Quantum corrections to the stress-energy tensor in thermodynamic equilibrium with acceleration

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Summary

We show that the stress-energy tensor has non-dissipative corrections of quantum origin with respect to its ideal form:

$$T^{\mu\nu}(x) = T^{\mu\nu}_{\rm id}(x) + \delta T^{\mu\nu}(x) = (\rho(x) + p(x))u^\mu u^\nu - g^{\mu\nu}p(x) + \delta T^{\mu\nu}(x)$$

The above corrections are proportional to terms quadratic in vorticity and acceleration [1, 2, 3, 4] and vanish - at least for free fields - in the limit $\hbar \to 0$. We calculate them by using a general equilibrium formalism and show that all relevant coefficients can be expressed as correlators of the stress-energy tensor operator and Poincaré group generators.

General covariant global equilibrium

The most general equilibrium distribution in relativistic quantum statistical mechanics can be expressed in a covariant fashion [5, 6]

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu\right)\right] \tag{1}$$

where β is the four-temperature vector and *defines* a hydrodynamical frame [7] $u = \beta/\sqrt{\beta^2} = T\beta$; $\zeta = \mu/T$ and Σ is an arbitrary spacelike 3D hypersurface, provided that β is a Killing vector:

$$\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu = 0 \qquad \partial_\mu\zeta = 0$$

In Minkowski spacetime the solution of the Killing equation reads:

$$\beta^\nu = b^\nu + \varpi^{\nu\mu}x_\mu \qquad \zeta = \text{const.} \qquad \to \qquad \varpi^{\mu\nu} = -\frac{1}{2}(\partial^\nu\beta^\mu - \partial^\mu\beta^\nu) = \text{const.}$$

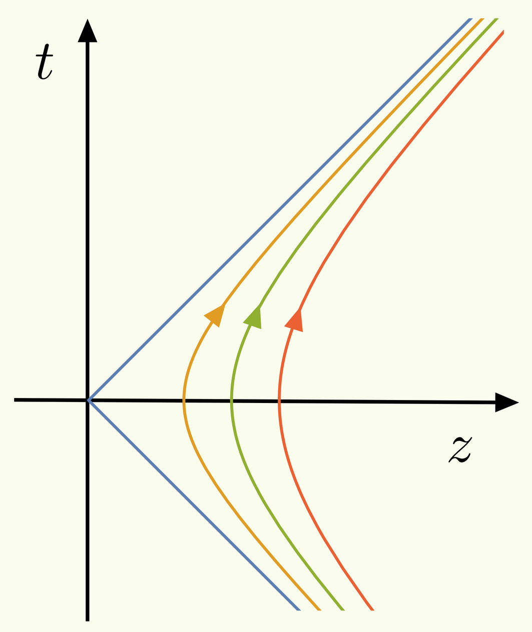
and (1) turns into:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-b_\mu\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu} + \zeta\widehat{Q}\right] \qquad \widehat{P}^\nu, \widehat{J}^{\mu\nu} \quad \text{Generators of the Poincaré group}$$

Special cases:

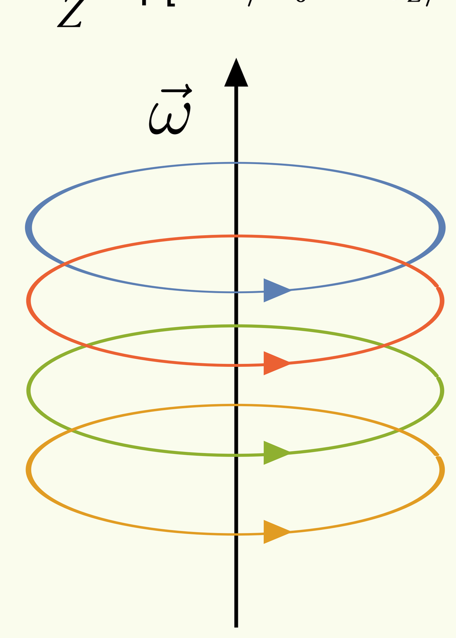
Pure acceleration

$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$



Pure rotation [8]

$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{H}/T_0 + \omega\widehat{J}_z/T_0]$$



Expansion in ϖ

$$\langle\widehat{T}^{\mu\nu}(x)\rangle = \frac{1}{Z}{\rm tr}\left[\widehat{T}^{\mu\nu}(x)\exp\left(-b_\mu\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu} + \zeta\widehat{Q}\right)\right]$$

The mean value of the stress-energy tensor in general equilibrium can be calculated through an expansion in ϖ if the thermal correlation length is much smaller than the length over which the fields β and ζ significantly vary (hydrodynamic limit), that is $\partial\beta/\beta \ll 1/\beta, 1/m$ and $\varpi \ll 1$

