Resumming large radiative corrections in the high-energy evolution of the Color Glass Condensate

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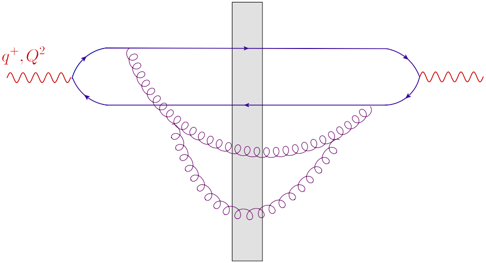
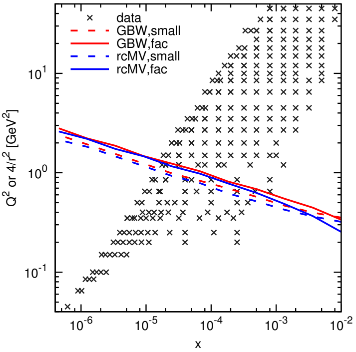




Introduction

- The CGC effective theory: the pQCD description for the wavefunction of an energetic hadron and its interactions
  - non-linear effects: gluon saturation, multiple scattering
- "Effective theory" : high-energy evolution described by pQCD
  - Balitsky-JIMWLK hierarchy $\approx$ BK equation (at large N_c)
  - non-linear generalizations of the BFKL equation
- The non-linear evolution has recently been promoted to NLO
  - essential for a realistic phenomenology

(Balitsky, Chirilli, 2008, 2013; Kovner, Lublinsky, Mulian, 2013)

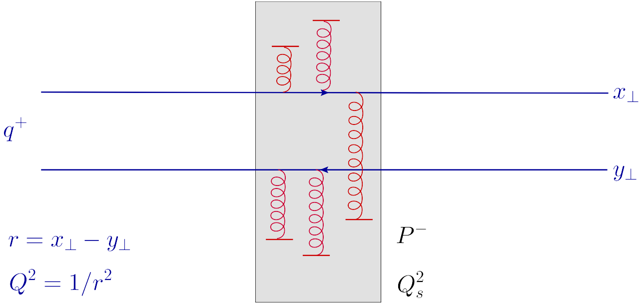
- Large corrections enhanced by double or single transverse logarithms
  - lack of convergence, unstable evolution at NLO
- Similar problems encountered & solved for NLO BFKL
  - collinear resummations, formulated in Mellin space

(Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03)

Introduction

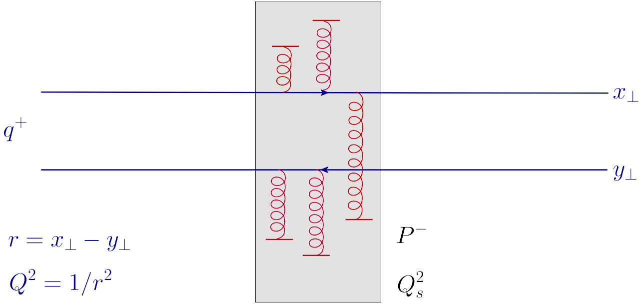
- Mellin representation is not suitable beyond the linear approximation
- Non-linear effects (multiple scattering) most naturally discussed in terms of transverse coordinates
  - eikonal approximation, Wilson lines
- Alternative resummation, formulated in transverse coordinates
  (E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, 2015)
  - direct calculation of Feynman graphs (light-cone perturbation theory)
    (arXiv:1502.05642, Phys.Lett. B744 (2015) 293)
  - transparent physical interpretation
  - promising phenomenology (so far, only DIS)
    (arXiv:1507.03651, Phys.Lett. B, to appear)
  - extension to pp and pA (at least) is straightforward

Dipole–hadron scattering ($\gamma^* p$, $\gamma^* A$, pA, ...)



