Lefschetz-thimble method for evading the mean-field sign problem

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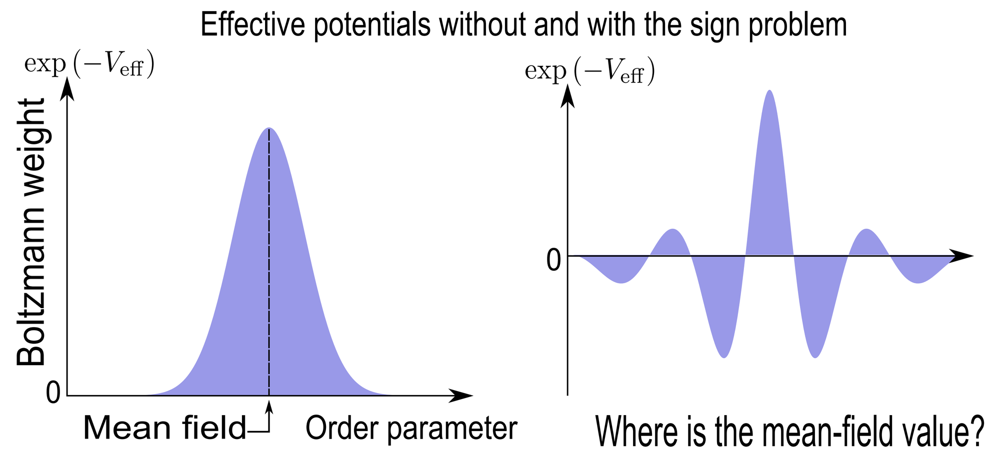
1. Introduction: Sign Problem in the MFA

Mean-field theory of the heavy-dense QCD is characterized by the action of Polyakov loops $\ell(\boldsymbol{x})$:

$$S_{\mathrm{eff}}(\ell) \simeq \int \mathrm{d}^3\boldsymbol{x}\left[e^{\mu}\ell(\boldsymbol{x}) + e^{-\mu}\overline{\ell}(\boldsymbol{x})\right] \notin \mathbb{R}.$$

Even after the MFA, the effective potential becomes complex!



The integration over the order parameter plays a pivotal role for $F = -T\ln Z$ being real.
(Fukushima, Hidaka, PRD75, 036002)

Question: Can we make a tied connection bet. the saddle-point approximation and the mean-field approximation with complex S?

2. Lefschetz-thimble path integral

Multiple oscillatory integration:

$$Z = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{e}^{-S(x)},$$

where $S(x)$ is a complex action functional of the real field $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
Morse equation & steepest descent path: Using Morse eq.,

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = \overline{\left(\frac{\partial S(z)}{\partial z^i}\right)},$$

the Lefschetz thimble $\mathfrak{J}_\sigma$ (=steepest descent path) is identified as (z_σ: saddle point)

$$\mathfrak{J}_\sigma = \{z(0)\,|\,z(-\infty) = z_\sigma\}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}S(z) = 0,$$

the complex phase is constant along $\mathfrak{J}_\sigma$.
Lefschetz-thimble decomposition: The partition function Z becomes

$$Z = \sum_{\sigma\in\Sigma}\langle \mathfrak{K}_\sigma, \mathbb{R}^n\rangle \int_{\mathfrak{J}_\sigma} \mathrm{d}^n z\, \mathrm{e}^{-S(z)},$$

which is the sum of non-oscillatory integrals ($\mathfrak{K}_\sigma = \{z(0)\,|\,z(+\infty) = z_\sigma\}$).
(Witten, arXiv:1001.2933 [hep-th])

3. Charge conjugation symmetry

Reality of observables, Charge conjugation symmetry:
Charge conjugation $C : (x_i) \mapsto (C_{ij}x_j)$ acts on the action as

$$\overline{S(x)} = S(C\cdot x).$$

This symmetry ensures $Z \in \mathbb{R}$:

$$\overline{Z} = \int \mathrm{d}x \mathrm{e}^{-\overline{S}(x)} = \int \mathrm{d}x \mathrm{e}^{-S(C\cdot x)} = Z.$$

