Functional Renormalization Group in fermionic systems

A. Jakovác

ELTE, Dept. of Atomic Physics

Literature

This talk is based on the papers:

A. Patkos, Mod.Phys.Lett. A27 (2012) 1250212
AJ and A. Patkos, Phys. Rev. D88, 065008 (2013)
AJ, A. Patkos and P. Posfay, arXiv:1406.3195 [hep-th]

Other papers we used:

B. Rosenstein, D. Warr and S.H. Park, Phys. Rev. Lett. 62, 1433 (1989)
H. Gies and C. Wetterich, Phys. Rev. D65:065001 (2002)
J. Jaeckel and C Wetterich, Phys. Rev. D68:025020 (2003)
J. Braun, H. Gies and D.D. Scherer, Phys. Rev. D83:085012 (2011)
. . .

Outlines

Outlines

Functional Renormalization Group for bosons

Computation of quantum n-point correlation functions

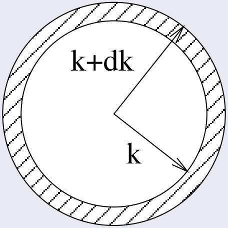
scalar model with action $S[\Phi]$

$$\left\langle T\hat{\Phi}(x_1)\ldots\hat{\Phi}(x_n)\right\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\,\Phi(x_1)\ldots\Phi(x_n)e^{iS[\Phi]}$$

Euclidean theory: $iS \to -S_E$ (Wick-rotation).

Gradual path integration

- in each step integrate out modes with $|p| \in [k+dk, k]$
- k starts at UV scale: classical (tree level)
- $k \to 0 \quad \Rightarrow \quad$ quantum



Gradual path integration with external self-energy

Introduce external kinetic term (regulator) $R_k(p)$

$$S_E \to S_E - \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \Phi_p^* R_k(p) \Phi_p$$

Sharp cutoff

- $R_k(p) \to \infty$: prohibits propagation of p momentum modes
- $R_k(p) = 0$: original action
- integration above scale k: $R_k(p) = R_0 \Theta(k-p)$ $\quad (R_0 \to \infty)$

Smooth version

In fact R_k can be chosen to be more general, provided:

- $R_{k\to\Lambda}$ is "large": suppress fluctuations
- $R_{k\to 0}(p) = 0$: recover full quantum result

popular choice: Litim's regulator $R_k(p) = (k^2 - p^2)\Theta(k-p)$

Wetterich equation

Generating functional

$$e^{-W_k[J]} = \int \mathcal{D}\Phi \, e^{-S_E[\Phi] - \frac{1}{2}\int \Phi^* R_k \Phi + \int J\Phi}$$

differentiate wrt. k:

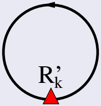
$$-\partial_k W_k[J] e^{-W_k[J]} = -\frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, \partial_k R_k(p) \int \mathcal{D}\Phi \, \Phi^*(p)\Phi(p) e^{-S_E[\Phi] - \frac{1}{2}\int \Phi^* R_k \Phi + \int J\Phi}.$$

RHS is a 2-point function: $\langle \Phi\Phi \rangle \to \frac{\partial^2 W}{\partial J \partial J}$

Equation for the quantum effective action $\Gamma_k[\varphi]$:

Wetterich equation

$$\partial_k \Gamma_k = \frac{1}{2}$$



$$= \frac{1}{2} \mathrm{Tr}\, \partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1}$$

exact equation!