- Dipole: large q^+, transverse size r, transverse resolution $Q^2 = 1/r^2$
- Target: large P^-, high gluon density (CGC), saturation momentum Q_s^2
- Elastic S-matrix $S(r)$ $\sim$ dipole survival probability
  - scattering is weak $(S(r) \simeq 1)$ when the dipole is small: $Q^2 \gg Q_s^2$
  - scattering amplitude $T(r) \equiv 1 - S(r)$: $T(r) \propto r^2$ as $r \ll 1/Q_s$

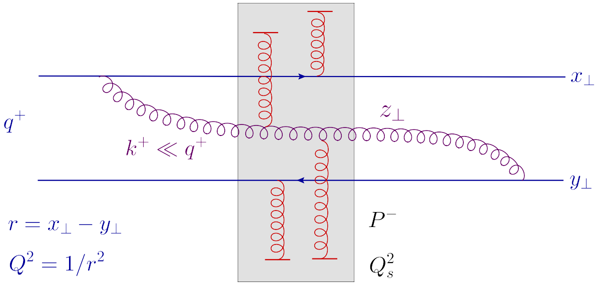
Dipole–hadron scattering ($\gamma^* p$, $\gamma^* A$, pA, ...)



- Dipole: large q^+, transverse size r, transverse resolution $Q^2 = 1/r^2$
- Target: large P^-, high gluon density (CGC), saturation momentum Q_s^2
- Elastic S-matrix $S(r)$ $\sim$ dipole survival probability
  - scattering is strong $(S(r) \ll 1)$ when the dipole is large: $Q^2 \gtrsim Q_s^2$
  - unitarity bound ('black disk limit') : $T(r) \leq 1$

High energy evolution

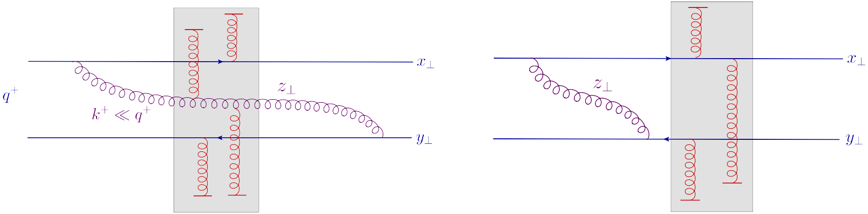
- Probability $\sim \alpha_s \ln \frac{1}{x}$ to radiate a soft gluon with $x \equiv \frac{k^+}{q^+} \ll 1$



$$\frac{2xq^+}{Q^2} \sim \frac{1}{P^-} \Longrightarrow x \simeq \frac{Q^2}{s} \qquad (s = 2q^+P^-)$$

- When $\alpha_s \ln \frac{1}{x} \sim 1$, need for resummation: $(\alpha_s Y)^n$ with $Y \equiv \ln \frac{1}{x}$
  - BFKL evolution of the dipole in the background of the dense target
  - multiple scattering $\Longrightarrow$ non-linear evolution $\Longrightarrow$ Balitsky-JIMWLK

The BK equation *(Balitsky, '96; Kovchegov, '99)*



- Large N_c : the original dipole splits into two new dipoles

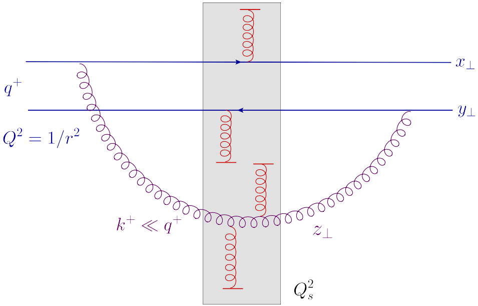
$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}} \right]$$

  - 'dipole kernel' : probability for the dipole to split *(Al Mueller, 1990)*
  - 'real term' : the soft gluons exists at the time of scattering
  - 'virtual term' : initial (or final) state evolution

- Mean field approximation to the Balitsky-JIMWLK hierarchy

Double-logarithmic approximation: DLA 1.0

- Large transverse separation between projectile and target: $Q^2 \gg Q_s^2$

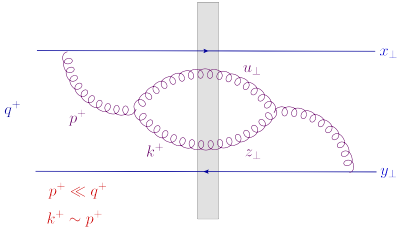


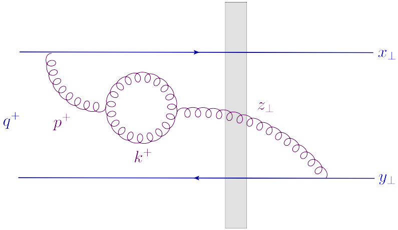
- Large transverse phase-space for gluon emission at $r \ll z \ll 1/Q_s$
  - scattering is still weak: $T \equiv 1 - S \ll 1$ for all dipoles
  - the daughter dipoles scatter stronger (since larger): $T(z) \gg T(r)$

$$\frac{\partial}{\partial Y}\frac{T(r^2)}{r^2} \simeq \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{\mathrm{d}z^2}{z^2}\frac{T(z^2)}{z^2} \implies \Delta T \sim \bar{\alpha}_s Y \ln\frac{Q^2}{Q_s^2}\, T$$

