**Introduction**

- Understanding the dynamics at the critical point is important to set the QCD critical point.
- Dynamic universality classes are much more complicated than static ones.
- Real-time functional RG is a new possibility to treat dynamic fluctuations in a systematic manner.

Closed-time path (CTP) formalism:

\[
Z = \text{tr} \left[ \rho_{\mathcal{D}} \right] = \text{tr} \left[ e^{-i \mathcal{H} t} \rho_{\mathcal{D}}(0) e^{i \mathcal{H} t} \right] = \int \mathcal{D} \varphi \mathcal{D} \varphi' \left( \langle \varphi(r) \varphi'(r) \rangle - \delta(r-r') \right) \exp \left\{ S[\varphi] - S[\varphi'] \right\}.
\]

Microscopic action is given by the relativistic O(N) model:

\[
S[\varphi] = \int dt \, \delta^4 \left( \frac{1}{2} (\partial_\mu \varphi_\mu)^2 - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4} (\varphi^2)^2 \right).
\]

Goals:

- Develop the renormalization group for CTP effective actions: Real-time FRG.
- Compute the dynamic critical phenomenon using the real-time FRG.

**Fluctuation-Dissipation Theorem & Scaling Operators**

Introduce classical and quantum fields

\[ \varphi = (\varphi_+ + \varphi_-)/2, \quad \tilde{\varphi} = \varphi_+ - \varphi_- \]

**Fluctuation-Dissipation Theorem:** Retarded and statistical propagators are closely related.

\[ \langle \varphi(p) \varphi(-p) \rangle = \coth \frac{\beta p^2}{2} \Im \left\{ i \tilde{\varphi}(p) \tilde{\varphi}(-p) \right\} \]

We require that the coarse-graining procedure respect FDT at any RG scales [1].

Take the derivative expansion of the inverse retarded propagator.

\[ \Gamma^{(2)}(x/y) = \begin{cases} \zeta^2 - \zeta^2 \Lambda^2 - \frac{\zeta^2}{2} & \text{if } \zeta \Lambda^2 \ll 1 \\ \zeta^2 & \text{if } \zeta \Lambda^2 \gg 1 \end{cases} \]

The same \( \Omega \) must appear thanks to FDT!

**Local interaction approximation:** The interaction part of the CTP effective action is assumed to be local,

\[ \mathcal{U} = m^2 \varphi_+ \varphi_- + \lambda_1 \left( \varphi_+ - \varphi_- \right)^2 + \lambda_2 \varphi_+ \varphi_- + \lambda_3 \varphi_+ \varphi_- \]

with the O(N) invariants,

\[ \sigma_1 = \frac{1}{2} \delta^4 \varphi_0, \quad \sigma_2 = \varphi_0^2 \varphi_0, \quad \sigma_3 = \frac{1}{2} \varphi_0^2 \varphi_0 \]

Another term, such as \( \lambda_1 \sigma_1 \sigma_2 \),

may be included as a local interaction from the viewpoint of symmetry, but FDT states that \( \lambda_1 = 0 \).

In the UV region (or \( T = 0 \)), furthermore, FDT gives another constraint:

\[ \lambda_3 = 4 \lambda_2 = \lambda/3. \]

**Functional Renormalization Group**

FRG modifies the CTP action as \( S[\varphi, \tilde{\varphi}] \to S[\varphi, \tilde{\varphi}] + \Delta S[\varphi, \tilde{\varphi}] \), where

\[ \Delta S[\varphi, \tilde{\varphi}] = - \int \frac{d^4 x}{(2\pi)^4} \varphi(x) R(x, y) \tilde{\varphi}(y). \]

The function \( R(y) \) is a momentum-dependent mass term, i.e.

\[ R_{\text{opt}}(y) = \left( \frac{1}{2} y^2 + m_0^2 \right)^2. \]

The initial density matrix must also be modified to respect FDT.

**Renormalization Group equation:**

The 1PI effective action \( \Gamma_k \) follows the one-loop exact RG equation [2]:

\[ \frac{\partial}{\partial \ln k} \Gamma_k = k H \left( \frac{\partial}{\partial \ln k} \Gamma_k(\rho_k) \right) \]

with \( s = \ln k/\Lambda \). It connects quantum and classical effective actions:

\[ \Gamma_k \to \Gamma_0 \quad \text{as} \quad s \to -\infty, \quad \Gamma_k \to S \quad \text{as} \quad s \to 0. \]

**Real-time RG Flow**

**Scaling properties:** At low energies and momenta, we assume the scaling property

\[ \omega \sim |p|^{z}. \]

Scale-invariance of the lowest order derivatives tells the scaling dimensions of classical/quantum fields:

\[ |\varphi| = \frac{1}{2} (d - 2 + \eta^2), \quad |\tilde{\varphi}| = |\varphi| + \eta. \]

The scaling dim. of quantum fields is larger than that of classical fields by \( \eta \).

\[ z = 1 \quad \text{UV fixed point} \]

\[ z = 2 + c \eta \quad \text{IR fixed point} \]

**RG Flow at the critical temperature:**

\[ T_{\lambda_{\text{opt}}}; \quad (N = 1 \text{ at } T \neq 0) \]

As the RG scale is lowered, \( Z \) goes inside the negative region. This behaviors is out of the scope of our truncation, and we put \( Z^{\text{IR}} = 0 \) at low-energy scales.

**Dynamic scaling exponent:**

\[ z \approx 2.05 \]

**Expansion around the upper critical dimension:**

Set \( d = 4 - \epsilon \). One can check the regulator dependence of static and dynamic scaling exponents:

\[ r_{\text{opt}}^{(\eta)} = \left( \frac{1}{2} - 3 \eta + \epsilon \right), \quad r_{\text{opt}}^{(\eta^{\text{eff}})} = \left( \frac{1}{2} - 3 \eta + \epsilon \right) \]

**Microscopic QFT and the macroscopic dynamics (model A) is connected within our ansatz.**

**What’s the next step?**

The time-development is governed by Hamiltonian. There are many conserved quantities:

- Energy-momentum tensor: \( T^{\mu\nu} \)
- Noether charge (Baryon number): \( J^0 \)
- Mode couplings between \( \varphi \) and \( J^0 \), \( T^{\mu\nu} J^0 \) affects the dynamic critical behavior significantly [3].

1. We must develop a new technique of FRG in order to treat the hydrodynamic modes governed by composite operators \( T^{\mu\nu} J^0 \). Can we describe model H in our approach?

2. If the experimental length scale \( \zeta \ll 1/m_0 \), pions behave as massless and affects critical behaviors. What happens if \( \zeta \sim 1/m_0 \)?

Developing microscopic approaches, we will be able to answer these questions.

**References**