Running couplings

Treatment of the Wetterich equation

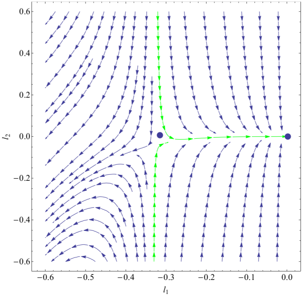
$O_n[\varphi]$ operator basis built on the scalar field φ

$$\Gamma_k[\varphi] = \sum_n g_n(k) O_n[\varphi]$$

From Wetterich equations, expanding the RHS in basis $O_n[\varphi]$:

$$\partial_k g_n(k) = \beta_n(g)$$

solution: $g_n(k)$ curves (running couplings)



- different curves: initial conditions
- $\beta_n = 0$ fixed points
- with one fixed point:
  asymptotically free, trivial models
- with more fixed points:
  phase transitions, asymptotic safety

Practical approach

General operator basis is too large...

we should include all operators, also the most exotic ones

In practice we need an Ansatz

- Local Potential Approximation (LPA)

$$\Gamma_{\mathrm{LPA}} = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + U_k(\varphi) \right]$$

  Wetterich equation $\Rightarrow$ equation for U_k

  with Litim's regulator $(Q_d^{-1} = (4\pi)^{d/2}\Gamma(\frac{d}{2}+1))$

$$\boxed{\partial_k U_k = \frac{Q_d k^{d+1}}{k^2 + U_k''}}$$

  partial differential equation

- LPA+ wave function renormalization (LPA′):

$$\frac{1}{2}(\partial_\mu \varphi)^2 \to \frac{Z_k(\varphi)}{2}(\partial_\mu \varphi)^2$$

Outlines

Motivation

- fundamental matter is fermionic (+ Higgs)
  $\Rightarrow$ bosonic matter are fermionic compounds (bound states)
  use original dof without auxiliary bosonic representants
- double representation of the same physical quantity?
  (cf. linear sigma model)
  understand the systematics behind bosonization
- scalar potential $\equiv$ potential for fermionic composite operators
  condensation to zero momentum mode (Bose-, chiral condensation, superfluidity)
  how can it be consitent with the fermionic nature?

Fermionic effective action

effective action is functional of fermionic (Grassmann) variables, too: $\Gamma[\bar{\psi}, \psi, \varphi]$

- φ bosonic fields – omit for this discussion
- fermions in Nambu representation $\Psi = \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}$

Wetterich equation:

$$\partial_k \Gamma_k[\Psi] = -\frac{1}{2} \mathrm{Tr}\, \partial_k R_k \left(\Gamma^{(1,1)}_{k,\Psi\Psi} + R_k \right)^{-1}$$

where

- minus sign from fermionic trace
- R_k fermionic regulator
- $(\Gamma^{(1,1)}_{k,\Psi\Psi})_{ij} = \frac{\overrightarrow{\partial}}{\partial \Psi_i} \Gamma_k[\Psi] \frac{\overleftarrow{\partial}}{\partial \Psi_j} = \begin{pmatrix} \Gamma_{\psi\psi} & \Gamma_{\psi\bar{\psi}} \\ \Gamma_{\bar{\psi}\psi} & \Gamma_{\bar{\psi}\bar{\psi}} \end{pmatrix}$

Local fermionic potential approximation

Does LPA' work in the fermionic case?

$$\Gamma_k[\Psi] \to \int d^d x \, \left[Z_k \bar{\psi} \partial\!\!\!/ \psi + U_k(\Psi_x)\right].$$

Problem with this approach

- $\psi_i^2(x) = 0$ because of the fermionic nature
  $\Rightarrow \quad (\bar{\psi} C \psi)^n = 0$ for large enough n!
  $\Rightarrow \quad U_k(\Psi)$ is a finite polynomial?

But...

- after bosonization $U_b(\varphi)$ is allowed in any form!
- with $\varphi \sim \bar{\psi}\psi$ we should have arbitrary fermionic potential

Key for the interpretation: $\bar{\psi}(x)\psi(x') \neq 0$ if x and x' are just "almost" at the same position.

Complete fermionic effective action

Note: $\sum_{n>N} f_n x^n = 0$ if $f_n = 0$ or $x = 0$.

