On η-η' complex and its temperature dependence*

D. Klabučar[a] (speaker), D. Horvatić[a], D. Kekez[c]

$\star$ 14$^{\rm th}$ Zimány winter school on heavy ion physics

Budapest, Hungary, December 1. – 5. 2014.

[a] Physics Department, University of Zagreb, Croatia

[b] Rudjer Bošković Institute, Zagreb, Croatia

Introducing η-η' complex into pseudoscalar nonet

- Pseudoscalar mesons of light quarks $q = u, d, s$ are (almost) Goldstone bosons of DChSB, so for $m_{u,d,s} \to 0$ also vanishing meson masses^2 $M^2_{u\bar{d}} = M^2_{\pi^+}, M^2_{u\bar{s}} = M^2_K, ..., \hat{M}^2_{NA} = \mathrm{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}})$
QCD chiral behavior reproduced correctly by Dyson-Schwinger-Bethe-Salpeter approach (DS) – except anomalously heavy η' !
- $|u\bar{d}\rangle = |\pi^+\rangle, |u\bar{s}\rangle = |K^+\rangle$, ... but $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ do not correspond to any physical particles (at $T = 0$ at least!), although in the isospin limit (adopted from now on) $M_{u\bar{u}} = M_{d\bar{d}} = M_{u\bar{d}} = M_\pi$. $I =$ good Q.no. $\Rightarrow$ recouple into "more physical" $I_3 = 0$ octet-singlet basis

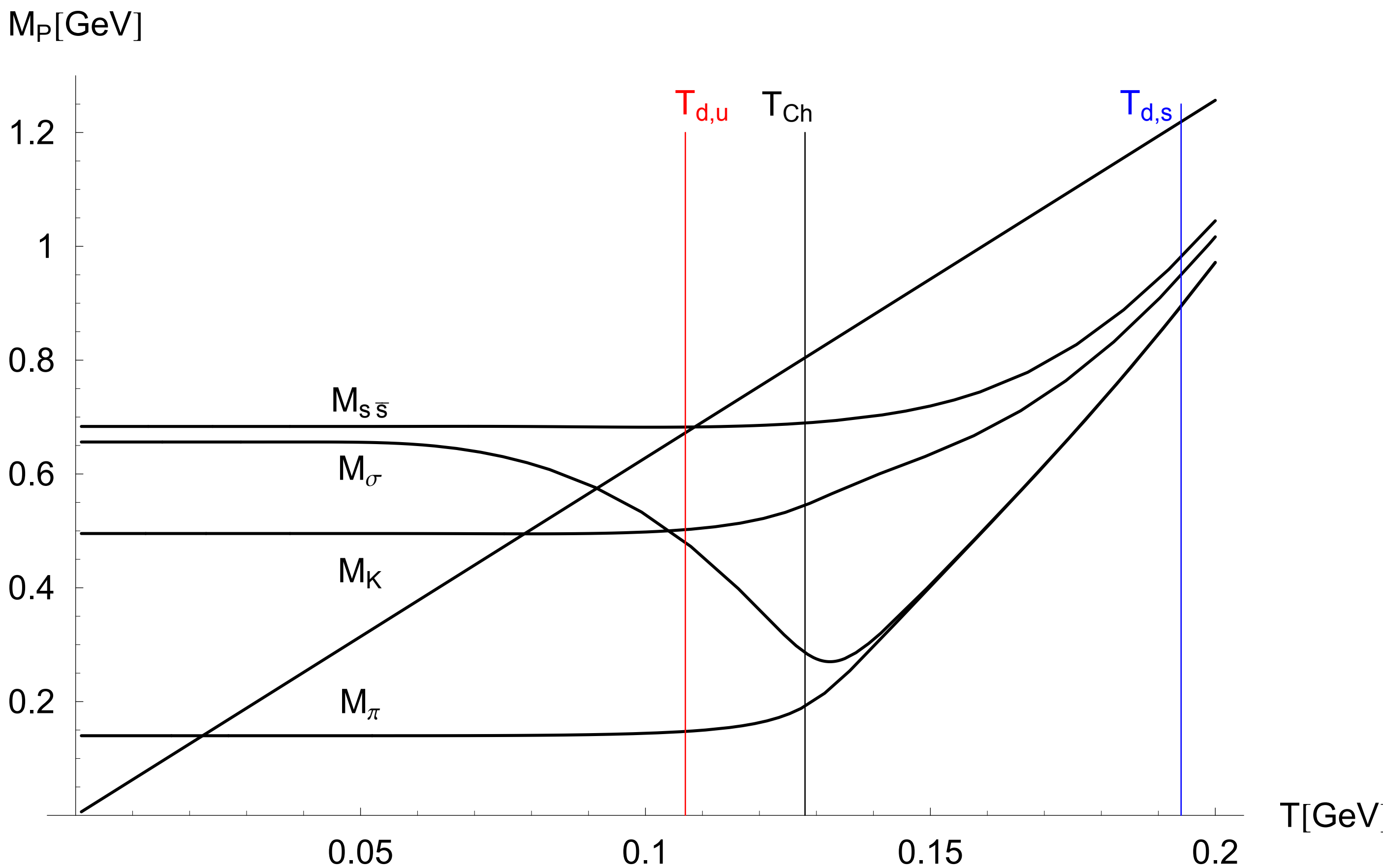
$$I = 1 \qquad |\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \,,$$

$$\text{but } I = 0 \qquad |\eta_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) \approx |\eta\rangle \quad \text{mixes with}$$

$$I = 0 \qquad |\eta_0\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \approx |\eta'\rangle \quad \text{seems too heavy for}$$

Except η-η', pseudoscalars qualitatively understood at both $T = 0$ and $T > 0$

- e.g., a simple DS model (so-called 'separable') yields:



- 'Deconfinement' $T_{d,q}$ from S_q pole - very different $T_{d,u}$, $T_{d,s}$ ... can be cured/synchronized with $T_{\rm Ch} (= T_{cri})$ by Polyakov loop
- But what about η and η' both at $T = 0$ and $T > 0$?

Physical η and η' must have a diagonal mass matrix

- the "non-anomalous" (chiral-limit-vanishing!) part of the mass-squared matrix of π^0 and η's is (in π^0-η_8-η_0 basis)

$$\hat{M}^2_{NA} = \begin{pmatrix} M_\pi^2 & 0 & 0 \\ 0 & M_{88}^2 & M_{80}^2 \\ 0 & M_{08}^2 & M_{00}^2 \end{pmatrix} \overset{\text{diagonalization}}{\underset{\text{U}_A(1)\text{ problem}}{\Longrightarrow}} \begin{pmatrix} M_\pi^2 & 0 & 0 \\ 0 & M_\pi^2 & 0 \\ 0 & 0 & M_{s\bar{s}}^2 \end{pmatrix}$$

$$M_{88}^2 \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_8 \rangle = \frac{2}{3}(M_{s\bar{s}}^2 + \frac{1}{2} M_\pi^2), \qquad M_{00}^2 \equiv \langle \eta_0 | \hat{M}^2_{NA} | \eta_0 \rangle = \frac{2}{3}(\frac{1}{2} M_{s\bar{s}}^2 + M_\pi^2),$$

$$M_{80}^2 \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_0 \rangle = M_{08}^2 = \frac{\sqrt{2}}{3}(M_\pi^2 - M_{s\bar{s}}^2)$$

- What reproduces M_π & M_K cannot also $M_\eta = 548$ & $M_{\eta'} = 958$ MeV!

- $\hat{M}^2_{NA}$ not enough! To avoid the U$_A$(1) problem, one must break the U$_A$(1) symmetry (as it is destroyed by the gluon anomaly) at least at the level of the masses.