(Nishimura, Ogilvie, Pangeni, PRD90, 045039 (2014))
The linear map C on $\mathbb{R}^n$ can be extended to an antilinear map on $\mathbb{C}^n$ by

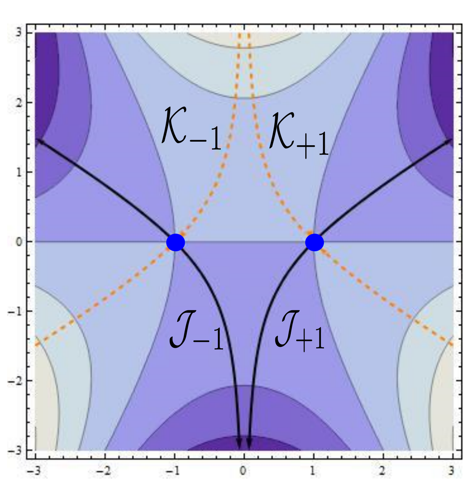
$$CK : (z_i) \mapsto (C_{ij}\overline{z_j}).$$

Morse equation for $\tilde{z} := CK(z)$ is

$$\frac{\mathrm{d}\tilde{z}_i}{\mathrm{d}t} = C_{ij}\cdot\overline{\left(\frac{\partial S(\tilde{z})}{\partial \tilde{z}_j}\right)} = \overline{\left(\frac{\partial S(\tilde{z})}{\partial \tilde{z}_i}\right)},$$

which is nothing but the original flow equation.
⇒ The Lefschetz thimble decomposition manifestly ensures the **real** partition function.
(Tanizaki, Nishimura, Kashiwa, PRD 91, 101701 (2015))

Example:
For $S(x) = \mathrm{i}(x^3/3 - x)$, the Lefschetz thimbles has the symmetry under $x \mapsto -\overline{x}$:



4. Polyakov-loop effective model

The fundamental Polyakov loop $\ell_{\mathbf{3}}$ is an order parameter of confinement;

$$\ell_{\mathbf{3}} = \frac{1}{3}\mathrm{tr}\,[\boldsymbol{L}], \quad \boldsymbol{L} = \mathcal{P}\exp\left(\mathrm{i}g\int_0^\beta A_4 \mathrm{d}x^4\right).$$

The Polyakov line $\boldsymbol{L}$ is diagonalized as

$$\boldsymbol{L} = \frac{1}{3}\mathrm{diag}\left[\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}, \mathrm{e}^{\mathrm{i}(-\theta_1+\theta_2)}, \mathrm{e}^{-2\mathrm{i}\theta_2}\right].$$

Let us consider the $SU(3)$ matrix model:

$$Z_{\mathrm{QCD}} = \int \mathrm{d}\theta_1\mathrm{d}\theta_2 H(\theta_1,\theta_2)\exp\left[-S_{\mathrm{eff}}(\theta_1,\theta_2)\right],$$

where $H = \sin^2\theta_1 \sin^2((\theta_1+3\theta_2)/2)\sin^2((\theta_1-3\theta_2)/2)$ and

$$S_{\mathrm{eff}}(\ell) \simeq V\left[e^{\mu}\ell(\theta_1,\theta_2) + e^{-\mu}\overline{\ell}(\theta_1,\theta_2)\right].$$