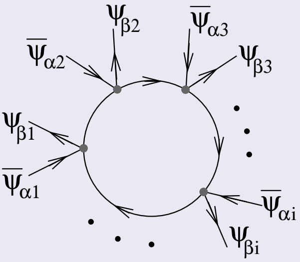
Here: Expansion of the exact effective potential for generic background:

$$\Gamma_k[\Psi] = \sum_{n\,\mathrm{even}} \frac{1}{n!} \sum_{indices} \int_{x_i} \Gamma^{(n)}_{k;i_1\ldots i_n}(x_1,\ldots,x_n)\Psi_{i_1}(x_1)\ldots\Psi_{i_n}(x_n)$$

- the proper vertices $\Gamma^{(n)} \not\equiv 0$
- in functional sense

  Γ_k is not a finite polynomial!

n-point function

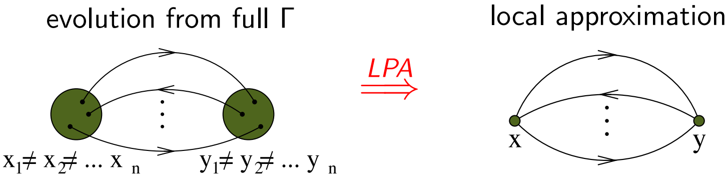


for 4-fermion interaction

Understanding fermionic LPA

Assumption behind the LPA

propagators vary in spacetime much slower than vertices



- numerical values of the diagrams are close
- vertex of the 1st diagram: $\Gamma_k^{(n)}(x_1,\dots,x_n)\Psi(x_1)\dots\Psi(x_{2n})$
- vertex of the 2nd diagram:

$$U_k^{(n)} \lim_{\Delta V\to 0}\left(\frac{1}{\Delta V}\int_{\Delta V}\bar\psi(x)\psi(x)\right)^n \xrightarrow{\text{notation}} U_k^{(n)}(\bar\psi(x)\psi(x))^n$$

- heuristically: position is not a point, but a patch (resolution/compositeness scale)

Evaluation of the tracelog

Computation: $\mathrm{Tr}\,\partial_k R_k(\Gamma^{(2)}+R)^{-1} = \hat{\partial}_k \,\mathrm{Tr}\ln(\Gamma^{(2)}+R)$

$\Rightarrow$ trace of a supermatrix.

Representation

$$\begin{pmatrix} \Gamma_{\psi\psi} & \Gamma_{\psi\bar\psi} \\ \Gamma_{\bar\psi\psi} & \Gamma_{\bar\psi\bar\psi} \end{pmatrix} = \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \begin{pmatrix} 1 & \bar C \\ 0 & 1 \end{pmatrix}$$

From consistency:

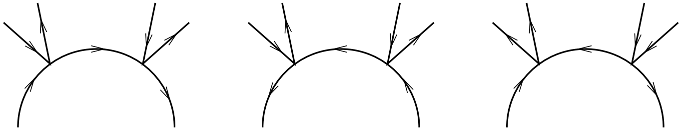
$$A = \Gamma_{\bar\psi\psi}, \quad B = M\Gamma_{\psi\bar\psi}, \quad C = \Gamma_{\psi\psi}\Gamma^{-1}_{\bar\psi\psi}, \quad \bar C = \Gamma^{-1}_{\psi\bar\psi}\Gamma_{\bar\psi\bar\psi}$$

$$M = 1 - \Gamma_{\psi\psi}\Gamma^{-1}_{\bar\psi\psi}\Gamma_{\bar\psi\bar\psi}\Gamma^{-1}_{\psi\bar\psi}.$$

Therefore

$$\mathrm{Tr}\log\Gamma^{(1,1)}_{k,\Psi\Psi} = \mathrm{Tr}\log\Gamma_{\bar\psi\psi} + \mathrm{Tr}\log\Gamma_{\psi\bar\psi} + \mathrm{Tr}\log(1 - \Gamma_{\psi\psi}\Gamma^{-1}_{\bar\psi\psi}\Gamma_{\bar\psi\bar\psi}\Gamma^{-1}_{\psi\bar\psi})$$

