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Emergence of Power Law
in Statistical Hadronization

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Theoretical equation of state: $S(E)$

Product of the spreads of energy and temperature

$$\Delta E \cdot \Delta\beta = 1 \tag{1}$$

Connection to the (absolute) temperature:
$C = dE/dT$ heat capacity, $\beta = 1/T$

$$|C|\Delta T \cdot \frac{\Delta T}{T^2} = 1 \tag{2}$$

The relative spread in temperature is the one over square root of the heat capacity!

$$\frac{\Delta T}{T} = \frac{\Delta\beta}{\beta} = \frac{1}{\sqrt{|C|}} \tag{3}$$

The heat capacity C is proportional to the size of the heat bath – mostly.

Gauss distributed reciprocial temperature, β

$$w(\beta) = \frac{1}{\sqrt{2\pi\sigma}} \, e^{-\frac{(\beta - 1/T_0)^2}{2\sigma^2}} \tag{4}$$

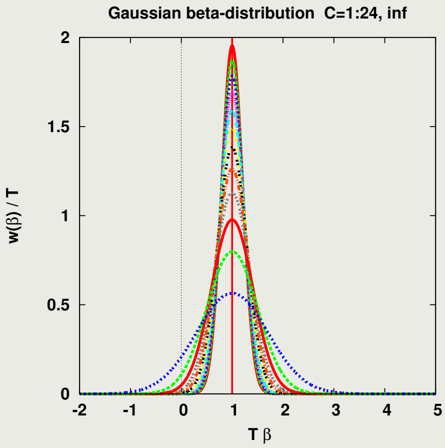
Average

$$\langle \beta \rangle = \frac{1}{T_0}$$

Spread (square root of variance)

$$\Delta\beta = \sigma = \frac{1}{T_0 \sqrt{|C|}}$$

Gaussian Fluctuations; Figure



Superstatistics: single particle energy distribution

Canonical distribution in additive thermodynamics:

$$p_i = p(E_i) = e^{\beta(\mu - E_i)}. \tag{5}$$

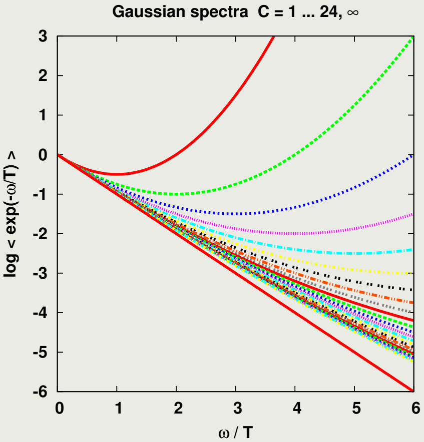
If β fluctuates according to Gauss, then the exponential weight factor averages to the **characteristic function**

$$\left\langle e^{-\beta\omega} \right\rangle = e^{-\omega/T_0} \, e^{\sigma^2\omega^2/2}. \tag{6}$$

Turning point: the largest single particle energy, where this can make a sense...

$$E_i^{\text{max}} - \mu = \omega^{\text{max}} = \frac{1}{\sigma^2 T_0} = |C| \, T_0. \tag{7}$$

Figure on Thermal Spectra with Gaussian β-distribution



Deficiences of the Gaussian Picture

1. There is a finite probability, $w(\beta) > 0$, for $\beta < 0$
2. $\langle e^{-\beta\omega} \rangle$ does not diminish for large ω (this cannot be a canonical spectrum)

Ideal Gas: Thermodynamics

EoS

$$p = nT, \qquad e = \frac{1}{3}p.$$

Heat Capacity:

$$E = \frac{1}{3}pV = \frac{1}{3}NT; \qquad C = \frac{dE}{dT} = \frac{1}{3}N.$$

fix N: $C(T)$ constant; $C(S)$ constant; We have to solve:

$$\frac{dT}{dE} = \left(\frac{1}{S'}\right)' = -\frac{S''}{(S')^2} = \frac{1}{C}. \tag{8}$$

