# Galilei relativistic hydrodynamics: the choice of frames

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# Outline

# Balances of simple fluids

## Local

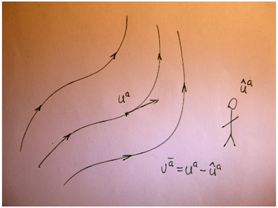
$$
\begin{aligned}
\partial_t \rho + \partial_k(\rho v^k) &= 0,\\
\partial_t(\rho v^i) + \partial_k(P^{ik} + \rho v^i v^k) &= 0^i,\\
\partial_t e + \partial_k(q^k + e v^k) &= 0.
\end{aligned}
$$

## Substantial

$$
\begin{aligned}
\dot{\rho} + \rho \partial_k v^k &= 0,\\
(\dot{\rho v^i}) + \rho v^i \partial_k v^k + \partial_k P^{ik} &= 0^i,\\
\dot{e} + e \partial_k v^k + \partial_k q^k &= 0.
\end{aligned}
$$

Notation:

- $\partial_t = \frac{\partial}{\partial_t}$, $\partial_i = \nabla$, $v^i = \mathbf{v}$
- Indeces are not coordinates.
- Total energy.




## Transformation

$v^i$ relative velocity,
$\partial_t + v^i \partial_i = \frac{d}{dt}$, comoving derivative,
$\hat{q}^i = q^i + e v^i$, conductive and convective

# Fluid thermodynamics

total - kinetic = internal , $e_{int} = e - \rho\frac{v^2}{2}$

$$\frac{d}{dt}\left(\rho\frac{v^2}{2}\right) + \rho\frac{v^2}{2}\partial_i v^i + \partial_i(P^{ik}v_k) - P^{ik}\partial_i v_k = 0.$$

$$\dot{e}_{int} + e_{int}\partial_k v^k + \partial_k\underbrace{(q^k - P^{ik}v_i)} + P^{ik}\partial_i v_k = 0.$$

Thermodynamics:

$$s(e_{int}, \rho), \quad de_{int} = Tds + \mu d\rho; \quad e_{int} + p = Ts + \mu\rho, \quad s^i = \frac{q^i_{int}}{T}$$

$$\dot{s} + s\partial_i v^i + \partial_i s^i = \frac{1}{T}\dot{e}_{int} - \frac{\mu}{T}\dot{\rho} + s\partial_i v^i + \partial_i\frac{q^i_{int}}{T} =$$

$$-\frac{1}{T}\left(e_{int}\partial_i v^i + \partial_i q^i_{int} + P^{ij}\partial_i v_j\right) + \frac{\mu}{T}\left(\rho\partial_i v^i\right) + s\partial_i v^i + \frac{\mu}{T}\partial_i q^i_{int} + q^i_{int}\partial_i\frac{1}{T} =$$

$$\boxed{q^i_{int}\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0}.$$

Basic fields: $\rho$, $e_{int}$, $v^i$; Constitutive functions: $q^i_{int}$, $P^{ij}$

# Problems

## Nonrelativistic

1. Transformation rules. Pressure, energy?
2. $v^i$ is a relative velocity. Is dissipation real/physical/objective ($\partial_i v_j$)?
3. What is moving? Mass? (Frames: Eckart or Landau-Lifsic)
4. Local equilibrium? Thermodynamics is comoving with what? Entropy current density?

## Material frame indifference

- Comoving time derivatives and more (Jaumann 1909).
- (Truesdell-)Noll formulation with transformation rules. Self contradictory (Matolcsi-VP 2006),
- Brenner problem.
- Beyond fluids: basic kinematics, many deformation measures, etc ... (Fülöp-VP 2012).

# Compatibility

## Relativistic

- What is ideal fluid?
- Flows. Eckart or Landau-Lifsic? Is there a choice? (Ván-Biró 2013)
- Thermodynamics?
- Generic stability. Israel-Stewart does not help. (Ván 2009, Ván-Biró, 2012).

## Kinetic theory and more

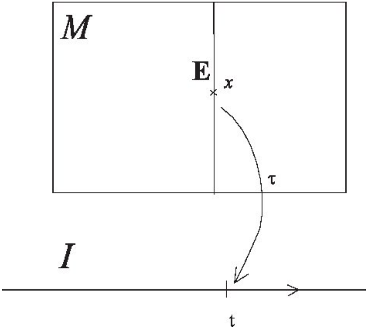
- Moment (or gradient) expansion: series of balances, increasing tensorial order. Transformation rules are inherited (Ruggeri 1981).
- Instabilities at second order.

Where are the balances with increasing tensorial order?
Kinetic energy is a transformation rule?