Diagrammatically:



Outlines

Gross-Neveu model and representation through invariants

Gross-Neveu model: matter content $\psi_i,\ i = 1 \dots N_f$

$$S[\Psi] = \int d^d x \left[\sum_{i=1}^{N_f} \bar{\psi}_i \partial\!\!\!/ \psi_i + \frac{g}{2N_f} I \right]$$

where $I = \left(\sum_{i=1}^{N_f} \bar{\psi}_i \psi_i \right)^2$.

- chiral symmetry: $\psi \to -\gamma_5 \psi,\ \bar{\psi} \to \bar{\psi}\gamma_5$
- $O(N_f)$ flavour symmetry
- symmetries $\Rightarrow$ Γ_k must depend on invariants

Ansatz for the effective action

$$\Gamma_k[\Psi] = \int d^d x \left[Z_k \sum_{i=1}^{N_f} \bar{\psi}_i \partial\!\!\!/ \psi_i + U(I) \right]$$

- could depend on other invariants (eg. $(\bar{\psi}\gamma_\mu \psi)^2$)
- assumption on dependence on I only $\Rightarrow$ self-consistent!

Computation in LPA' goes as described in general case

$\Rightarrow$ lot of cancellations! (expected also in general case)

Finally we get a considerably simple expression:

$$\mathrm{STr}\log\Gamma = -2\,\mathrm{Tr}\log G_0^{-1} + \int_q \log\left(1 - \frac{2m\bar{\psi}\psi}{Z^2P_k^2(q)+m^2}\right).$$

Evaluating the integrals using Litim's regulator we find

$$\boxed{\partial_k U_k = k^{d+1}Q_d\left[\frac{4N_f+1}{Z^2k^2+4I\,U'^2} - \frac{1}{Z^2k^2+4IU'(U'+\tilde{U})}\right]}$$

where $Q_d^{-1} = (4\pi)^{d/2}\Gamma(\frac{d}{2}+1)$.

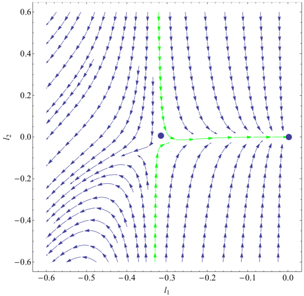
Two terms with different sign $\Rightarrow$

Bosonic type terms appear, too! (without explicit bosonic dof)

Flow and fixed points

For $2 < d < 4$: one nontrivial fixed point for any N_f:

Flow pattern $d = 3$



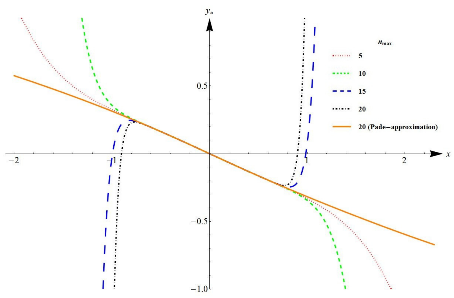
- scaling exponents (agree with bosonized version for $N_f = \infty$)

$$\Theta_n = d - 2n\left(1 + \frac{(n-1)(n-2)}{2N_f - 1}\right)$$

- for $d = 3$ only $\Theta_{n=1} = 1$ is relevant.
- $d = 2$ only Gaussian fixed point
  $d = 4$ non-Gaussian FP$\rightarrow \infty$ for $N_f = \infty$.

Fixed point potential

$N_f = 2$ fixed point potential



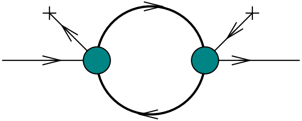
- exact asymptotics can be obtained: $U_k \sim I^{\frac{d}{2(d-1)}} \to I^{3/4} \quad (d = 3)$
- for finite I values: nth order potential & Padé resummation
- Physical regime $x > 0 \quad \Rightarrow \quad U < 0$ as we expected.
- physical point is at $\Psi = 0$.