Why $\eta_0 \approx \eta'$ has an anomalous piece of mass:

$U_A(1)$ symmetry is broken by nonabelian ("gluon") axial anomaly: even in the chiral limit (ChLim, where $m_q \to 0$),

$$\partial_\alpha \bar\psi(x)\gamma^\alpha\gamma_5 \frac{\lambda^0}{2}\psi(x) \propto F^a(x)\cdot \widetilde{F}^a(x) \equiv \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \neq 0\,.$$

This breaks the $U_A(1)$ symmetry of QCD and precludes the 9th Goldstone pseudoscalar meson $\Rightarrow$ very massive η': even in ChLim, where $m_\pi, m_K, m_\eta \to 0$, still ('ChLim WVR')

$$0 \neq \Delta M^2_{\eta_0} = \Delta M^2_{\eta'} = \frac{(A = \mathrm{qty.dim.mass})^4}{(\text{“}f_{\eta'}\text{”})^2} = \frac{6\,\chi_{\mathrm{YM}}}{f_\pi^2} + O(\frac{1}{N_c})$$

$$\mathrm{Out\ of\ ChLim:}\quad {M_{\eta'}}^2 + {M_\eta}^2 - 2\,{M_K}^2 = \frac{2N_f}{f_\pi^2}\,\chi_{\mathrm{YM}} \quad \left(+O(\frac{1}{N_c})\right)$$

Anomalous part of η_0 mass: $\Delta M_{\eta_0}^2 = \chi_{\rm YM} \frac{2N_f}{f_\pi^2} + O(\frac{1}{N_c})$

QCD chiral behavior (reproduced by DS approach) of the non-anomalous parts of masses of light $q\bar{q}'$ pseudoscalars (i.e., all parts except ΔM_{η_0}) : $M_{q\bar{q}'}^2 = \text{const}\,(m_q + m_{q'}), \;\; (q, q' = u, d, s)$.

$\Rightarrow$ non-anomalous parts of the masses in WVR cancel:
$M_{\eta'}{}^2 + M_\eta{}^2 - 2\, M_K{}^2 \approx \Delta M_{\eta_0}{}^2$, approx. as in ChLim WVR

$$\chi = \int d^4x \, \langle 0|Q(x)Q(0)|0\rangle \; , \qquad Q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

- $Q(x)$ = topological charge density operator
- In WV rel., χ is the pure-glue, YM one, $\chi_{\rm YM} \leftrightarrow \chi_{quench}$.

Lattice: good $\chi_{\rm YM}$, subtleties with χ of light-flavor QCD [Bernard et al., JHEP 1206 (2012) 051] where $\chi = -\frac{\langle\bar{q}q\rangle_0}{\sum\limits_{q=u,d,s}\frac{1}{m_q}} + \mathcal{C}(\text{higher}\,\mathcal{O}\,\text{in}\,m)$.

Gluon anomaly is not accessible to ladder approximation

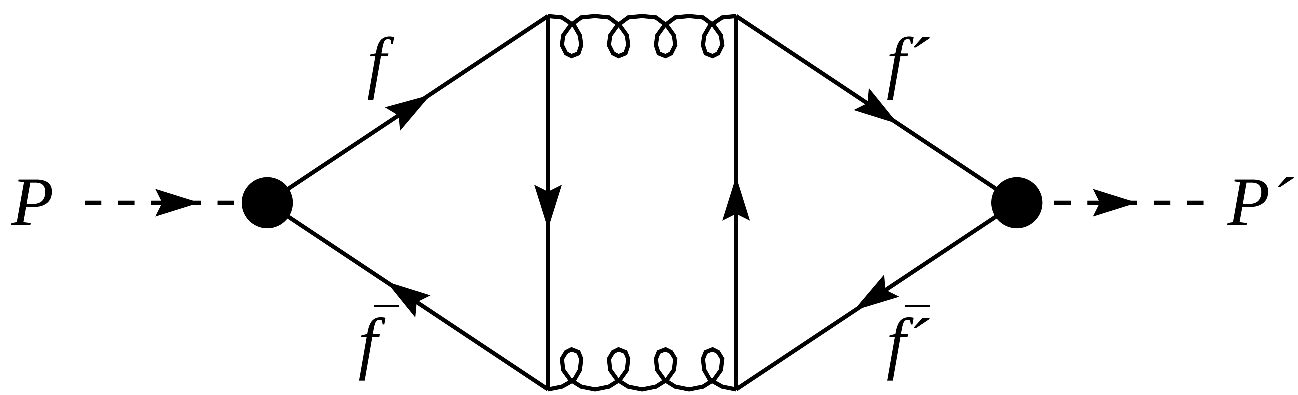
- All masses in $\hat{M}^2_{NA}$ are calculated in the ladder approx., which cannot include the gluon anomaly contributions.
- Large N_c: the gluon anomaly suppressed as $1/N_c$! $\to$ Include its effect just at the level of masses: break the $U_A(1)$ symmetry and avoid the $U_A(1)$ problem by shifting the η_0 (squared) mass by anomalous contribution 3β.
- complete mass matrix is then $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$ where

$$\hat{M}^2_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\beta \end{pmatrix} \quad \text{does not vanish in the chiral limit.}$$

$3\beta = \Delta M^2_{\eta_0}$ = the anomalous mass^2 of η_0 [in SU(3) limit incl. ChLim] is related to the YM topological susceptibility. Fixed by phenomenology or (here) taken from the lattice.

Transitions related to the $U_A(1)$ anomaly

- Transitions between hidden flavors $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ ($q, q' = u, d, s$)



- Diamond graph: **just the simplest example** of a transition $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ ($q, q' = u, d, s$), contributing to the anomalous masses in the η-η' complex, but not included in the interaction kernel in the ladder approximation.

Anomalous mass matrix in $q\bar{q}$ and octet-singlet bases

- we can also rewrite $\hat{M}_A^2$ in the $q\bar{q}$ basis $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$

$$\hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{breaking}]{\text{flavor}} \hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{pmatrix}$$

- We introduced the effects of the flavor breaking on the anomaly-induced transitions $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ ($q, q' = u, d, s$). $s\bar{s}$ transition suppression estimated by $X \approx f_\pi / f_{s\bar{s}}$.
- Then, $\hat{M}_A^2$ in the octet-singlet basis is modified to

$$\hat{M}_A^2 = \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(2-X-X^2) \\ 0 & \frac{\sqrt{2}}{3}(2-X-X^2) & \frac{1}{3}(2+X)^2 \end{pmatrix}$$

- $\to$ In the isospin limit, one can always restrict to 2×2 submatrix of etas $(I=0)$, as π^0 $(I=1)$ is decoupled then.

Anomalous mass matrix and mixing in *NS*–*S* basis

- nonstrange (*NS*) – strange (*S*) basis

$$|\eta_{NS}\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle \, ,$$

$$|\eta_S\rangle = |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle \, .$$

- the η–η' mass matrix in this basis is

$$\hat{M}^2 = \begin{pmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S\eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{pmatrix} = \begin{pmatrix} M^2_{u\bar{u}} + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} M^2_\eta & 0 \\ 0 & M^2_{\eta'} \end{pmatrix}$$

- *NS*–*S* mixing relations – states rotation diagonalizing $\hat{M}^2$:

$$|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle \, , \quad |\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle \, .$$

$$\theta = \phi - \arctan\sqrt{2}$$

Finally, fix anomalous contribution to η-η':

- Equal traces of diagonalized & non-diagnlz. $\hat{M}^2$ demand 1^{st} eqality in

$$\beta(2+X^2) = M_\eta^2 + M_{\eta'}^2 - 2M_K^2 = \frac{2N_f}{f_\pi^2}\chi_{\text{YM}} \quad (2^{\text{nd}}\text{equality} = \text{WV rel}$$

- requiring that the experimental trace $(M_\eta^2 + M_{\eta'}^2)_{exp}$ ≈ 1.22 GeV^2 be reproduced by the theoretical $\hat{M}^2$, yields $\beta_{\text{fit}} = \frac{1}{2+X^2}[(M_\eta^2 + M_{\eta'}^2)_{exp} - (M_{u\bar{u}}^2 + M_{s\bar{s}}^2)]$
- Or, get β from lattice χ_{YM} ! Then no free parameters!
- then, can calculate the NS-S mixing angle ϕ

$$\tan 2\phi = \frac{2\, M_{\eta_S \eta_{NS}}^2}{M_{\eta_S}^2 - M_{\eta_{NS}}^2} = \frac{2\sqrt{2}\beta X}{M_{\eta_S}^2 - M_{\eta_{NS}}^2} \qquad \text{and}$$

$$M_{\eta_{NS}}^2 = M_{u\bar{u}}^2 + 2\beta = M_\pi^2 + 2\beta, \quad M_{\eta_S}^2 = M_{s\bar{s}}^2 + \beta X^2 = M_{s\bar{s}}^2 + \beta\frac{f_\pi^2}{f_{s\bar{s}}^2}$$

Physical η, η' eigenmasses – of the two-level type:

- The diagonalization of the NS-S mass matrix then finally gives us the *calculated* η and η' masses:

$$
\begin{aligned}
M_\eta^2 &= \cos^2\phi\, M_{\eta_{NS}}^2 - M_{\eta_S \eta_{NS}}^2 \sin 2\phi + \sin^2\phi\, M_{\eta_S}^2 \quad (\text{note } M_{\eta_S\eta_{NS}}^2 = \sqrt{2}\beta X \\
M_{\eta'}^2 &= \sin^2\phi\, M_{\eta_{NS}}^2 + M_{\eta_S \eta_{NS}}^2 \sin 2\phi + \cos^2\phi\, M_{\eta_S}^2
\end{aligned}
$$

- Equivalently, secular determinant $\Rightarrow$ the eigenvalues of 2×2 matrix:

$$
\begin{aligned}
M_\eta^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 - \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 4\, M_{\eta_S\eta_{NS}}^4} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) - \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right] \\
M_{\eta'}^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 + \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 4\, M_{\eta_S\eta_{NS}}^4} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) + \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right]
\end{aligned}
$$

Separable model results on η and η' at $T = 0$

	$\beta_{\rm fit}$	$\beta_{\rm latt.}$	Exp.
θ	-12.22°	-13.92°	
M_η [MeV]	548.9	543.1	547.75
$M_{\eta'}$ [MeV]	958.5	932.5	957.78
X	0.772	0.772	
3β [GeV^2]	0.845	0.781	

- $X = f_\pi / f_{s\bar{s}}$ as well as the whole $\hat{M}^2_{NA}$ (consisting of M_π and $M_{s\bar{s}}$) are calculated model quantities.
- $\beta_{\rm latt.}$ was obtained from $\chi_{\rm YM}(T=0) = (175.7\ {\rm MeV})^4$
- But is an extension to high T possible, as there is a large mismatch of characteristic temperature scales of the pure-gauge YM ($T_c \sim 270$ MeV) vs. full QCD ($T_c \sim 160$ MeV) with quarks?
- Concretely in WVR, $\chi_{\rm YM}$ is more T-resistant than QCD quantities $M_{\eta,\eta',K}$ and f_π. Does WVR become unusable as T approaches the (pseudo-)critical temperatures of full QCD, such as $T \sim T_{\rm Ch}$?

Solution: another relation connecting YM and QCD

Early work by Di Vecchia & Veneziano ... Leutwyler & Smilga [Phys. Rev. D46 (1992) 5607] derived, up to $O(\frac{1}{N_c})$,

$$(\mathrm{at}\, T = 0), \qquad \chi_{\mathrm{YM}} = \frac{\chi}{1 + \chi \frac{N_f}{m \langle \bar{q}q \rangle_0}} \equiv \widetilde{\chi}$$

$\Rightarrow$ relates χ_{YM} to the full-QCD topological susceptibility χ, chiral condensate $\langle \bar{q}q \rangle_0$ and $m \equiv N_f \times$ the reduced mass. Presently $N_f = 3$, *i.e.*, $N_f/m = \sum_{q=u,d,s}(1/m_q)$.

- in the limit of very heavy quarks, $m_q, m \to \infty$, it confirms expectations that $\chi_{\mathrm{YM}} =$ value of topolog. susceptibility in *quenched* QCD, $\chi_{\mathrm{YM}} = \chi(m_q = \infty)$
- It shows $\chi \leq \min(-m \langle \bar{q}q \rangle_0 / N_f, \chi_{\mathrm{YM}})$

LS relation also holds in the oposite limit!

In the (presently pertinent!) regime of light quarks there is Di Vecchia-Veneziano result for small m_q:

$$\chi = -\frac{m\,\langle \bar{q}q\rangle_0}{N_f} + \mathcal{C}(m)\,,$$

- $\mathcal{C}(m)$ = small corrections of higher orders in small m_q, ... but $\mathcal{C}(m)$ should not be neglected, since $\mathcal{C}(m) = 0$ would imply that $\chi_{\rm YM} = \infty$.
- LS relation fixes the value of the correction at $T = 0$:

$$\frac{1}{\mathcal{C}(m)} = \frac{N_f}{m\,\langle \bar{q}q\rangle_0} - \chi_{\rm YM}(0)\left(\frac{N_f}{m\,\langle \bar{q}q\rangle_0}\right)^2 .$$

T-dependence of $\widetilde{\chi}$

- LS relation also must break down as T approaches the (pseudo-)critical temperatures of full QCD ($\sim T_{\rm Ch}$) since YM quantity $\chi_{\rm YM}$, is much more T-resistant than $\widetilde{\chi}$.
- $\widetilde{\chi}$ consists of the full-QCD quantities χ and $\langle\bar{q}q\rangle_0$, characterized by $T_{\rm Ch}$, just as $f_\pi(T)$.
- Thus, the troublesome mismatch in T-dependences of $f_\pi(T)$ and the pure-gauge $\chi_{\rm YM}(T)$ is expected to disappear if $\chi_{\rm YM}(T)$ is replaced by $\widetilde{\chi}(T)$, the T-extended RHS of LS relation
- The usual, successful zero-T WV relation is thereby retained, since $\chi_{\rm YM} = \widetilde{\chi}$ *at* $T = 0$.

T-dependence of χ and $\widetilde{\chi}$

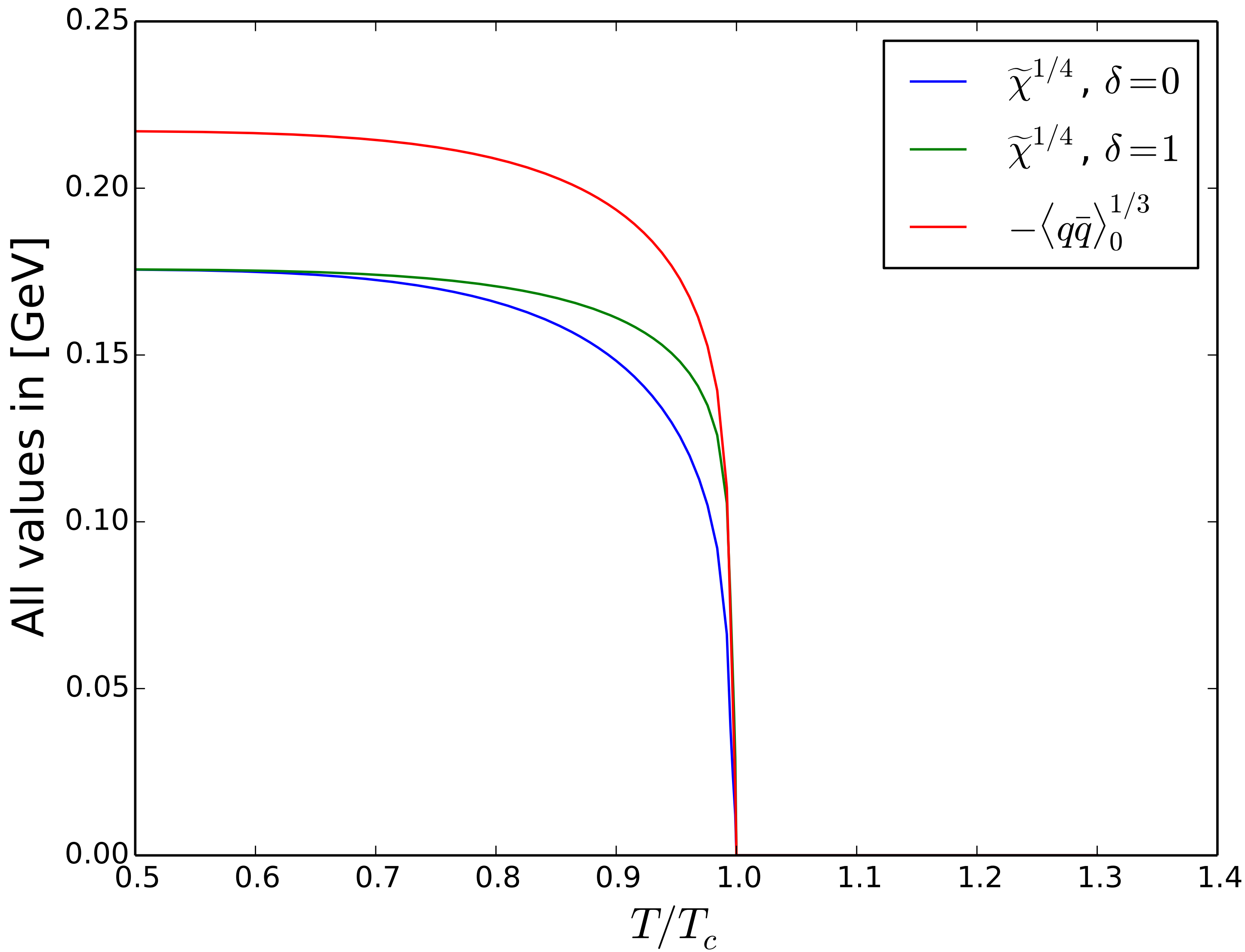
- Extending the light-quark full-QCD topol. susceptibility χ is somewhat uncertain, as there is no guidance from lattice [unlike for $\chi_{\rm YM}(T)$].
- The leading term in Di Vecchia-Veneziano relation $\propto \langle \bar{q}q \rangle_0(T)$ very plausibly, but for the correction term we have to explore a range of Ansätze, i.e.,