# Galilei-relativity

Time is absolute, space is relative.
Event:

$$(t, x^i) \longrightarrow (t, x^i + v^i t)$$




## Transfer of relativistic knowledge (Matolcsi 1993)

- World line: $x^a(t) \to (t, x^i(t))$
- Velocity: $u^a \to (1, v^i)$, $\tau_a u^a = 1$ $(u^a u_a = 1)$
- Fields: density-current: $A^a = au^a + a^{\bar{a}} \to (a, a^i)$
- Splitting according to a velocity. Observer = velocity field.
- Spacetime vectors and covectors cannot be identified.

# Balances

Covariant

$$\partial_a A^a = 0$$

Relative

$$\begin{aligned} u^a: \quad & \partial_t a + \partial_i a^i = 0, \\ \hat{u}^a: \quad & \hat{\partial}_t a + \partial_i \hat{a}^i = 0. \end{aligned}$$

$$(\partial_t - v^i \partial_i) a + \partial_i (a^i + a v^i) = \partial_t a + a \partial_i v^i + \partial_i a^i = 0$$

Substantial: comoving physical quantities, space (and time derivative) of the observer: $(a(t, \underline{\hat{x}^i}))$.

Expectation: objective energy. Total or internal: transformation rules.

## Basic field:

$$\boxed{Z^{abc} = z^{bc} u^a + z^{\bar{a}bc} : \qquad \text{Mass-energy-momentum density tensor}}$$

$$z^{bc} \to \begin{pmatrix} \rho & p^i \\ p^i & e^{ij} \end{pmatrix}, \qquad z^{ab} \to \begin{pmatrix} \rho & p^i \\ j^i & P^{ij} \end{pmatrix}, \qquad e = \frac{e^i_i}{2}$$

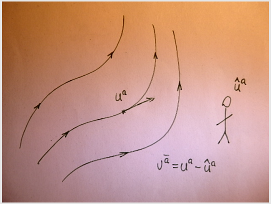
# Absolute and relative fields

## Basic field:

$$\boxed{Z^{abc} = z^{bc}u^a + z^{\bar{a}bc} : \qquad \text{Mass-energy-momentum density tensor}}$$

## Transformation rules:

$$\begin{aligned}
\tau_a\tau_b\tau_c Z^{abc} = \hat{\rho} &= \rho, \\
\ldots = \hat{p}^i &= p^i + \rho v^i, \\
\ldots = \hat{e} &= e + p^i v_i + \rho\frac{v^2}{2}, \\
\ldots = \hat{j}^i &= j^i + \rho v^i, \\
\ldots = \hat{P}^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\
\ldots = \hat{q}^i &= q^i + ev^i + P^{ij}v_j + p^j v_j v^i + (j^i + \rho v^i)\frac{v^2}{2}.
\end{aligned}$$



# Absolute and relative balances

## Absolute

$$\partial_a Z^{abc} = \dot{z}^{bc} + z^{bc}\partial_a u^a + \partial_a z^{\bar{a}bc} = 0$$

$$\begin{aligned}
\dot{\rho} + \rho\partial_a u^a + \partial_a j^{\bar{a}} &= 0,\\
\dot{p}^{\bar{b}} + p^{\bar{b}}\partial_a u^a + \rho\dot{u}^b + j^{\bar{a}}\partial_a u^b + \partial_a P^{\bar{a}\bar{b}} &= 0^{\bar{b}},\\
\dot{e} + e\partial_a u^a + \partial_a q^{\bar{a}} + p^{\bar{b}}\dot{u}_b + P^{\bar{a}\bar{b}}\partial_a u_b &= 0.
\end{aligned}$$

## Relative, substantial

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0,\\
\dot{p}^i + p^i\partial_k v^k + \rho\dot{v}^i + j^k\partial_k v^i + \partial_k P^{ik} &= 0^i,\\
\dot{e} + e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

# Termodynamics I. Absolute forms

## Analogy

$$s = \frac{1}{T}(e + p - \mu\rho) = \beta(e - \mu\rho) - \beta p \quad = \quad (\beta \quad -\beta\mu)\begin{pmatrix} e \\ \rho \end{pmatrix} + \beta p$$

$$ds = \beta(de_{int} - \mu d\rho) \to \underline{s(e_{int}, \rho)}, \quad ds \quad = \quad (\beta \quad -\beta\mu)\begin{pmatrix} de \\ d\rho \end{pmatrix}$$

## Absolute

$$S^a = su^a + s^{\bar{a}}, \qquad \boxed{s(z^{bc})}$$

$$\begin{aligned} S^a - \beta_{bc}Z^{abc} &= p^a \\ ds &= \beta_{bc}dz^{bc} \end{aligned}$$