Wave function renormalization

Formula for determining the wave function renormalization

$$\partial_k Z_k \frac{\delta(0)}{(2\pi)^d} = \frac{1}{N_f}\frac{d}{dq^2}\left\{-i\,\mathrm{Tr}\,\not{q}\overrightarrow{\partial}_{\bar\psi(-q)}\partial_k\Gamma_k\overleftarrow{\partial}_{\psi(q)}\right\}$$

Diagrammatically we have:



- proportional to the fermionic background $\bar\psi\psi$
- at the physical point $\bar\psi\psi = 0$
- no IR divergences occur

Result

$$Z = 1$$

Outlines

Conclusions

fermionic local potential approximation

- sensible: points $\rightarrow$ patches of resolution scale size
- feasible: explicit formulae, without series
- required for consistency: appear also in bosonic representation

Gross-Neveu model

- boson-like contribution in the scaling equations
- Gaussian & one nontrivial fixed point
- scaling exponents agree with the bosonized results for $N_f = \infty$
- fixed point potential (Padé resummation)
- wave function renormalization: $Z = 1$

Backup slides

Fermionic vs. bosonic potential

Compare the fermionic LPA' with the bosonized potential version.
Simplest case with just one scalar field:

$$\Gamma_k[\Psi] = \int d^d x \, \left[Z_k \bar{\psi} \partial\!\!\!/ \psi + U_k(\Psi) \right]$$

$$\Gamma_k^{aux}[\Psi, \sigma] = \int d^d x \, \left[Z_k \bar{\psi} \partial\!\!\!/ \psi + U_{k,aux}(\sigma) + \sigma \bar{\psi} \psi \right]$$

Constant background, use EoM: $\bar{\psi}\psi = -U'_{k,aux}(\sigma)$ to arrive at:

$$U_k(\Psi) = U_{k,aux}(\sigma) - \sigma U'_{k,aux}(\sigma)$$

Legendre transformation.

$\Rightarrow$ if asymptotically $U_{k,aux}$ increases faster than σ, then the $U_{k,aux} < \sigma U'_{k,aux}$, so asymptotically $U_k(\Psi) < 0$

inverse asymptotic behaviour

with the convention $\bar{\psi} \to i\bar{\psi}$ we would get positive potential

Functional Renormalization Group (FRG)

Basic idea: gradually eliminate degrees of freedom

Regulated effective action: for scalar fields, with regulator $R_k(p)$

$$Z_k[J] = \int \mathcal{D}\varphi e^{-S_E[\varphi] - \int_p \varphi^*(p) R_k(p) \varphi(p) - \int_p J^*(p)\varphi(p)} \quad \Rightarrow \quad \Gamma_k[\Phi]$$

- $R_{k=0} = 0 \quad \Rightarrow \quad$ full effective action for $k = 0$
- $R_{k=\Lambda}$ large $\quad \Rightarrow \quad$ fluctuations suppressed, classical action
- heuristically Γ_k effective action at scale k.

Technique: differentiate wrt. to k: $\quad \Rightarrow \quad$ **Wetterich equation**:
exact evolution equation for the effective action:

$$\partial_k \Gamma_k = \frac{1}{2} \mathrm{Tr}\, \partial_k R_k \left[\frac{\delta^2 \Gamma_k}{\delta \Phi^2} + R_k\right]^{-1} = \frac{1}{2} \hat{\partial}_k \,\mathrm{Tr} \log \left[\frac{\delta^2 \Gamma_k}{\delta \Phi^2} + R_k\right]$$