Next-to-leading order

- Any effect of $\mathcal{O}(\bar{\alpha}_s^2 Y) \Longrightarrow \mathcal{O}(\bar{\alpha}_s)$ correction to the BFKL kernel





- The prototype: two successive emissions, one soft and one non-soft
- The maximal correction thus expected: $\mathcal{O}(\bar{\alpha}_s \rho)$ with $\rho \equiv \ln(Q^2/Q_s^2)$
- But one finds an even larger effect: $\mathcal{O}(\bar{\alpha}_s \rho^2)$ ('double collinear log')
- Originally found as a NLO correction to the BFKL kernel

  (Fadin, Lipatov, Camici, Ciafaloni ... 95-98; Balitsky, Chirilli, 07)

- Very complicated in full generality
- Here: $N_{\rm f} = 0$, large $N_{\rm c}$, tiny fonts

$$\frac{{\rm d}S_{\boldsymbol{xy}}}{{\rm d}Y} = \frac{\bar\alpha_s}{2\pi}\int {\rm d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

$$+\,\bar\alpha_s\left[\bar b\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar b\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\,\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}\right.$$

$$\left.+\,\frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

$$+\,\frac{\bar\alpha_s^2}{8\pi^2}\int\frac{{\rm d}^2\boldsymbol{u}\,{\rm d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\,\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\,\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

Deconstructing NLO BK

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

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$$+\,\frac{\bar{\alpha}_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- blue : leading-order (LO) terms
- red : NLO terms enhanced by (double or single) transverse logarithms
- black : pure $\bar{\alpha}_s$ corrections (no logarithms)

Deconstructing NLO BK

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

$$+\,\bar{\alpha}_s\Bigg[\bar{b}\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

$$+\,\frac{67}{36}-\frac{\pi^2}{12}-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\Bigg]\Bigg\}$$

$$+\,\frac{\bar{\alpha}_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

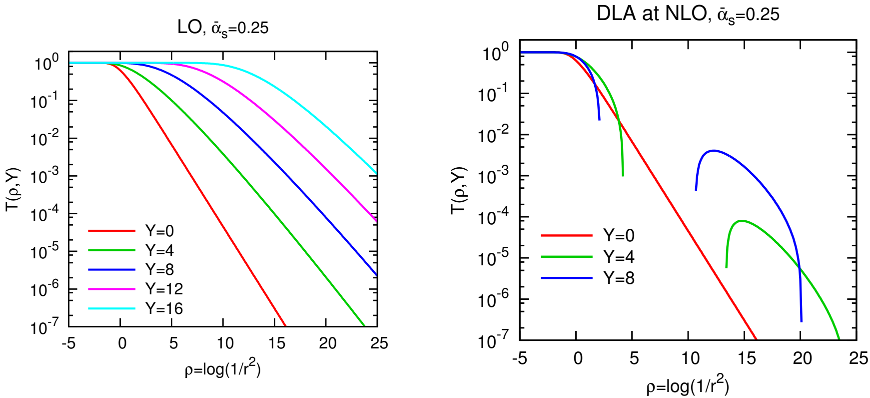
$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- Keeping just the logarithmically enhanced terms ($z \gg r$, weak scattering)

$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} \simeq \bar{\alpha}_s\int \mathrm{d}z^2\,\frac{r^2}{z^4}\left\{1-\bar{\alpha}_s\left(\frac{1}{2}\ln^2\frac{z^2}{r^2}+\frac{11}{12}\ln\frac{z^2}{r^2}-\bar{b}\,\ln r^2\mu^2\right)\right\}T(z)$$

NLO : unstable numerical solutions

- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2 Q_0^2)$ with increasing Y



- Left: leading-order BK : the saturation front
  - weak scattering at large ρ (small r) : $T \propto r^2 = \mathrm{e}^{-\rho}$
  - unitarity limit at small ρ (large r) : $T = 1$
  - transition at the saturation scale: $T(Y, r) \sim 1$ when $r = 1/Q_s(Y)$