Ideal Gas: Thermodynamics

Constant Heat Capacity $$-\frac{S''}{(S')^2} = \frac{1}{C}.$$

Integrals: temperature and entropy

$$T = \frac{1}{S'} = \frac{E}{C} + T_0, \qquad S = C \ln\left(1 + \frac{E}{CT_0}\right) + S_0.$$

Mutual info based probability (phase volume product)

$$\mathfrak{P}(E_1) = e^{S_1+S_2-S_{12}} \propto \left(1 + \frac{E_1}{C_1 T_0}\right)^{C_1} \left(1 + \frac{E - E_1}{C_2 T_0}\right)^{C_2}$$

Ideal Gas: temperature distribution

It is an **Euler-Beta distribution**

$$\mathfrak{P}(T_1) \propto T_1^{C_1} \left(T_* - \frac{C_1}{C_2}(T_1 - T_*) \right)^{C_2}$$

in the scaling variable: $x = C_1 T_1 / (C_1 + C_2) T_* = C_* T_1 / C_2 T_*$

$$\mathfrak{B}(x) = \frac{\Gamma(C_1 + C_2 + 2)}{\Gamma(C_1 + 1)\Gamma(C_2 + 1)} x^{C_1} (1 - x)^{C_2}$$

Beta distribution in x, binomial in C_1 at fix $C_1 + C_2$, NBD at fix C_2.

Ideal Gas: limits

- Huge reservoir ($C_2 \to \infty$: with $t = C_1 T_1 / T_*$)
  Euler-Gamma

$$\lim_{C_2 \to \infty} \mathfrak{B}(x)\, dx = \frac{1}{\Gamma(C_1+1)}\, t^{C_1}\, e^{-t}\, dt.$$

Euler fitted to Gaussian Uncertainty

Average: $\langle \beta \rangle = \frac{v}{a} = \frac{1}{T}$, Spread: $\frac{\Delta\beta}{\langle \beta \rangle} = \frac{1}{\sqrt{v}} = \frac{\Delta T}{T} = \frac{1}{\sqrt{|C|}}$

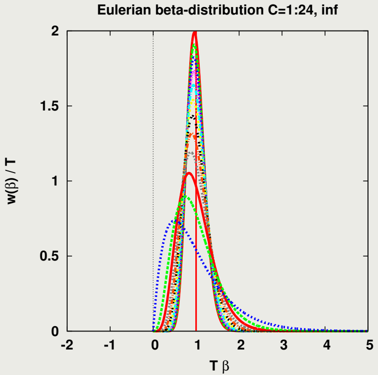
The corresponding Euler-Gamma distribution for $\beta = 1/T_*$:

$$w(\beta) = \frac{(|C|\,T)^{|C|}}{\Gamma(|C|)}\,\beta^{|C|-1}\,e^{-|C|\,T\beta}. \tag{9}$$

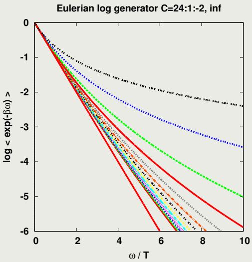
Characteristic function = **spectrum**

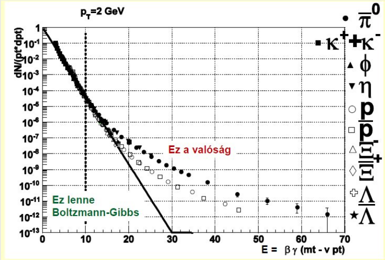
$$\left\langle e^{-\beta\omega} \right\rangle = \left(1 + \frac{\omega}{|C|\,T}\right)^{-|C|} \xrightarrow[|C|\to\infty]{} e^{-\omega/T}. \tag{10}$$

Plot Eulerian Fluctuations



Plot Eulerian Spectra and RHIC results as blast wave





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Finite Reservoirs

- Avogadro number (atoms in classical matter) $\sim 10^{24}$
- Neurons in human brain $\sim 10^{12}$
- Internet users in 2014 $\sim 10^{9}$
- New particles from heavy ion collisons $\sim 10^{3}$
- From elementary high energy collisions (pp) $\sim 10^{1}$

General expectation:

smaller size $\rightarrow$ larger *relative* fluctuations.