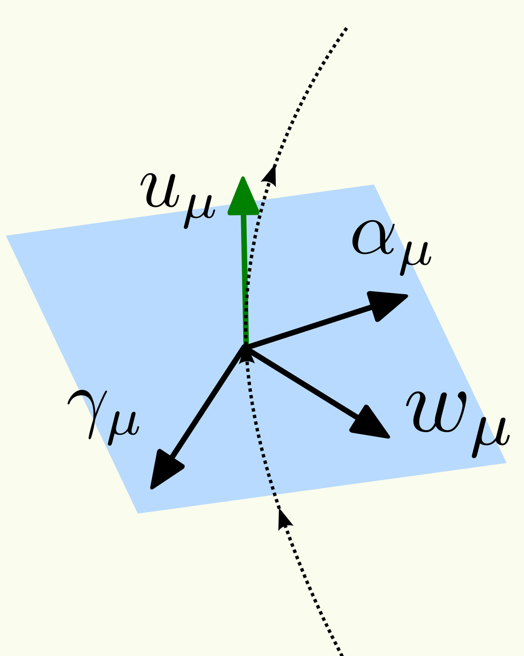
$$T^{\mu\nu}(x) \simeq \langle\widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} + \frac{1}{2}\varpi_{\rho\sigma}{\rm Re}\langle\widehat{J}^{\rho\sigma}_x;\widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} + \varpi_{\rho\sigma}\varpi_{\lambda\tau}\left[\frac{1}{8}{\rm Re}\langle\widehat{J}^{\rho\sigma}_x\widehat{J}^{\lambda\tau}_x;\widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)}\right.$$
$$\left.+\frac{1}{8}\beta^\rho(x)\beta^\lambda(x)\langle\widehat{P}^\sigma\widehat{P}^\tau;\widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} - \frac{1}{12}\beta^\rho(x)g^{\lambda\sigma}\langle\widehat{P}^\tau;\widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)} - \frac{1}{4}{\rm Re}\langle\widehat{J}^{\rho\sigma}_x;\widehat{T}^{\mu\nu}(x)\rangle_{\beta(x)}\langle\widehat{J}^{\sigma\tau}_x\rangle_{\beta(x)}\right]$$

with $\langle\cdots\rangle_{\beta(x)}$ is the mean value at familiar homogeneous thermodynamic equilibrium with a constant four-temperature equal to $\beta(x)$ in the point x, that is with the density operator: $\rho = 1/Z\exp\left[-\beta(x)\cdot\widehat{P} + \zeta\widehat{Q}\right]$

Decomposition into acceleration and vorticity

One can decompose ϖ into two spacelike vectors proportional to acceleration and vorticity by projecting onto the four-velocity $u = \beta/\sqrt{\beta^2}$

$$\varpi^{\mu\nu} = \alpha^\mu u^\nu - \alpha^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma}w_\rho u_\sigma$$



Frame

- $u^\mu = \beta^\mu/\sqrt{\beta^2}$ four-velocity
- $\alpha^\mu = u_\nu\varpi^{\mu\nu} = a^\mu/T$ (a^μ = acceleration)
- $w^\mu = \frac{1}{2}u_\rho\epsilon^{\rho\sigma\nu\mu}\varpi_{\sigma\nu} = \omega^\mu/T$ (ω^μ = vorticity)
- $\gamma^\mu = w_\nu\alpha_\rho u_\sigma\epsilon^{\mu\nu\rho\sigma}$

Final expression

The final expression of the stress-energy tensor up to second order in ϖ:

$$T^{\mu\nu}(x) \simeq (\rho - \alpha^2U_\alpha - w^2U_w)u^\mu u^\nu - (p - \alpha^2D_\alpha - w^2D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + Ww^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu)$$

Restoring the natural constants, the additional terms are quadratic in $\hbar$:

$$\alpha \to \frac{\hbar a}{cKT} \qquad w \to \frac{\hbar\omega}{cKT} \qquad \gamma \to \frac{\hbar^2 a\omega}{c^2K^2T^2}$$

The coefficients U, D, A, W are:

$$U_\alpha = \frac{1}{24}T\frac{\partial\rho}{\partial T} + \frac{1}{4}h + \frac{1}{2}k_t \qquad U_\lambda = \frac{1}{2}j_t \qquad D_\alpha = \frac{1}{24}h + \frac{1}{2}k_\theta - \frac{1}{3}k_s$$
$$D_\lambda = \frac{1}{2}j_\theta - \frac{1}{3}j_s \qquad A = \frac{1}{4}h + k_s \qquad W = j_s \qquad G = \frac{1}{2}l_v - \frac{1}{12}h$$

where ρ is the energy density, h the enthalpy and:

$$k_t(T,\zeta) = \langle\widehat{K}^3\widehat{K}^3;\widehat{T}^{00}(0)\rangle_T \qquad k_\theta(T,\zeta) = \frac{1}{3}\langle\widehat{K}^3\widehat{K}^3;\widehat{T}^{ii}(0)\rangle_T \qquad k_s(T,\zeta) = \langle\widehat{K}^1\widehat{K}^2;\widehat{T}^{12}(0)\rangle_T\,,$$
$$j_t(T,\zeta) = \langle\widehat{J}^3\widehat{J}^3;\widehat{T}^{00}(0)\rangle_T \qquad j_\theta(T,\zeta) = \frac{1}{3}\langle\widehat{J}^3\widehat{J}^3;\widehat{T}^{ii}(0)\rangle_T \qquad j_s(T,\zeta) = \langle\widehat{J}^1\widehat{J}^2;\widehat{T}^{12}(0)\rangle_T\,,$$
$$l_v(T,\zeta) = \langle\{\widehat{K}^1,\widehat{J}^2\};\widehat{T}^{03}(0)\rangle_T \qquad \langle\widehat{A};\widehat{B}\rangle_T = \langle\widehat{AB}\rangle_T - \langle\widehat{A}\rangle_T\langle\widehat{B}\rangle_T$$