$\beta_{bc}$ chemical potential-thermovelocity-temperature cotensor

# Thermodynamics II. Intensives.

$\beta_{bc}$ chemical potential-thermovelocity-temperature cotensor

## Transformation rules

$$\begin{aligned} \hat{\beta} &= \beta, \\ \hat{w}_i &= w_i + v_i, \qquad \textit{vector!} \\ \hat{\mu} &= \mu - w_i v^i - \frac{v^2}{2}. \end{aligned}$$

## Absolute-relative ($p^a = \beta p(u^a + w^{\bar{a}})$)

$$\begin{aligned} S^a = \beta_{bc} Z^{abc} - p^a \quad &\to \quad Ts = e + p - \mu\rho + w_i p^i, \\ &\to \quad Ts^i = q^i - \mu j^i - P^{ij} w_j + p w^i, \\ ds = \beta_{bc} dz^{bc} \quad &\to \quad de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i) dv^i. \end{aligned}$$

# Termodynamics III. Entropy balance.

$\partial_a S^a = \partial_a(su^a + s^{\bar{a}}) = \sigma \geq 0$, condition: $\partial_a Z^{abc} = 0$

## Entropy production

$$
\begin{aligned}
\partial_a S^a &= \ldots \\
&= -(j^{\bar{a}} - \rho w^{\bar{a}})\partial_a \left(\beta\mu + \beta\frac{w^2}{2}\right) + \\
&\left(q^{\bar{a}} - w^{\bar{a}}(e - p^{\bar{b}} w_{\bar{b}}) + (j^{\bar{a}} - \rho w^{\bar{a}})\frac{w^2}{2} - P^{\bar{a}\bar{b}} w_{\bar{b}}\right)\partial_a\beta - \\
&\beta\left(P^{\bar{a}}_{\bar{b}} + w^{\bar{a}}(\rho w_{\bar{b}} - p_{\bar{b}}) - j^{\bar{a}} w_{\bar{b}} - p\delta^{\bar{a}}_{\bar{b}}\right)\partial_a(u^b + w^{\bar{b}}) \geq 0
\end{aligned}
$$

Distinguished frame of reference $\boxed{v^i = -w^i}$.

$$
-\hat{j}^i\partial_i\frac{\mu_{int}}{T} + \hat{q}^i\partial_i\frac{1}{T} - \frac{1}{T}(\hat{P}^{ij} - p\delta^{ij})\partial_i w_j \geq 0.
$$

# Thermostat(odynam)ics.

Gibbs relation: $de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i)dv^i$

## Maxwell relations

$s(e, \rho, p^i, v^i)$

$$\frac{\partial s}{\partial p^i} = \frac{w^i}{T}, \qquad \frac{\partial s}{\partial v^i} = \frac{\rho w^i - p^i}{T}$$

$$\frac{\partial^2 s}{\partial v^i p^j} = \frac{\partial^2 s}{\partial p^i v^j} = \boxed{\frac{\partial w_i}{\partial v^j} = \delta_{ij} - \rho \frac{\partial w_i}{\partial p^j}}$$

Megoldás:

$$w_i = \frac{p_i}{\rho} + A_{ij}\left(v^j + \frac{p^j}{\rho}\right) + \overline{w}_i$$

Simple and invariant(!) part:

$$\boxed{p^i = \rho w^i}$$

# What is flowing?

## Gibbs relation:

$$de = Tds + \mu d\rho + w_i d(\rho w^i) \longrightarrow d\left(e - \rho\frac{w^2}{2}\right) = Tds + \left(\mu + \frac{w^2}{2}\right) d\rho$$

Internal energy and chemical potential.

## The momentum flows

The invariant EOS:

$$\hat{p}^i = \underbrace{p^i}_{=0} + \rho v^i = \rho \hat{w}^i$$

## Is there a second choice?

The relative velocity is the thermovelocity for every laboratory observer.
May we chose conveniently?

$$\tilde{v}^i = \frac{j^i}{\rho}$$

# Complete relative system

$$\dot{\rho} + \rho\partial_i w^i + \partial_i j^i = 0,$$
$$\rho\dot{w}^i + \partial_k P^{ik} + j^k\partial_k v^i = 0,$$
$$\dot{e}_{int} + e_{int}\partial_i w^i + \partial_i q^i + P^{ij}\partial_i w_j = 0,$$
$$-j^i\partial_i\frac{\mu_{int}}{T} + q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i w_j \geq 0.$$

Generic stability is ok.

# Conclusions

- Fluid mechanics is absolute (Galilei-covariant).
- Thermodynamics and dissipation are absolute.
- Usually the momentum flows. Because
  - $p^i = \rho w^i$ is an EOS.
  - $v^i = -w^i$ leads to a simple form of the dissipation.

  The usual balances are obtained (?).
- Relative velocity field ($v^i$) may be chosen?
- Generic stability.
- Special relativistic hydro?

Thank you for the attention!