Charge conjugation symmetry is established by replacement

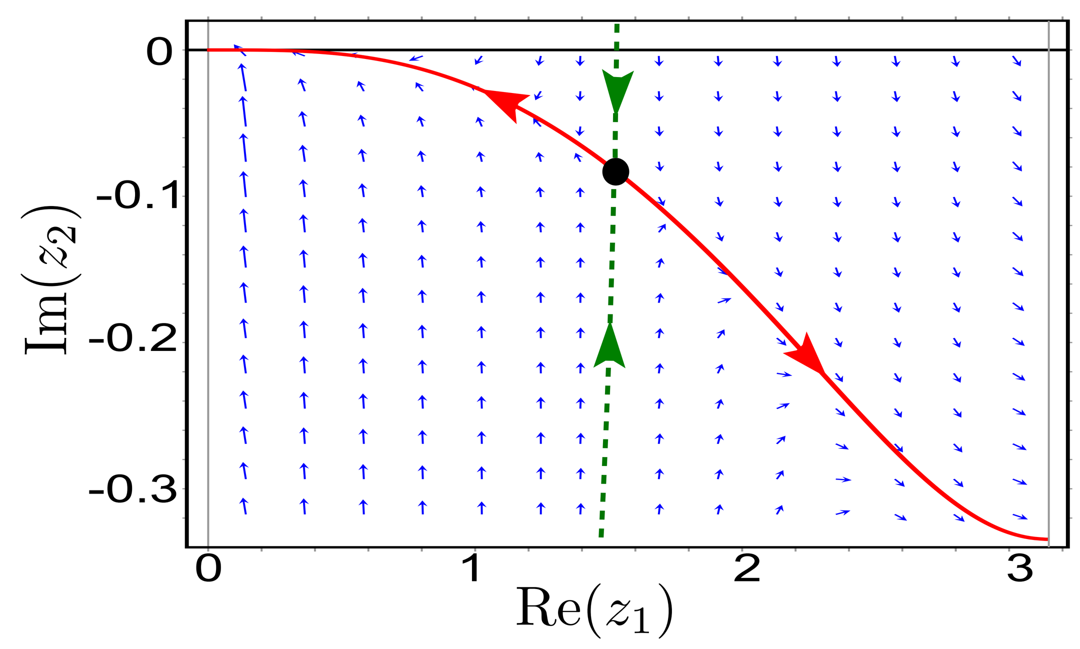
$$\boldsymbol{L}(\theta_1,\theta_2) \leftrightarrow \boldsymbol{L}^\dagger(\theta_1,-\theta_2).$$

In this parametrization,

$$\begin{aligned} S_{\mathrm{eff}} - \ln H &= -\frac{8h}{3}(2\cos\theta_1\cos(\theta_2 - \mathrm{i}\mu) + \cos(2\theta_2 + \mathrm{i}\mu)) \\ &\quad - \ln\left[\sin^2\theta_1 \sin^2\left(\frac{\theta_1+3\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-3\theta_2}{2}\right)\right]. \end{aligned}$$

5. Polyakov loops at finite densities

Behaviors of Morse flow:



In this model, the saddle point itself is charge-conjugation invariant.

$$\mathrm{Im}(z_1) = 0,\ \mathrm{Re}(z_2) = 0.$$

Complex saddle-point approximation:
Polyakov loops are CK-invariant, and thus $\langle\ell\rangle$, $\langle\overline{\ell}\rangle$ are real.

$$\ell \simeq \frac{1}{3}(2e^{\mathrm{i}z_2}\cos\theta_1 + e^{-2\mathrm{i}z_2}),$$

Saddle-point approximation gives

$$\langle\ell_{\overline{\mathbf{3}}}\rangle - \langle\ell_{\mathbf{3}}\rangle \simeq \frac{2}{3}(\sinh 2\mathrm{i}z_2^* - 2\cos z_1^* \sinh \mathrm{i}z_2^*) > 0.$$

Difference between two Polyakov loops at finite chemical potential can be captured correctly.
(Tanizaki, Nishimura, Kashiwa, PRD 91, 101701 (2015))

6. Conclusions & Future Prospects

Conclusion

- General framework of Lefschetz thimbles is established to ensure the reality of observables.
- This theorem applies to finite-density QCD and its effective models, since

$$\begin{aligned} \overline{\det\gamma_\mu D_\mu(\mu_{\mathrm{qk}}, A)} &= \det\gamma_\mu D_\mu(-\mu_{\mathrm{qk}}, A^\dagger) \\ &= \det\gamma_\mu D_\mu(\mu_{\mathrm{qk}}, -\overline{A}). \end{aligned}$$

  It is CK-symmetric, and the Lefschetz-thimble decomp. respects it.

Future Prospects

- Studying phase diagram of Polyakov-loop models using Lefschetz thimbles.
- We must ensure not only reality but also positivity of Z to satisfy thermodynamics.
- Silver Blaze phenomenon at finite densities is a big goal for ab initio approach.
- Two aspects of Lefschetz thimbles seem to be important for Silver Blaze:
  1. Stokes jumps of Lefschetz thimbles.
  2. Interference of multiple Lefschetz thimbles.