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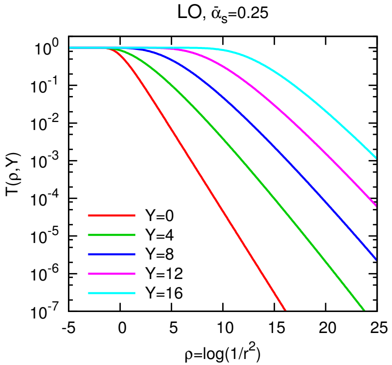
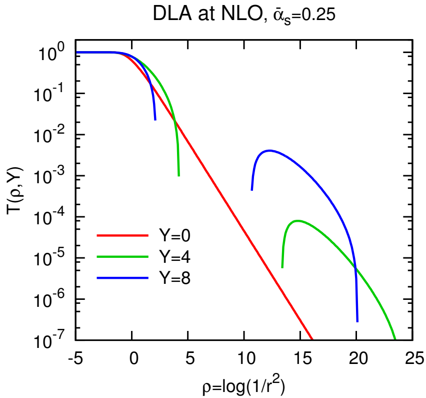
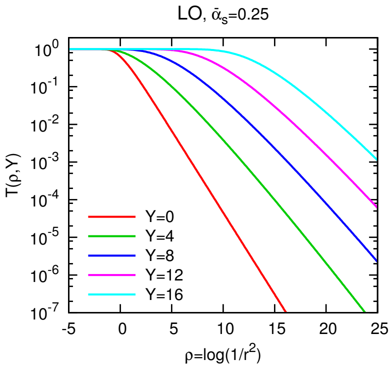


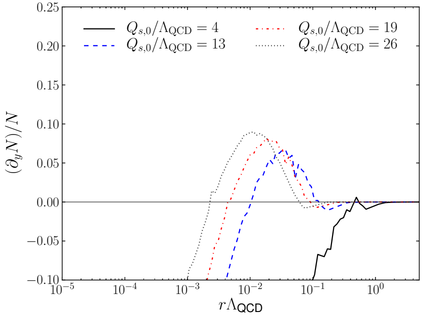


- Left: leading-order BK : the saturation front
- Right: LO BK + the double collinear logarithm at NLO
  (our calculation, arXiv:1502.05642)

NLO : unstable numerical solutions

- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2 Q_0^2)$ with increasing Y





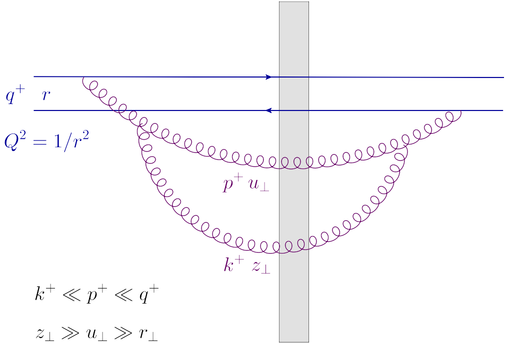
- Left: leading-order BK : the saturation front
- Right: full NLO BK : evolution speed $(\partial_Y T)/T$

  (Lappi, Mäntysaari, arXiv:1502.02400)

- The main source of instability: the double collinear logarithm

The double collinear logarithm

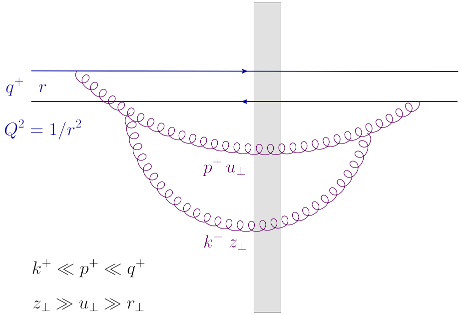
- Two successive emissions which are strongly ordered in both ...



