_1, \mathbf{k}_2) = C_2(\mathbf{k}_1, \mathbf{k}_2) - 1.$$

Let us assume, that the function $g(t) = R_2(t)/w(t)$ is also an element of the Hilbert space H. This is possible, if

$$\int dt\, w(t) g^2(t) = \int dt\, \left[R_2^2(t)/w(t)\right] < \infty, \quad (6)$$

Then the function *g* can be expanded as

$$g(t) = \sum_{n=0}^{\infty} g_n h_n(t),$$

$$g_n = \int dt\, R_2(t) h_n(t).$$

From the completeness of the Hilbert space and from the assumption that *g*(*t*) is in the Hilbert space:

$$R_2(t) = w(t) \sum_{n=0}^{\infty} g_n h_n(t).$$

MODEL - INDEPENDENT SHAPE ANALYIS III.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)},$$

$$C_2(t) = \mathcal{N} \left\{ 1 + \lambda_w \, w(t) \sum_{n=0}^{\infty} g_n h_n(t) \right\}$$

Model-independent AND experimentally testable:

- method for any approximate shape *w*(*t*)
- the core-halo intercept parameter of the CF is $$\lambda_* = \lambda_w \sum_{n=0}^{\infty} g_n h_n(0)$$
- coefficients by numerical integration (fits to data) $$g_n = \int dt \, R_2(t) h_n(t)$$
- condition for applicability: experimentally testabe $$\int dt \left[R_2^2(t) / w(t) \right] < \infty$$

EDGEWORTH EXPANSION: ~ GAUSSIAN

$$t = \sqrt{2} Q R_E,$$
$$w(t) = \exp(-t^2/2),$$
$$\int_{-\infty}^{\infty} dt \, \exp(-t^2/2) H_n(t) H_m(t) \propto \delta_{n,m},$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_E \exp(-Q^2 R_E^2) \times \left[1 + \frac{\kappa_3}{3!} H_3(\sqrt{2} Q R_E) + \frac{\kappa_4}{4!} H_4(\sqrt{2} Q R_E) + \ldots \right] \right\}.$$

$$H_n(t) = \exp(t^2/2) \left(-\frac{d}{dt} \right)^n \exp(-t^2/2).$$

$$H_1(t) = t,$$
$$H_2(t) = t^2 - 1,$$
$$H_3(t) = t^3 - 3t,$$
$$H_4(t) = t^4 - 6t^2 + 3, \ \ldots$$

3d generalization straightforward

- Applied by NA22, L3, STAR, PHENIX, ALICE, CMS (LHCb?)

LAGUERRE EXPANSIONS: ~ EXPONENTIAL

Model-independent but experimentally tested:

- *w*(*t*) exponential
- *t*: dimensionless
- Laguerre polynomials

$$t = QR_L,$$
$$w(t) = \exp(-t)$$
$$\int_0^\infty dt\, \exp(-t) L_n(t) L_m(t) \propto \delta_{n,m},$$

$$L_n(t) = \exp(t) \frac{d^n}{dt^n} (-t)^n \exp(-t).$$

$$L_0(t) = 1,$$
$$L_1(t) = t - 1,$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_L \exp(-QR_L) \left[1 + c_1 L_1(QR_L) + \frac{c_2}{2!} L_2(QR_L) + \dots \right] \right\}$$