$$\chi(T) = -\frac{m \langle \bar{q}q \rangle_0(T)}{N_f} + \mathcal{C}(m) \left[\frac{\langle \bar{q}q \rangle_0(T)}{\langle \bar{q}q \rangle_0(T=0)} \right]^\delta , \quad (0 \leq \delta < 2).$$

Then, $\widetilde{\chi}(T) =$

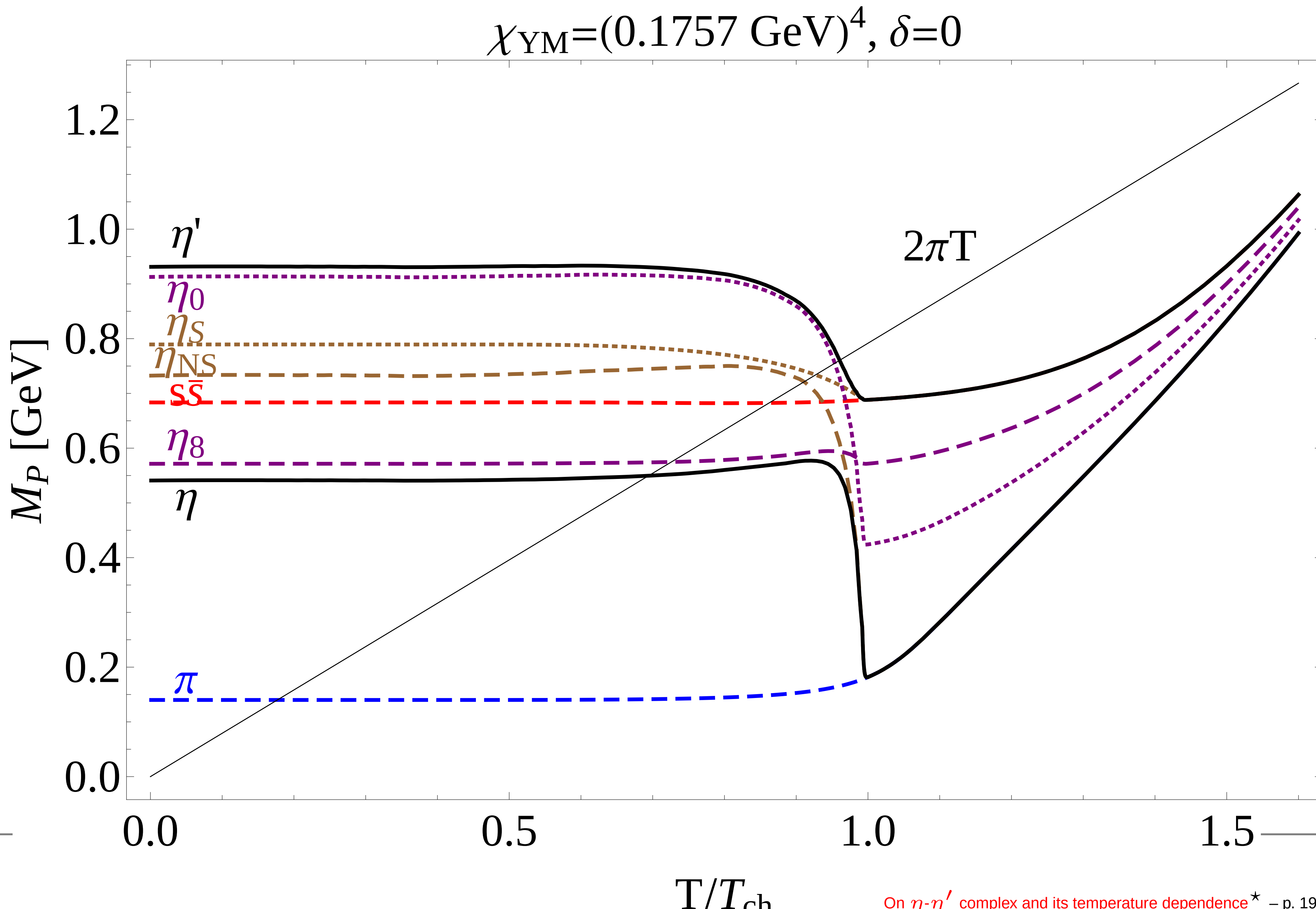
$$= \frac{\langle \bar{q}q \rangle_0(T)}{\sum_{q=u,d,s} \left(\frac{1}{m_q}\right)} \left\{ 1 - \frac{\langle \bar{q}q \rangle_0(T)}{\sum_{q=u,d,s} \left(\frac{1}{m_q}\right)} \frac{1}{\mathcal{C}(m)} \left[\frac{\langle \bar{q}q \rangle_0(T=0)}{\langle \bar{q}q \rangle_0(T)} \right]^\delta \right\} .$$

Chiral condensate $\langle q\bar{q}\rangle_0(T)$ and resulting $\widetilde{\chi}(T)$