Ideal Gas: microcanonical statistical weight

The one-particle energy, ω, out of total energy, E, is distributed in a one-dimensional relativistic jet according to a statistical weight factor which depends on the number of particles in the reservoir, n:

$$P_1(\omega) = \frac{\Omega_1(\omega)\,\Omega_n(E-\omega)}{\Omega_{n+1}(E)} = \rho(\omega) \cdot \frac{(E-\omega)^n}{E^n} \qquad (11)$$

HEP Superstatistics: E fix, n has a distribution (based on the physical model of the reservoir and on the event by event detection of the spectra).

Phase Volume Ratio is a $q < 1$ Tsallis–Pareto

Thermodynamic limit:

$$\lim_{n\to\infty} \lim_{E\to nT} \left(1 - \frac{\omega}{E}\right)^n = \mathrm{e}^{-\omega/T}. \tag{12}$$

Compare with Tsallis distribution:

$$\left(1 - \frac{\omega}{E}\right)^n = \left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}, \tag{13}$$

If and only if

$$T = \frac{E}{n}, \qquad q = 1 - \frac{1}{n}. \tag{14}$$

Ideal Reservoir: (Negative) binomial n-distribution

n particles among k cells: bosons $\binom{n+k}{n}$ fermions $\binom{k}{n}$

A subspace (n,k) out of (N,K)

Limit: $K \to \infty$, $N \to \infty$; average occupancy $f = N/K$ is fixed.

$$B_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{n+k}{n}\binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n (1+f)^{-n-k-1}. \tag{15}$$

$$F_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}} = \binom{k}{n} f^n (1-f)^{k-n}. \tag{16}$$

Norm and Pascal triangle

Binomial expansion:

$$(a+b)^k = \sum_{n=0}^{\infty} \binom{k}{n} a^n b^{k-n} \tag{17}$$

Replace k by $-k-1$ and a by $-a$, noting that

$$\binom{-k-1}{n} = \frac{(-k-1)(-k-2)\ldots(-k-n)}{n!} = (-1)^n \frac{(k+1)(k+2)\ldots(k+n)}{n!} = (-1)^n \binom{n+k}{n}.$$

we arrive at

$$(b-a)^{-k-1} = \sum_{n=0}^{\infty} \binom{n+k}{n} a^n b^{-n-k-1} \tag{18}$$

BD recursion: Pascal Triangle

$$F_{n,k} = f\,F_{n-1,k-1} + (1-f)\,F_{n,k-1} \tag{19}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

NBD recursion: Tilted Pascal Triangle

$$B_{n,k} = \frac{f}{1+f} B_{n-1,k} + \frac{1}{1+f} B_{n,k-1} \tag{20}$$

1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1				

Bosonic reservoir

Reservoir in hep: E is fixed, n fluctuates according to NBD.

$$\sum_{n=0}^{\infty}\left(1-\frac{\omega}{E}\right)^{n}B_{n,k}(f)=\left[(1+f)-f\left(1-\frac{\omega}{E}\right)\right]^{-k-1}=\left(1+f\frac{\omega}{E}\right)^{-k-1} \tag{21}$$

Note that $\langle n\rangle=(k+1)f$ for NBD. Then with $T=E/\langle n\rangle$ and $q-1=\frac{1}{k+1}$ we get

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q>1$ Tsallis-Pareto distribution.

Fermionic reservoir

E is fixed, n is distributed according to BD:

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n F_{n,k}(f) = \left[(1-f) + f\left(1 - \frac{\omega}{E}\right)\right]^k = \left(1 - f\frac{\omega}{E}\right)^k \tag{22}$$

Note that $\langle n \rangle = kf$ for BD. Then with $T = E/\langle n \rangle$ and $q - 1 = -\frac{1}{k}$ we get

$$\left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q < 1$ Tsallis-Pareto distribution.