  - longitudinal momenta : $q^+ \gg p^+ \gg k^+$
  - ... and transverse sizes (or momenta): $r_\perp^2 \ll u_\perp^2 \ll z_\perp^2 \ll 1/Q_s^2$
- "Two iterations of DLA 1.0 $\Longrightarrow \mathcal{O}((\bar{\alpha}_s Y \rho)^2)$" ... not exactly !
  - additional constraint due to time ordering: $\tau_p > \tau_k$

Time ordering

- Heisenberg: fluctuations have a finite lifetime $\tau_k \sim \frac{k^+}{k_\perp^2} \sim k^+ z_\perp^2$



$$p^+ u_\perp^2 > k^+ z_\perp^2 \implies \Delta Y \equiv \ln \frac{p^+}{k^+} > \Delta\rho \equiv \ln \frac{z_\perp^2}{u_\perp^2}$$

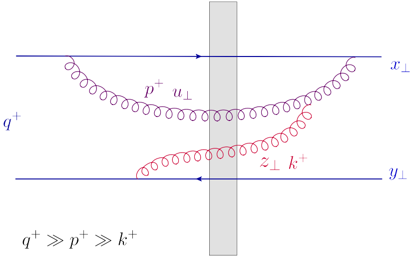
- Additional restriction on the DLA phase-space ⇒ double collinear logs

$$Y > \rho \implies \bar{\alpha}_s Y \rho \longrightarrow \bar{\alpha}_s (Y - \rho)\rho = \bar{\alpha}_s Y \rho - \bar{\alpha}_s \rho^2$$

- Time ordering enters perturbation theory via energy denominators

Time-ordered (light-cone) perturbation theory

- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions
- The time (x^+) integrals yield energy denominators



- time-ordered graphs

$$\frac{1}{p^- + k^-} = \frac{\tau_p \tau_k}{\tau_p + \tau_k}$$

- to have double logs, one needs

$$\tau_p \gg \tau_k$$

- Integrate out the harder gluon ($p^+, u_\perp$) to DLA :

$$\bar{\alpha}_s \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+} \Theta(p^+ u^2 - k^+ z^2) = \bar{\alpha}_s Y \rho - \frac{\bar{\alpha}_s \rho^2}{2}$$

DLA 2.0

- The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within DLA 1.0

  (E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1502.05642)

$$\frac{\partial T(Y,r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{\mathrm{d}z^2}{z^2}\, \frac{r^2}{z^2}\, T(Y,z^2)$$

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- Non-local in Y

- DLA 2.0: Resums all the powers of $\bar{\alpha}_s Y \rho$ and $\bar{\alpha}_s \rho^2$

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- The importance of time-ordering had already been recognized

  Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)

- The diagrammatic foundation in pQCD was not properly appreciated

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- The diagrammatic foundation in pQCD was not properly appreciated
- So far, no change in the kernel: double-logs come from non-locality

DLA 2.0

- The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within DLA 1.0

  (E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1502.05642)

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- Non-local in Y
- DLA 2.0: Resums all the powers of $\bar{\alpha}_s Y \rho$ and $\bar{\alpha}_s \rho^2$
- The importance of time-ordering had already been recognized

  Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)
- The diagrammatic foundation in pQCD was not properly appreciated
- Equivalently: a local equation but with an all-order resummed kernel

The collinearly improved BK equation

- The argument extends beyond DLA, that is, to BFKL/BK equations
- The generalized, non-local, BK equation (with the usual kernel, but with time-ordering) is equivalent to the following, local, equation:

$$\frac{\mathrm{d}\tilde{S}_{\boldsymbol{xy}}}{\mathrm{d}Y} = \bar{\alpha}_s \int \frac{\mathrm{d}^2\boldsymbol{z}}{2\pi} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \mathcal{K}_{\mathrm{DLA}}\big(\bar{\rho}^2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\big) \big(\tilde{S}_{\boldsymbol{xz}}\tilde{S}_{\boldsymbol{zy}} - \tilde{S}_{\boldsymbol{xy}}\big)$$

... with the all-order resummed kernel: *(see also Sabio-Vera, 2005)*

$$\mathcal{K}_{\mathrm{DLA}}(\rho^2) \equiv \frac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s\rho^2}\big)}{\sqrt{\bar{\alpha}_s\rho^2}} = 1 - \frac{\bar{\alpha}_s\rho^2}{2} + \frac{(\bar{\alpha}_s\rho^2)^2}{12} + \cdots$$

... and the symmetrized version of the collinear double-log:

$$\bar{\rho}^2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

- The first correction, of $\mathcal{O}(\bar{\alpha}_s\bar{\rho}^2)$, coincides with the NLO double-log