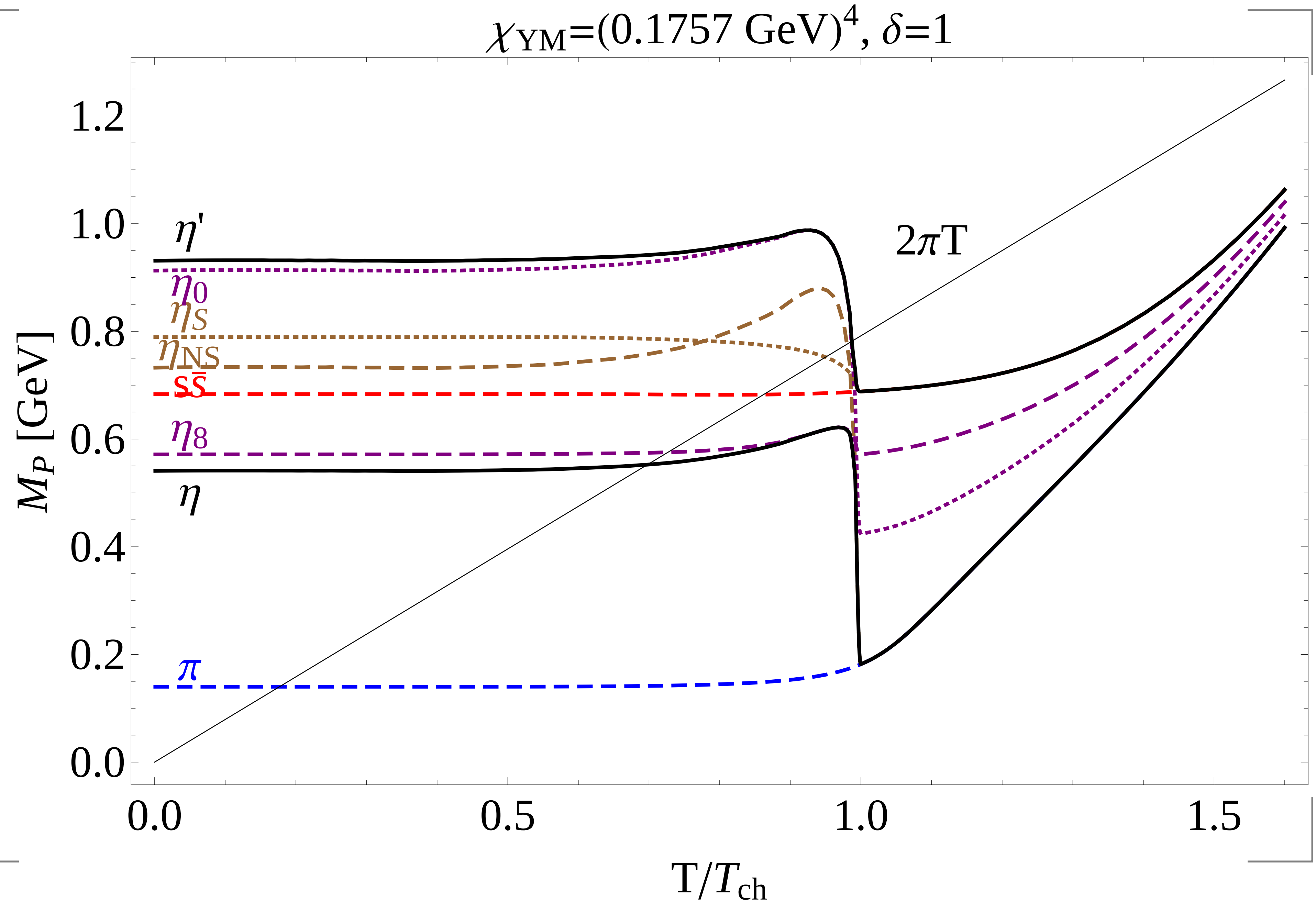


Case 1: T-independent correction term in χ

[Benić, Horvatić, Kekez and Klabučar, Phys. Rev. D **84** (2011) 016006.]



Case 2: Strongly T-dependent correction term $\propto \langle\bar{q}q\rangle_0(T)$



Recapitulation of so far, & what follows

- Leutwyler-Smilga and Di Vecchia-Veneziano relations 1.) enable one to retain unchanged WV relation, with $\chi_{\rm YM}$, for $T = 0$ (in fact, any T sufficiently below $T_{\rm Ch}$) and 2.) to replace the T-dependence of $\chi_{\rm YM}$ by that of $\widetilde{\chi}$ which is essentialy that of the chiral condensate. This provides an explanation for the η' mass drop and thus for the data on increased η' multiplicities, and indicates how chiral restoration may be linked with the $U_A(1)$ one.
- We shall show our exact solutions to Shore's generalization of WVR support the above

Shore's generalization of WV valid to all orders in $1/N_c$

- WV rel. – lowest order in $1/N_c$ – improved like this:

$$(f^0_{\eta'})^2 M^2_{\eta'} + (f^0_{\eta})^2 M^2_{\eta} = \frac{1}{3}\left(f^2_\pi M^2_\pi + 2 f^2_K M^2_K\right) + 6A \qquad (1)$$

$$f^0_{\eta'} f^8_{\eta'} M^2_{\eta'} + f^0_{\eta} f^8_{\eta} M^2_{\eta} = \frac{2\sqrt{2}}{3}\left(f^2_\pi M^2_\pi - f^2_K M^2_K\right) \qquad (2)$$

$$(f^8_{\eta'})^2 M^2_{\eta'} + (f^8_{\eta})^2 M^2_{\eta} = -\frac{1}{3}\left(f^2_\pi M^2_\pi - 4 f^2_K M^2_K\right) \qquad (3)$$

A is the full QCD topological charge parameter

$$A = \frac{\chi}{1 + \chi\left(\frac{1}{\langle \bar{u}u\rangle\, m_u} + \frac{1}{\langle \bar{d}d\rangle\, m_d} + \frac{1}{\langle \bar{s}s\rangle\, m_s}\right)} \qquad (4)$$

= hard to calculate on lattice ...
However, it is known that $A = \chi_{\rm YM} + \mathcal{O}(\frac{1}{N_c})$

Reduction to the standard WV relation (= large N_c result)

Replacement 3 different condensates $\rightarrow \langle \bar{q}q \rangle_0$ reduces the full QCD topological charge A (4) to the combination $\widetilde{\chi}$ on the RHS of Leutwyler-Smilga relation (lowest $\mathcal{O}(\frac{1}{N_c})$):

$$\chi_{\rm YM} = \frac{\chi}{1 + \frac{\chi}{\langle \bar{q}q \rangle_0} \sum\limits_{q=u,d,s} \frac{1}{m_q}} \rightarrow \widetilde{\chi}(T,\mu) = \frac{\langle \bar{q}q(T,\mu) \rangle_0}{\sum\limits_{q=u,d,s} \frac{1}{m_q}} + corr's \approx A(T$$

Previously, we only **conjectured** $\chi_{\rm YM}(T) \rightarrow \widetilde{\chi}(T)$ [Benić et al, Phys. Rev. D84 (2011) 016006] , to explain increased η' multiplicity at RHIC noted by Csörgő et al.

Also note (1)+(3) $\Rightarrow$

$$(f^0_{\eta'})^2 M^2_{\eta'} + (f^0_\eta)^2 M^2_\eta + (f^8_\eta)^2 M^2_\eta + (f^8_{\eta'})^2 M^2_{\eta'} - 2 f^2_K M^2_K = 6A$$

- Then, large N_c limit and ‘off-diagonal’ $f^0_\eta, f^8_{\eta'} \rightarrow 0$, as well as $f^0_{\eta'}, f^8_\eta, f_K \rightarrow f_\pi$, recovers the standard WV.

η' and η have 4 independent decay constants

$$f^0_{\eta'}, f^8_{\eta}, f^0_{\eta}, f^8_{\eta'} : \qquad \langle 0|A^{a\,\mu}(x)|P(p)\rangle = i f^a_P\, p^\mu e^{-ip\cdot x}, \;\; a = 8, 0; \;\; P = \eta, \eta' .$$

- Equivalently, one has 4 related but different constants $f^{NS}_{\eta'}, f^{NS}_{\eta}, f^{S}_{\eta}, f^{S}_{\eta'}$ if instead of octet and singlet axial currents ($a = 8, 0$) one takes this matrix element of the nonstrange-strange axial currents ($a = NS, S$)

$$A^\mu_{NS}(x) = \frac{1}{\sqrt{3}} A^{8\,\mu}(x) + \sqrt{\frac{2}{3}} A^{0\,\mu}(x) = \frac{1}{2}\left(\bar{u}(x)\gamma^\mu\gamma_5 u(x) + \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right) \;,$$

$$A^\mu_{S}(x) = -\sqrt{\frac{2}{3}} A^{8\,\mu}(x) + \frac{1}{\sqrt{3}} A^{0\,\mu}(x) = \frac{1}{\sqrt{2}} \bar{s}(x)\gamma^\mu\gamma_5 s(x) \;,$$

$$\begin{bmatrix} f^{NS}_{\eta} & f^{S}_{\eta} \\ f^{NS}_{\eta'} & f^{S}_{\eta'} \end{bmatrix} = \begin{bmatrix} f^{8}_{\eta} & f^{0}_{\eta} \\ f^{8}_{\eta'} & f^{0}_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} ,$$

$$a, P = NS, S : \qquad \langle 0|A^\mu_{NS}(x)|\eta_{NS}(p)\rangle = i f_{NS}\, p^\mu e^{-ip\cdot x} \;, \qquad \langle 0|A^\mu_{NS}(x)|\eta_{S}(p)\rangle = 0 \;,$$

$$a, P = NS, S : \qquad \langle 0|A^\mu_{S}(x)|\eta_{S}(p)\rangle = i f_{S}\, p^\mu e^{-ip\cdot x} \;, \qquad \langle 0|A^\mu_{S}(x)|\eta_{NS}(p)\rangle = 0 \;,$$

- Note: in a DS approach, $f_{NS} = f_{u\bar{u}} = f_{d\bar{d}} = f_\pi$, $f_S = f_{s\bar{s}}$ are calculated quantities

Two Mixing Angles and FKS one-angle scheme

- Any 4 η-η' decay constants conveniently parametrized in terms of two decay constants and two angles:

$$f_\eta^8 = \cos\theta_8 \, f_8 \,, \qquad f_\eta^0 = -\sin\theta_0 \, f_0 \,, \qquad f_\eta^{NS} = \cos\phi_{NS} \, f_{NS} \,, \qquad f_\eta^S = -\sin\phi_S \, f_S \,,$$

$$f_{\eta'}^8 = \sin\theta_8 \, f_8 \,, \qquad f_{\eta'}^0 = \cos\theta_0 \, f_0 \,, \qquad f_{\eta'}^{NS} = \sin\phi_{NS} f_{NS} \,, \qquad f_{\eta'}^S = \cos\phi_S f_S$$

- Big practical difference between 0-8 and NS-S schemes:
- while θ_8 and θ_0 differ a lot from each other and from $\theta \approx (\theta_8 + \theta_0)/2$, FKS showed that $\phi_{NS} \approx \phi_S \approx \phi$.

$$\begin{bmatrix} f_\eta^{NS} & f_\eta^S \\ f_{\eta'}^{NS} & f_{\eta'}^S \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} .$$

For four decay constants, can use FKS one-angle scheme!