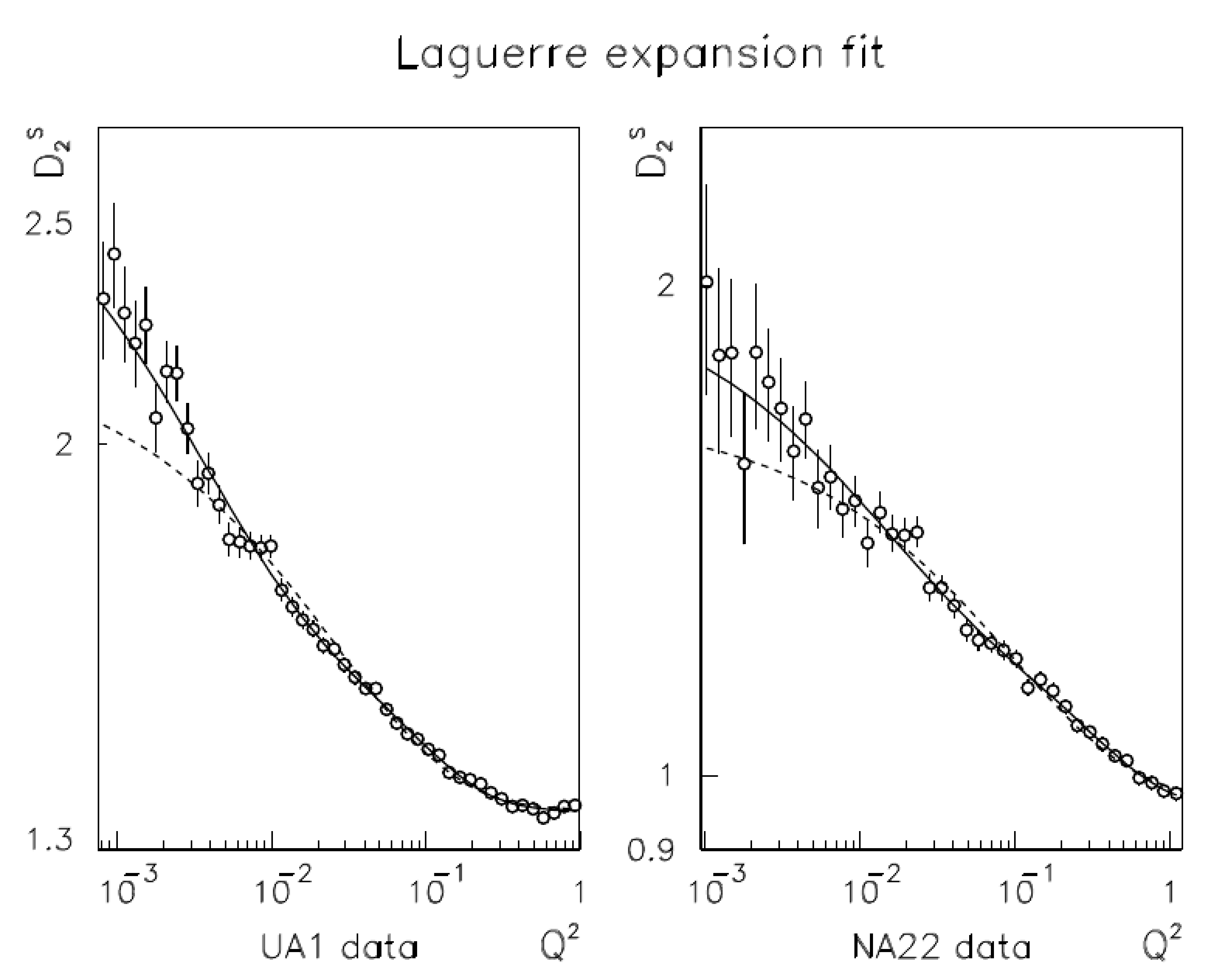
First successful tests

- NA22, UA1 data
- convergence criteria satisfied
- intercept parameter ~ 1

$$\int_0^\infty dt\, R_2^2(t) \exp(+t) < \infty,$$

$$\lambda_* = \lambda_L [1 - c_1 + c_2 - \dots],$$
$$\delta^2 \lambda_* = \delta^2 \lambda_L \left[1 + c_1^2 + c_2^2 + \dots \right] + \lambda_L^2 \left[\delta^2 c_1 + \delta^2 c_2 + \dots \right]$$

LAGUERRE EXPANSIONS: ~ superEXPONENTIAL



T. Csörgő and S: Hegyi, hep-ph/9912220, T. Csörgő, hep-ph/001233

LEVY EXPANSIONS: ~ 1d LEVY

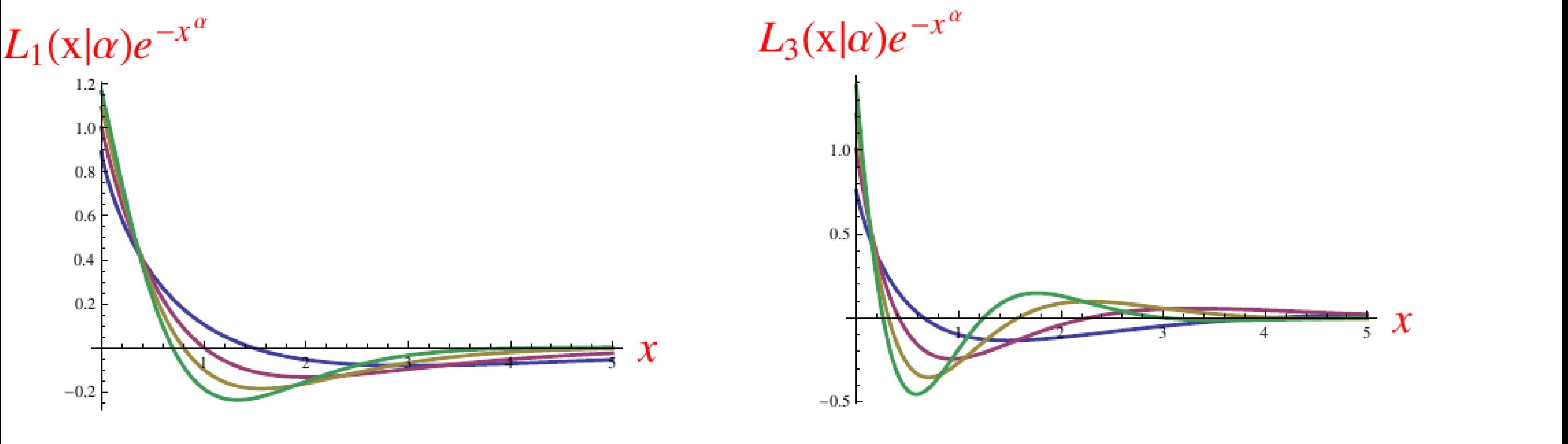
Model-independent but:

- Levy generalizes Gaussians and exponentials
- ubiquoutous in nature
- How far from a Levy?
- Need new set of polynomials orthonormal to a Levy weight

$$L_1(x\,|\,\alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$L_2(x\,|\,\alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$\mu_{r,\alpha} = \int_0^\infty dx \;\; x^r f(x\,|\,\alpha) = \tfrac{1}{\alpha}\Gamma(\tfrac{r+1}{\alpha})$$



Lévy polynomials of first and third order times the weight function e^{-x^α} for $\alpha = 0.8, 1.0, 1.2, 1.4$.

1st-order Lévy polynomial	$\gamma\left[1+\lambda e^{-R^\alpha Q^\alpha}\left[1+c_1 L_1(Q\vert\alpha,R)\right]\right]$
3rd-order Lévy polynomial	$\gamma\left[1+\lambda e^{-R^\alpha Q^\alpha}\left[1+c_1 L_1(Q\vert\alpha,R)+c_3 L_3(Q\vert\alpha,R)\right]\right]$

M. de Kock, H. C. Eggers, T. Cs: arXiv:1206.1680v1 [nucl-th]

MINIMAL MODEL ASSUMPTION: LEVY

experimental conditions:
(i) The correlation function tends to a constant for large values of the relative momentum Q.
(ii) The correlation function deviates from its asymptotic, large Q value in a certain domain of its argument.

(iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

Model-independent but:

- Assumes that Coulomb can be corrected
- No assumptions about analyticity yet
- For simplicity, consider 1d case first
- For simplicity, consider factorizable *x k*
- Normalizations :
  - density
  - multiplicity
  - single-particle spectra

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)}$$

$$S(x, k) = f(x)\, g(k)$$

$$\int \mathrm{d}x\, f(x) = 1, \qquad \int \mathrm{d}k\, g(k) = \langle n \rangle,$$

$$N_1(k) = \int \mathrm{d}x\, S(x, k) = g(k).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MINIMAL MODEL ASSUMPTION: LEVY/GAUSS

$$\psi_{k_1,k_2}(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\exp(\mathrm{i}k_1x_1 + \mathrm{i}k_2x_2) + \exp(\mathrm{i}k_1x_2 + \mathrm{i}k_2x_1)\right].$$

$$N_2(k_1,k_2) = \int \mathrm{d}x_1\mathrm{d}x_2\, S(x_1,k_1)\, S(x_2,k_2)\, |\psi_{k_1,k_2}(x_1,x_2)|^2,$$

Model-independent but:

- Assumes plane-wave propagation
- C_2 measures a modulus squared Fourier-transform vs relative momentum
- Analyticity assumptions is an extra!
- Resulting correlations Gaussian
- Radius interpreted as variances

$$C_2(k_1,k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int \mathrm{d}x\, \exp(\mathrm{i}q_{12}x)\, f(x),$$

$$q_{12} = k_1 - k_2.$$

$$\tilde{f}(q) \approx 1 + \mathrm{i}q\langle x\rangle - q^2\langle x^2\rangle/2 + \ldots,$$

$$C(q) = 1 + |\tilde{f}(q)|^2 \approx 2 - q^2(\langle x^2\rangle - \langle x\rangle^2) \approx 1 + \exp(-q^2R^2),$$

$$R = \sqrt{\langle x^2\rangle - \langle x\rangle^2}.$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MINIMAL MODEL ASSUMPTION: LEVY

Model-independent but:

- not assumes analyticity
- C_2 measures a modulus squared Fourier-transform vs relative momentum
- Levy: Generalized Central Limit Theorems
- Stable for adding one more small random source
- Correlations non-Gaussian
- Radius not a variance
- $0 < \alpha \leq 2$

$$C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int \mathrm{d}x \, \exp(\mathrm{i} q_{12} x) \, f(x),$$

$$x = \sum_{i=1}^{n} x_i,$$

$$f(x) = \int \Pi_{i=1}^{n} \mathrm{d}x_i \, \Pi_{j=1}^{n} f_j(x_j) \, \delta(x - \sum_{k=1}^{n} x_k).$$

$$\tilde{f}_i(q) = \exp\left(iq\delta_i - |\gamma_i q|^\alpha\right), \qquad \prod_{i=1}^{n} \tilde{f}_i(q) = \exp\left(iq\delta - |\gamma q|^\alpha\right),$$

$$\gamma^\alpha = \sum_{i=1}^{n} \gamma_i^\alpha, \qquad \delta = \sum_{i=1}^{n} \delta_i.$$

$$\tilde{f}(q) = \prod_{i=1}^{n} \tilde{f}_i(q)$$

$$C(q; \alpha) = 1 + \lambda \exp\left(-|qR|^\alpha\right).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

UNIVARIATE LEVY EXAMPLES

Include some well known cases:

- $\alpha = 2$
  - Gaussian source, Gaussian C_2

$$f(x) = \frac{1}{(2\pi R^2)^{1/2}} \exp\left[-\frac{(x-x_0)^2}{2R^2}\right]$$

$$C(q) = 1 + \exp\left(-q^2 R^2\right)$$

- $\alpha = 1$
  - Lorentzian source, exponential C_2

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x-x_0)^2},$$

$$C(q) = 1 + \exp\left(-|q\, R|\right).$$

- asymmetric Levy:
  - asymmetric support
  - Streched exponential

$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x-x_0)^{3/2}} \exp\left(-\frac{R}{8(x-x_0)}\right)$$

$$x_0 < x < \infty,$$

$$C(q) = 1 + \exp\left(-\sqrt{|q\, R|}\right).$$

- Radius not a variance,
- Variance infinite for Lorentz
  - infinite for any $\alpha < 2$

T. Cs, hep-ph/0001233, T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MULTIVARIATE LEVY DISTRIBUTIONS

The characteristic function is $f(t) = e^{-t^{\alpha}}$, where $t = \left(\sum_{i,j=1,3} {R_{i,j}}^2 q_i q_j\right)^{1/2}$

$$C_2(k_1, k_2) = 1 + \lambda \exp\left[-\left(\sum_{i,j=1}^{3} R_{ij}^2 q_i q_j\right)^{\alpha/2}\right]$$

Model-independent but:

- A new parameter alpha generalizes Gauss
- Solved only for symmetric Levy distributions $\left({R_{i,j}}^2 = {R_{j,i}}^2\right)$
- Deep open problems in mathematical statistics

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MULTIVARIATE LEVY EXPANSIONS

$$L_1(x\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$L_2(x\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$L_1(t|\alpha) = \frac{t}{\alpha}\Gamma\left(\frac{1}{\alpha}\right) - \frac{1}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)$$

$$\mu_{r,\alpha} = \int_0^\infty dx\ x^r f(x\,|\,\alpha) = \tfrac{1}{\alpha}\Gamma(\tfrac{r+1}{\alpha})$$

$$\begin{aligned} L_2(t|\alpha) \; = \; & \frac{1}{\alpha}\left[\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\frac{4}{\alpha}\right) - \frac{1}{\alpha}\Gamma^2\left(\frac{3}{\alpha}\right)\right] - \\ & -\frac{t}{\alpha}\left[\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{4}{\alpha}\right) - \Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\frac{3}{\alpha}\right)\right] + \\ & +\frac{t^2}{\alpha}\left[\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{3}{\alpha}\right) - \frac{1}{\alpha}\Gamma^2\left(\frac{2}{\alpha}\right)\right] \end{aligned}$$

1st-order Levy expansion

$$t = \left(\sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j\right)^{1/2}$$

$$C_2(Q) = N\left\{1 + \lambda \exp\left(-\left(\sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j\right)^{\alpha/2}\right)\left[1 + c_1 \frac{\left(\sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j\right)^{1/2}}{\alpha}\left(\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{2}{\alpha}\right)\right)\right]\right\}$$

M. de Kock, H. C. Eggers, T. Cs: arXiv:1206.1680v1 [nucl-th]

SUMMARY AND CONCLUSIONS

Several model-independent methods:

- Based on matching an abstract measure in H to the approximate shape of data
- Gaussian: Edgeworth expansions
- Exponential: Laguerre expansions
- Levy: Generalized Central Limit Theorems
- Levy expansions $0 < \alpha \leq 2$
- New directions: multivariate Levy expansions