Boltzmann limit

In the $k \gg n$ limit (low occupancy in phase space)

$$\binom{n+k}{n} f^n (1+f)^{-n-k-1} \longrightarrow \frac{k^n}{n!} \left(\frac{f}{1+f} \right)^n \ldots$$
$$\binom{k}{n} f^n (1-f)^{k-n} \longrightarrow \frac{k^n}{n!} \left(\frac{f}{1-f} \right)^n \ldots \quad (23)$$

After normalization this is the **Poisson** distribution:

$$\Pi_n = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \quad \text{with} \quad \langle n \rangle = k \frac{f}{1 \pm f} \quad (24)$$

The result is **exactly** the Boltzmann-Gibbs weight factor:

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E} \right)^n \Pi_n(\langle n \rangle) = e^{-\omega/T}. \quad (25)$$

Summary of reservoir fluctuation models

$$\left\langle \left(1-\frac{\omega}{E}\right)^n \right\rangle_{\textbf{Bernoulli}} = \textbf{Tsallis}(q<1)$$

$$\left\langle \left(1-\frac{\omega}{E}\right)^n \right\rangle_{\textbf{Possion}} = \textbf{Boltzmann}(q=1)$$

$$\left\langle \left(1-\frac{\omega}{E}\right)^n \right\rangle_{\textbf{NBD}} = \textbf{Tsallis}(q>1) \qquad (26)$$

In all the three above cases

$$T = \frac{E}{\langle n \rangle}, \quad \text{and} \quad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2} \qquad (27)$$

Ideal gas with general *n*-fluctuations

Canonical approach: expansion for small $\omega \ll E$.
Tsallis-Pareto distribution as an approximation:

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \ldots \qquad (28)$$

Ideal reservoir phase space up to the subleading canonical limit:

$$\left\langle \left(1-\frac{\omega}{E}\right)^n \right\rangle = 1 - \langle n \rangle \frac{\omega}{E} + \langle n(n-1) \rangle \frac{\omega^2}{2E^2} - \ldots \qquad (29)$$

To subleading in $\omega \ll E$

$$\mathbf{T} = \frac{\mathbf{E}}{\langle \mathbf{n} \rangle}, \qquad \mathbf{q} = \frac{\langle \mathbf{n}(\mathbf{n}-1) \rangle}{\langle \mathbf{n} \rangle^2} = 1 - \frac{1}{\langle n \rangle} + \frac{\Delta n^2}{\langle n \rangle^2}. \qquad (30)$$

General system with general reservoir fluctuations

Canonical approach: expansion for small $\omega \ll E$.

$$\left\langle \frac{\Omega_n(E-\omega)}{\Omega_n(E)} \right\rangle = \left\langle e^{S(E-\omega)-S(E)} \right\rangle = \left\langle e^{-\omega S'(E)+\omega^2 S''(E)/2-\dots} \right\rangle$$

$$= 1 - \omega \langle S'(E) \rangle + \frac{\omega^2}{2} \left\langle S'(E)^2 + S''(E) \right\rangle - \dots \tag{31}$$

Compare with expansion of Tsallis

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \dots \tag{32}$$

Interpret the parameters

$$\frac{1}{T} = \langle S'(E) \rangle, \qquad q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2} \tag{33}$$

$\langle S''(E) \rangle = -1/CT^2$ expressed via the heat capacity of the reservoir, $1/C = dT/dE$

Understanding the parameter q

in terms fluctuations

Opposite sign contributions from $\langle S'^2 \rangle - \langle S' \rangle^2$ and from $\langle S'' \rangle$.

In all cases approximately

$$q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2}.$$

- $q > 1$ and $q < 1$ are both possible
- for any relative variance $\Delta T / T = 1/\sqrt{C}$ it is exactly $q = 1$
- for $nT = E/\text{dim} = \text{const}$ it is $\Delta T / \langle T \rangle = \Delta n / \langle n \rangle$.
- for ideal gas and n distributed as NBD or BD, the Tsallis form is exact

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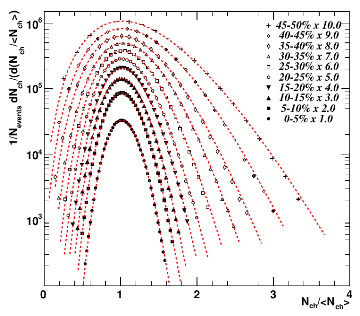
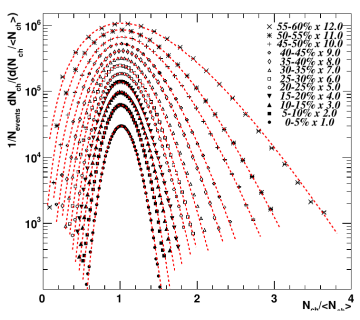
Experimental NBD distributions PHENIX PRC 78 (2008) 044902