- ϕ relates $\{f_\eta^8, f_{\eta'}^8, f_\eta^0, f_{\eta'}^0\}$ with $\{f_{NS}, f_S\} = \{f_\pi, f_{s\bar{s}}\}$:

$$\begin{bmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

- Some other useful relations between quantities of NS-S (FKS) and 0-8 schemes:

$$f_8 = \sqrt{\frac{1}{3}f_{NS}^2 + \frac{2}{3}f_S^2}\,, \qquad \theta_8 = \phi - \text{arctan}\left(\frac{\sqrt{2}f_S}{f_{NS}}\right)\,,$$

$$f_0 = \sqrt{\frac{2}{3}f_{NS}^2 + \frac{1}{3}f_S^2}\,, \qquad \theta_0 = \phi - \text{arctan}\left(\frac{\sqrt{2}f_{NS}}{f_S}\right)\,.$$

Solve numerically Shore's Eqs. (1)-(3) for $M_{\eta'}$, M_η, and ϕ:

Inputs:	$M_\pi, M_K, f_\pi = f_{\rm NS}, f_{s\bar{s}} = f_{\rm S}$ and f_K, calculated in 3 different DS models					
$\chi_{\rm YM}$	191^4	175.7^4	191^4	175.7^4	191^4	175.7^4
M_η	499.8	485.7	496.7	482.8	526.2	507.0
$M_{\eta'}$	931.4	815.8	934.9	818.4	983.2	868.7
ϕ	52.01°	46.11°	51.85°	46.07°	47.23°	40.86°
θ	-2.72°	-8.62°	-2.89°	-8.67°	-7.51°	-13.87°
θ_0	7.74°	1.84°	7.17°	1.39°	-0.33°	-6.69°
θ_8	-12.00°	-17.90°	-11.85°	-17.6°	-14.12°	-20.47°
f_0	108.8	108.8	107.9	107.9	101.8	101.8
f_8	122.6	122.6	121.1	121.1	110.7	110.7
f_η^0	-14.7	-3.5	-13.5	-2.6	0.6	11.9
$f_{\eta'}^0$	107.9	108.8	107.1	107.9	101.8	101.1
f_η^8	119.9	116.7	118.5	115.4	107.4	103.7
$f_{\eta'}^8$	-25.5	-37.7	-2.49	-37.6	-27.0	-38.7

(in D. Horvatić et al., Eur. Phys. J. A **38** (2008) 257.) $M_{\eta,\eta'}$ and f's in MeV, $\chi_{\rm YM}$ is in MeV^4.

The same is now reproduced analytically:

- Eqs. (1)-(3) $\Rightarrow$ two closed-form solutions for M_η, $M_{\eta'}$ and $\tan\phi$ in terms of $f_\pi, f_{s\bar{s}}, M_\pi$, M_K and A.
  The set reproducing the previous numerical results is:

$$\tan\phi = \frac{-2Af_\pi^2 + 4Af_{s\bar{s}}^2 - 2f_K^2 f_\pi^2 M_K^2 + f_\pi^4 M_\pi^2 + f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 + \Delta}{4\sqrt{2}\,A\,f_\pi\,f_{s\bar{s}}}$$

$$M_{\eta,\eta'}^2 = \frac{2Af_\pi^2 + 4Af_{s\bar{s}}^2 + 2f_K^2 f_\pi^2 M_K^2 - f_\pi^4 M_\pi^2 + f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 \mp \Delta}{2f_\pi^2 f_{s\bar{s}}^2}$$

where $\Delta^2 =$

$$32\,A^2\,f_\pi^2 f_{s\bar{s}}^2 + \left\{2A(f_\pi^2 - 2f_{s\bar{s}}^2) + f_\pi^2\,\left[2f_K^2 M_K^2 - (f_\pi^2 + f_{s\bar{s}}^2)M_\pi^2\right]\right\}^2$$

[Benić, Horvatić, Kekez & Klabučar, Phys. Lett. B738 (2014) 113]

Find matrix elem's in NS-S basis from these $M_\eta, M_{\eta'}, \phi$:

$$M^2_{\eta_{NS}} \equiv M^2_{\text{NS}} = \cos^2\phi\, M^2_\eta + \sin^2\phi\, M^2_{\eta'}$$

$$M^2_{\eta_S} \equiv M^2_{\text{S}} = \sin^2\phi\, M^2_\eta + \cos^2\phi\, M^2_{\eta'}$$

$$M^2_{\eta_{NS}\eta_S} \equiv M^2_{\text{NSS}} = \sin\phi\,\cos\phi\,(M^2_\eta - M^2_{\eta'})$$

to use $$M^2_{\eta,\eta'} = \frac{1}{2}\left[M^2_{\text{NS}} + M^2_{\text{S}} \mp \sqrt{(M^2_{\text{NS}} - M^2_{\text{S}})^2 + 4M^4_{\text{NSS}}}\right]$$

Mathematica leads to surprisingly simple results:

$$M^2_{\text{NS}} = M^2_\pi + \frac{4A}{f^2_\pi}, \qquad M^2_{\text{NSS}} = \frac{2\sqrt{2}A}{f_\pi f_{s\bar{s}}}$$

$$M^2_{\text{S}} = \frac{1}{f^2_{s\bar{s}}}[2\, f^2_K\, M^2_K - f^2_\pi\, M^2_\pi] + \frac{2A}{f^2_{s\bar{s}}} = M^2_{s\bar{s}} + \frac{2A}{f^2_{s\bar{s}}}$$

$f^2_\pi\, M^2_\pi = -m_u\langle u\bar{u}\rangle - m_d\langle d\bar{d}\rangle$ and $f^2_K\, M^2_K = -m_u\langle u\bar{u}\rangle - m_s\langle s\bar{s}\rangle$

$\Rightarrow \quad 2\, f^2_K\, M^2_K - f^2_\pi\, M^2_\pi = f^2_{s\bar{s}}\, M^2_{s\bar{s}}$ "eq. (23)"

Compare $M_{\rm NS}$, $M_{\rm NSS}$ and $M_{\rm S}$ with NS-S mass matrix:

$$\begin{bmatrix} M^2_{\rm NS} & M^2_{\rm NSS} \\ M^2_{\rm NSS} & M^2_{\rm S} \end{bmatrix} = \begin{bmatrix} M^2_\pi + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{bmatrix}$$

$\Rightarrow$ Very similar formulas in WV case and "Shore case":

$$1.) \quad \beta_{\rm WV} = \frac{6\chi_{YM}}{f_\pi^2(2+X^2)}\,, \qquad \beta_{\rm Shore+FKS} = \frac{2A}{f_\pi^2} \approx \frac{2\chi_{YM}}{f_\pi^2}$$