Results for the scalar field

The coefficient for a free real scalar massive field with a stress-energy tensor given by:

$$\widehat{T}^{\mu\nu}_\xi = \partial^\mu\widehat{\psi}\partial^\nu\widehat{\psi} - \frac{1}{2}g^{\mu\nu}\left(\partial_\lambda\widehat{\psi}\,\partial^\lambda\widehat{\psi} - m^2\widehat{\psi}^2\right) + 2\xi\,\partial_\lambda\left(g^{\mu\nu}\widehat{\psi}\,\partial^\lambda\widehat{\psi} - g^{\lambda\mu}\widehat{\psi}\,\partial^\nu\widehat{\psi}\right)$$

Massless case

$$U = \kappa T^4$$

Massive fully relativistic

$$U = \frac{m^4}{(2\pi)^4}\sum_{r=1}^\infty a_r(x = m/T, \xi)$$

Massive non relativistic limit (n particles density)

$$U = nf(m, T)$$

	$\kappa(\xi)$	$a_r(x,\xi)$	$f(m,t)$
U_α	$\frac{1}{12}(1-6\xi)$	$\frac{1}{24}\left[(r^2+24\xi x^{-2})K_2(rx) + 3(1-8\xi)rx^{-1}K_3(rx)\right]$	$\frac{1}{24}m^2T^{-1} + \frac{1}{8}m(1-8\xi) + (\frac{5}{16} - \frac{3}{2}\xi)T + o(T)$
U_w	$\frac{1}{12}(1-4\xi)$	$\frac{1}{2}(1-4\xi)x^{-2}K_2(rx)$	$(\frac{1}{2} - 2\xi)T + o(T)$
D_α	$\frac{1}{18}(6\xi-1)$	$\frac{1}{24}\left[(12-48\xi)x^{-2}K_2(rx) + (24\xi-5)rx^{-1}K_3(rx)\right]$	$m(\xi - \frac{5}{24}) + (\frac{1}{2}\xi - \frac{1}{48})T + o(T)$
D_w	$\frac{1}{6}\xi$	$\xi x^{-2}K_2(rx)$	$\xi T + o(T)$
A	$\frac{1}{12}(1-6\xi)$	$\frac{1}{4}\left[(4\xi-2)x^{-2}K_2(rx) + (1-4\xi)rx^{-1}K_3(rx)\right]$	$m(\frac{1}{4} - \xi) + (\frac{1}{8} - \frac{3}{2}\xi)T + o(T)$
W	$\frac{1}{12}(2\xi-1)$	$\frac{1}{2}(2\xi-1)x^{-2}K_2(rx)$	$(\xi - \frac{1}{2})T + o(T)$
G	$\frac{1}{36}(1+6\xi)$	$\frac{1}{6}\left[(6\xi-3)x^{-2}K_2(rx) + rx^{-1}K_3(rx)\right]$	$\frac{1}{6}m + (\xi - \frac{1}{12})T + o(T)$

Comparison with previous expansions

The coefficients denoted by D_α, D_w, A and W are in the following relation with those known as $\xi_3, \xi_4, \lambda_3, \lambda_4$ defined in [1, 2, 3]

$$\frac{A}{T^2} = \lambda_4 \qquad \frac{W}{T^2} = \lambda_3$$
$$\frac{D_w}{T^2} = \left(\frac{\lambda_3}{3} - 2\xi_3\right) \qquad \frac{D_\alpha}{T^2} = (3\lambda_4 - 9\xi_4)$$

We found that our calculated W is in agreement with the coefficient λ_3 found in [2] for the massless case with $\xi = 0$.

Consequences and conclusions

- The stress-energy tensor has non-dissipative quantum corrections if the fluid is rotating or accelerating. Such corrections may be phenomenologically relevant in relativistic heavy ion collisions especially in the early stage where acceleration is very large.
- These corrections depend on the explicit form of the quantum stress-energy tensor operator (dependence on ξ for the free scalar field). Thus, different tensors are thermodynamically inequivalent [9]
- In the β frame, the energy density also gets corrections, as well as the relation between pressure and energy density. In the non-relativistic Boltzmann limit:

$$p_{\it eff} \simeq \rho_{\it eff}\frac{KT}{m}\left[1 + \left(\frac{2}{3}\xi - \frac{1}{12}\right)\frac{m\hbar^2|a|^2}{KT}\right]$$

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