Au + Au collisons at $\sqrt{s_{NN}} = 62$ (left) and 200 GeV (right). Total charged multiplicities.





$$k \approx 10 - 20 \quad \to \quad q \approx 1.05 - 1.10.$$

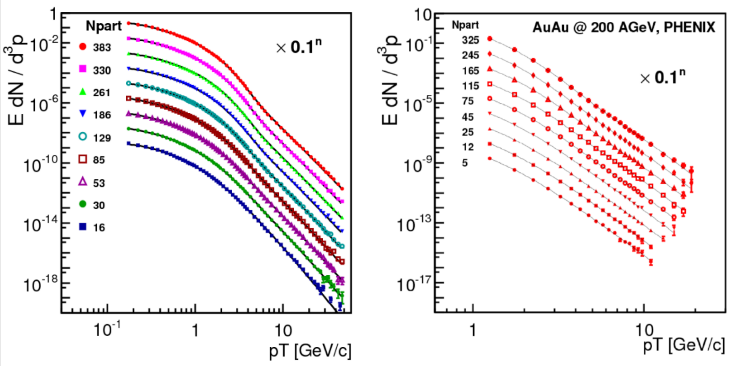
Statistical vs QCD power-law

The experimental fact for hadrons is **NBD**!

- QCD power-law: size-independent power $(k+1) > 4$
- statistical power: $(k+1) = \langle n \rangle / f \propto$ reservoir size
- consider data fits: $k+1$ powers vs $N_{\rm part}$
- soft and hard power-laws should differ for large $N_{\rm part}$

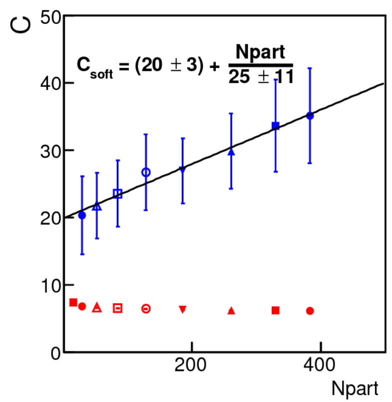
Soft and Hard Tsallis fits:

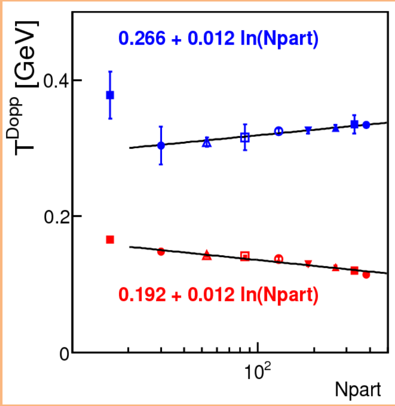
ALICE PLB 720 (2013) 52; PHENIX PRL 101 (2008) 232301



the knick is around $p_T \approx 4 - 5$ GeV.

Hard and Soft Trends with N_{part} arxiv: 1405.3963

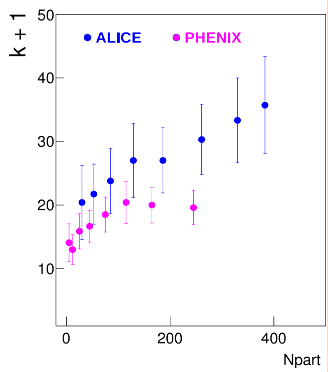




$C = k + 1$ powers of the power law and fitted T parameters (ALICE).

Soft Powers vs N_{part} arxiv: 1405.3963

Only the soft (*"statistical"*) branches for **PHENIX** and **ALICE:**



Summary

- The (measured and calculated) temperature surely fluctuates, but is **not Gaussian**.
- Ideal Gas prefers the **Beta or Gamma** distribution for T estimators.
- **NBD** particle number fluctuations generate ***exact Tsallis*** distribution with $q > 1$.