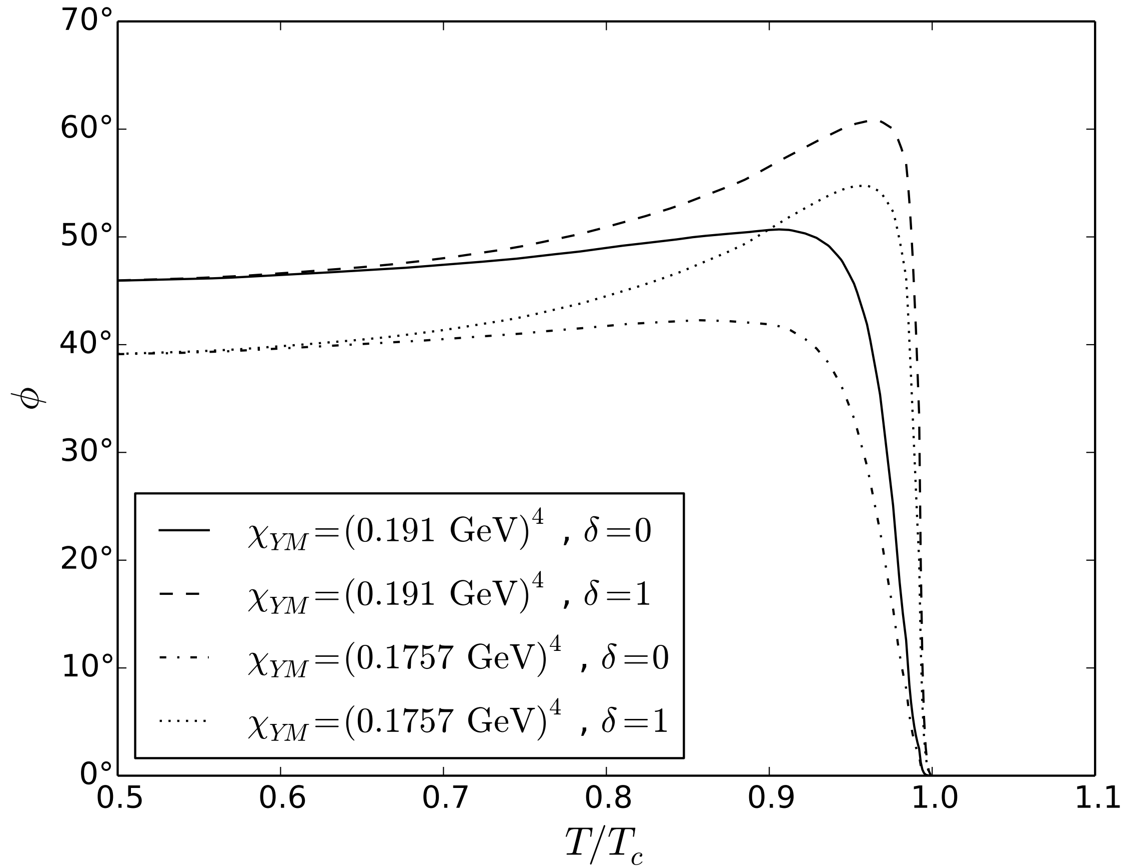
Explains why Shore's scheme needs higher values of χ_{YM} than WV, to approach empirical masses.

$2.) \quad X = \frac{f_\pi}{f_{\rm ss}}$ the SAME in the both WV and Shore cases ...

... but in the "Shore case", it follows from equations! Before, incl. WV, it was an input – estimate, educated guess.

Extending Shore + FKS scheme to $T > 0$

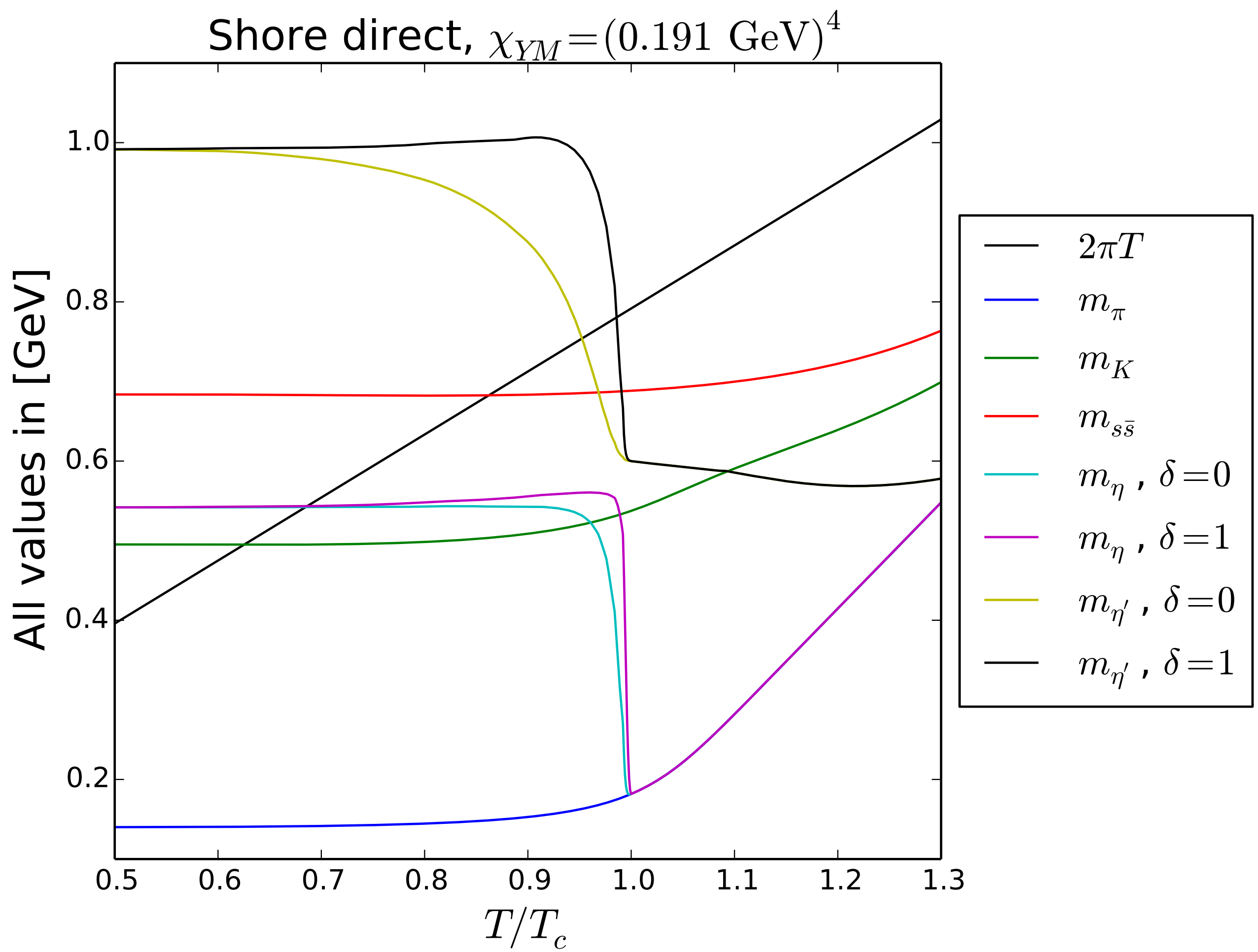
Presently, all results of the Shore + FKS scheme at $T > 0$ are obtained with the approximation $A(T) \approx \widetilde{\chi}(T)$



The T-dependence of the mixing angle $\phi(T)$ for the cases of the T-independent correction term in $\chi(T)$ ($\delta = 0$) and the correction term in $\chi(T)$ behaving like the leading term, i.e., like the chiral condensate ($\delta = 1$), and for the two values of $\widetilde{\chi}(T = 0) = \chi_{YM}$.

T-dependence of pseudoscalar masses without GMOR

Results are identical from direct evaluation of solutions and from the mass matrix (but where GMOR was not used at $T = 0$):