$\hat{\partial}_k$ acts only on k in R_k

Approximations

- Wetterich equation in an operator basis yields differential equations for the coefficients
- We need an Ansatz to treat the Wetterich equation: most popular are the Local Potential Approximation (LPA) and LPA with wave function renormalization (LPA′)

$$\Gamma_k[\Phi] = \int d^d x \left[\frac{Z}{2}(\partial_\mu \Phi)^2 + U_k(\Phi) \right]$$

- Wetterich equation $\Rightarrow$ partial diff. eq. for U; eg. for Φ^4 model

$$\partial_k U_k = \frac{Q_d k^{d+1}}{k^2 + U''}$$

  - $R_k(p) = \Theta(k-p)(k^2-p^2)$ Litim's optimized regulator
  - $Q_d^{-1} = (4\pi)^{d/2}\Gamma(\frac{d}{2}+1)$.

Understanding fermionic LPA

In formulae

- assume that $\Gamma_k^{(n>2)}$ localized within (or sufficiently small outside) L resolution (compositeness) scale

$$\Gamma_k^{(n)}(x_1,\dots,x_n) \neq 0 \text{ only for } |x_i - x_j| < L \text{ (ie. } x_i \in \Delta V_x)$$

- assume the most important configurations for the k-evolution are slowly varying on this scale, ($L\partial\Psi \approx 0$)

The n-th term

$$\int_{x_i\in\Delta V_x} \prod_i dx_i\, \Gamma^{(n)}(x_1,\dots,x_n)\, \bar\psi(x_1)\psi(x_2)\dots\bar\psi(x_{n-1})\psi(x_n)$$

use average value

for correlators $x_{2i} \approx x_{2i-1}$, and

$$\int_{x_i\in\Delta V_x} dx_i \bar\psi(x_i)\psi(x_i) \to \Delta V_x \bar\psi(x)\psi(x)$$

$\Rightarrow \int dx U^{(n)}(x)(\bar\psi(x)\psi(x))^{n/2}$ $\rightarrow$ just for notation!

Computation overview

General structure of the expressions: $\Gamma^{(2)} \sim G_0^{-1} - \#\Psi \otimes \Psi^T$

$\Rightarrow$ Inverse, tracelog can be computed

- no flavour mixing $\Rightarrow$ use background $\psi = (\zeta, \ldots, \zeta)$
- use static background

The most complicated expressions:

$$\mathrm{Tr}\log \Gamma_{\bar\psi\psi} = \mathrm{Tr}\log G_0^{-1} - \int_q \log\left(1 + \tilde U N_f(\zeta^T G_0 \zeta)\right)$$

$$\mathrm{Tr}\log M = \int_q \log\left(1 - \frac{\tilde U^2 N_f(\zeta^T G_0\zeta)(\zeta^T \hat G_0 \zeta)}{(1+\tilde U N_f(\zeta^T G_0\zeta))(1+\tilde U N_f(\zeta^T \hat G_0\zeta))}\right)$$

where

$$\tilde U = 2U' + 4IU'', \quad G_0^{-1} = Z\not{q}(1 + r_k(q)) + m$$

$$\hat f(q) = f(-q), \quad m = 2U'\bar\psi\psi$$

Cancellations! …also in more complicated setup

Fixed point analysis

Rescaling:

$$x = \frac{k^{2(1-d-\eta)} I}{(4Q_d N_f)^2} \quad y = \frac{k^{-d} U(I)}{4Q_d N_f} \quad t = \log k$$

we find

$$\partial_t y = -dy + (2d-1)y - \frac{1 - 1/(4N_f)}{1 + 4xy'^2} + \frac{1}{4N_f} \frac{1}{1 + 12xy'^2 + 16x^2 y' y''}$$

For fixed points: $\partial_t y = 0$

polynomial Ansatz $y_*(x) = \sum_n \frac{1}{n} \ell_{*n} x^n$

This yields:

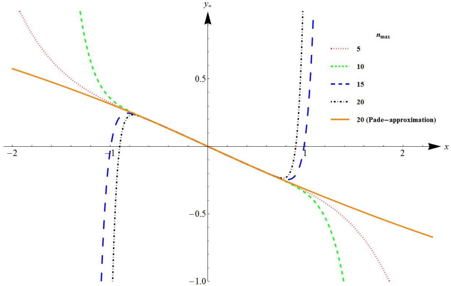
$$0 = (d-2)\ell_{*1} + \left(1 - \frac{1}{2N_f}\right) 4\ell_{*1}^2$$

$$0 = \left[-\frac{d}{n} + 2(d-1) + 8\ell_{*1}\left(1 - \frac{n}{2N_f}\right)\right] \ell_{n*} + \mathcal{F}(\ell_{*i<n})$$

Solvable: $\ell_{*1} \to \ell_{*2} \to \cdots \to \ell_{*n} \to \ldots.$

Fixed point potential

- exact asymptotics can be obtained: $U_k \sim I^{\frac{d}{2(d-1)}}$
- for finite I values: nth order potential & Padé resummation
- Physical regime $x > 0$ $\Rightarrow$ $U < 0$ as we expected.
- physical point is at $\Psi = 0$.



$N_f = 2$ potentials

Resummation:

$$y_*(x) = (1+x^2)^{\frac{d}{4(d-1)}} \lim_{N\to\infty} \mathrm{Pade}_N^N \left[\frac{\sum_{n=1}^{2N} \frac{1}{n} \ell_{*n} x^n}{(1+x^2)^{\frac{d}{4(d-1)}}} \right],$$

Pade_N^N: resum polynomials with degree $2N$ to ratio of polynomials with degree N.

Auxiliary bosonic fields

- Fermions are the fundamental degrees of freedom
  $\Rightarrow \quad \Gamma[\Psi]$ describes physics
- Introduce bosonic auxiliary variables for easier treatment
  $\Rightarrow \quad \bar{\Gamma}[\Psi, \Phi]$
- Condition to describe the same physics:

$$\left.\frac{\delta\Gamma}{\delta\Phi}\right|_{\Phi_{phys}} = 0 \quad \Rightarrow \quad \bar{\Gamma}[\Psi, \Phi_{phys}[\Psi]] = \Gamma[\Psi]$$

- equivalence class of $\bar{\Gamma}$ bosonic theories, which describe the same fermionic physics!
- representant in the LPA' approximation: Yukawa-type model

$$\Gamma[\Psi, \Phi] = \Gamma_{kin}[\Psi] + \Gamma_{kin}[\Phi] + \int d^d x \left[\bar{U}(\Phi) + h\Phi_i \bar{\psi} L^{(i)} \psi\right]$$

$L^{(i)}$ are flavour and Dirac-matrices

Evolution equation

- After FRG step the Yukawa form is not preserved:

$$\partial_k U = \frac{1}{2}\hat{\partial}_k \,\mathrm{Tr}\log\left[\Gamma_k^{(2)}[\Psi,\Phi] + R_k\right] \equiv \mathcal{R}(\Psi,\Phi),$$

  depends also on Ψ!
- technique:

$$\Gamma_k^{(repr)}[\Psi,\Phi] \quad \overset{FRG}{\Longrightarrow} \quad \Gamma_{k-dk}[\Psi,\Phi] \quad \overset{equiv.rel.}{\Longrightarrow} \quad \Gamma_{k-dk}^{(repr)}[\Psi,\Phi].$$

- result for GN-Yukawa theory (after rescaling)

$$\partial_t u + du + \frac{2-\eta-d}{2}\varphi u' = Q_d\left[\frac{-5}{1+(h\varphi)^2} + \frac{2+u''+(h\varphi)^2}{(1+(h\varphi)^2)(1+u'')+2h^2\varphi u'}\right].$$

  bosonic& fermionic modes