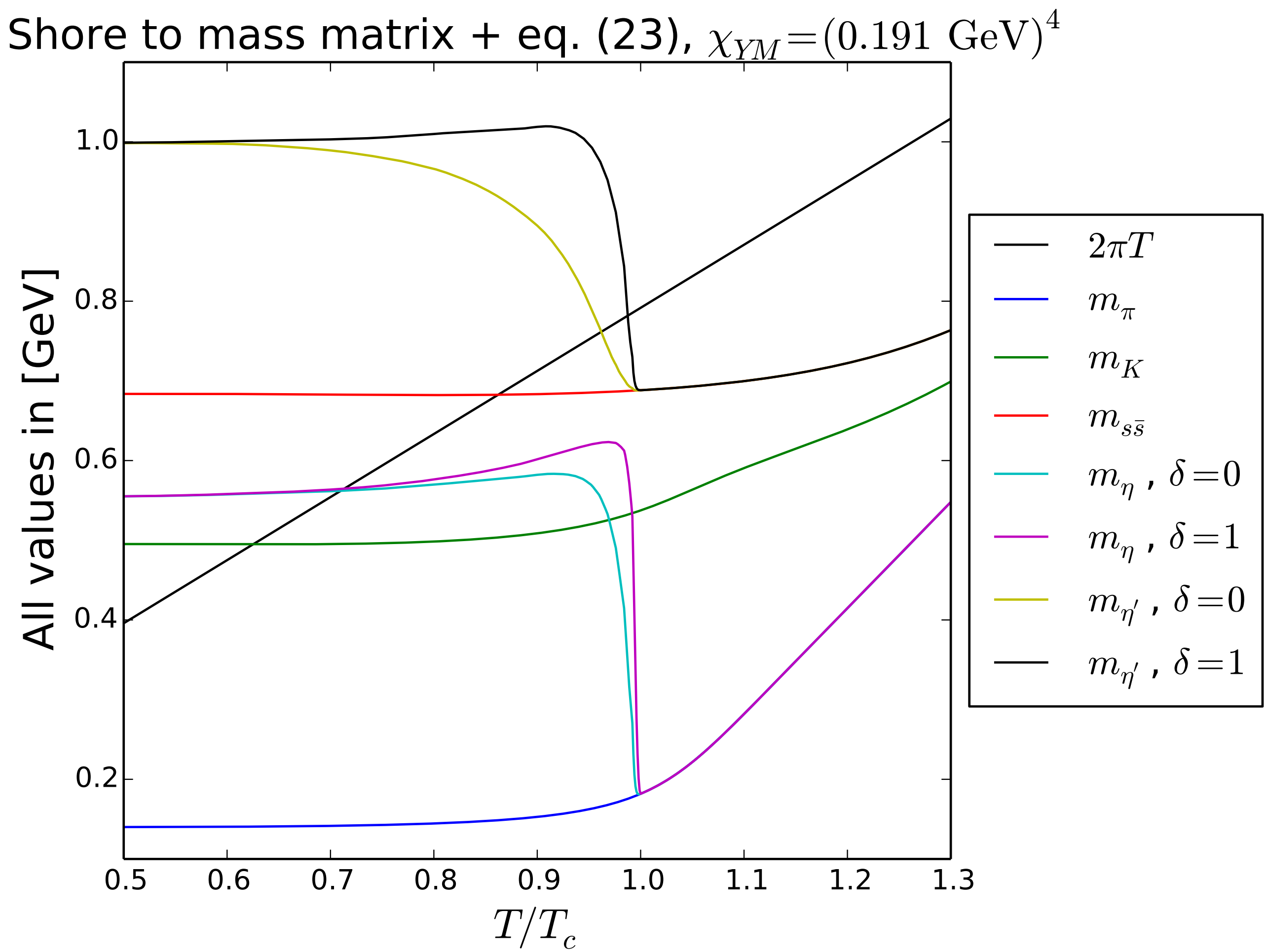


The behavior of $M_{\eta'}(T)$ after $T \approx T_c$ results from our model-calculated π & K starting to violate GMOR there.

T-dependence of pseudoscalar masses with GMOR

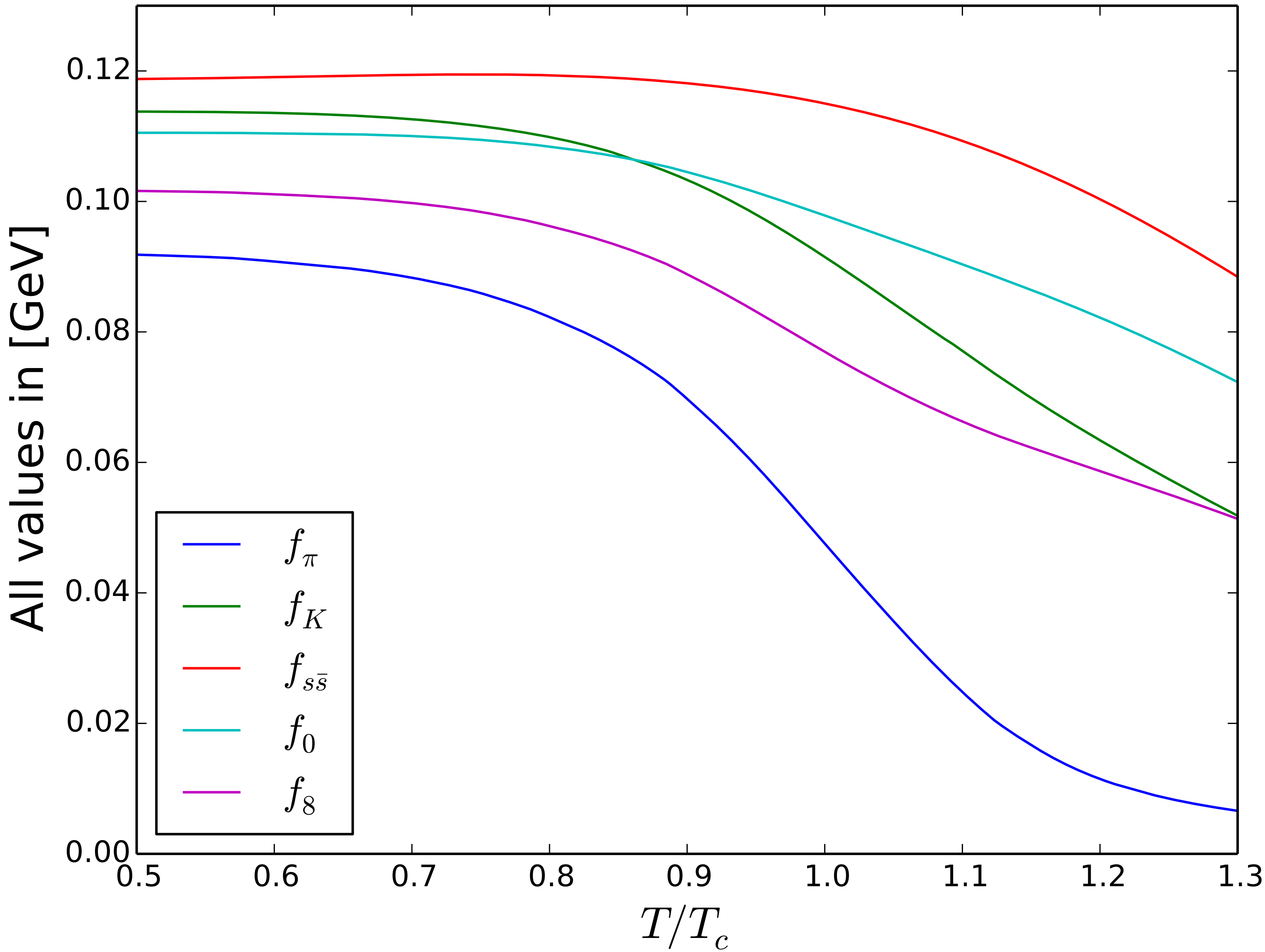
Results where GMOR was used to identify

$$2\, f_K^2\, M_K^2 \;-\; f_\pi^2\, M_\pi^2 = \; f_{s\bar{s}}^2\, M_{s\bar{s}}^2$$ ("eq. (23)")



The behavior of $M_{\eta'}(T)$ after $T = T_c$ is the same as $M_{s\bar{s}}(T)$ due to using GMOR.

T-dependence of pseudoscalar decay constants



Summary

- The results of the approach through Witt.-Ven. rel. + η-η' mass matrix and Shore's rels. + FKS were shown to be similar numerically.
- The results for Shore's approach (with FKS 1-angle scheme) are also available as analytic, closed-form expressions, and they explain both the similarities and differences in it the results on the η-η' complex.
- The full QCD topological charge parameter A (to which χ_{YM} appears only as a numerical approximation at $T = 0 = \mu$) is not a pure-gauge quantity, but a full QCD quantity. The Leutwyler-Smilga quantity $\widetilde{\chi}$ is the approximation of A with $\langle u\bar{u}\rangle, \langle d\bar{d}\rangle, \langle s\bar{s}\rangle \to \langle q\bar{q}\rangle$.
  This fact refines and gives support to our earlier explanation of the data on η' enhanced multiplicity in RHIC experiments at $T > 0$, where we replace the T-dependence of $\chi_{\rm YM}$ by that of the Leutwyler-Smilga quantity $\widetilde{\chi}(T) \propto$ chiral condensate.
  It also motivates additionally our work on extending the same approach to $\mu > 0$ for RHIC, NICA, GSI/FAIR, compact stars ...
- $\Rightarrow$ Increased motivation for lattice to calculate A and χ of full QCD