Extending to single-logs & running coupling

- Recall the NLO equation with all the transverse logs

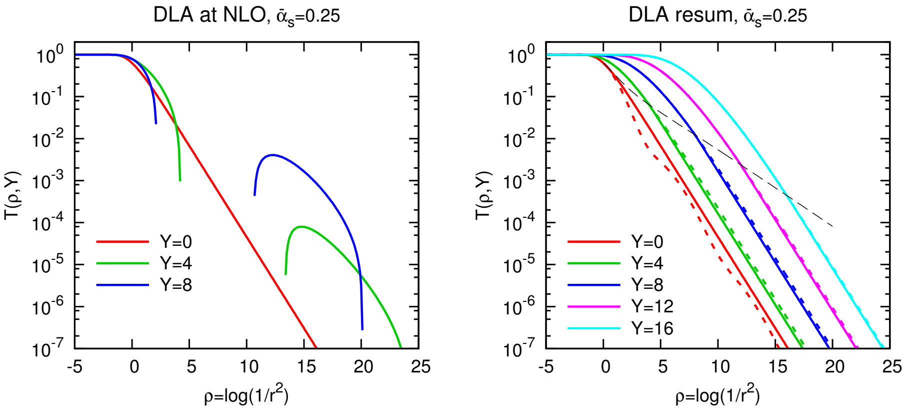
$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} = \bar{\alpha}_s \int \mathrm{d}z^2 \, \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left(\frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} - \bar{b} \, \ln r^2 \mu^2 \right) \right\} T(z)$$

  - the double-logarithm is already included within $\mathcal{K}_{\text{DLA}}(\rho)$ ✓
  - the collinear single-log is part of the DGLAP anomalous dimension ✓
  - the running coupling log is resummed by replacing $\bar{\alpha}_s \to \bar{\alpha}_s(r_{\min})$ ✓

$$\frac{\mathrm{d}\tilde{S}_{\boldsymbol{xy}}}{\mathrm{d}Y} = \int \frac{\mathrm{d}^2 \boldsymbol{z}}{2\pi} \, \bar{\alpha}_s(r_{\min}) \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2 (\boldsymbol{z}-\boldsymbol{y})^2} \left[\frac{r^2}{z_<^2} \right]^{\pm A_1 \bar{\alpha}_s} \mathcal{K}_{\text{DLA}}\big(\bar{\rho}^2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\big) \times \big(\tilde{S}_{\boldsymbol{xz}} \tilde{S}_{\boldsymbol{zy}} - \tilde{S}_{\boldsymbol{xy}} \big)$$

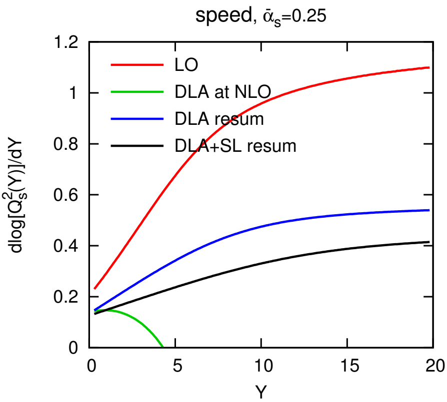
$$A_1 \equiv \frac{11}{12}, \qquad z_<^2 \equiv \min\{(\boldsymbol{x}-\boldsymbol{z})^2, (\boldsymbol{y}-\boldsymbol{z})^2\}$$

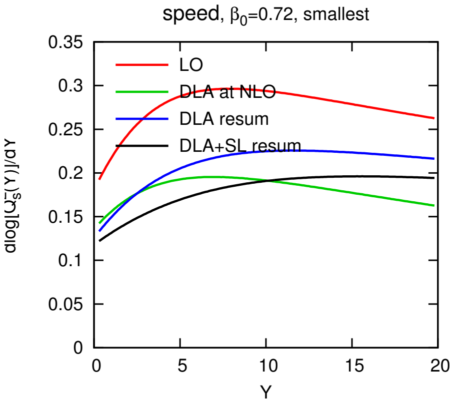
Numerical solutions: saturation front



- Fixed coupling $\bar{\alpha}_s = 0.25$, double collinear logs alone
  - left: expanded to NLO
  - right: resummed to all orders
- The resummation stabilizes & slows down the evolution

Saturation exponent $\lambda_s \equiv \mathrm{d}\ln Q_s^2/\mathrm{d}Y$





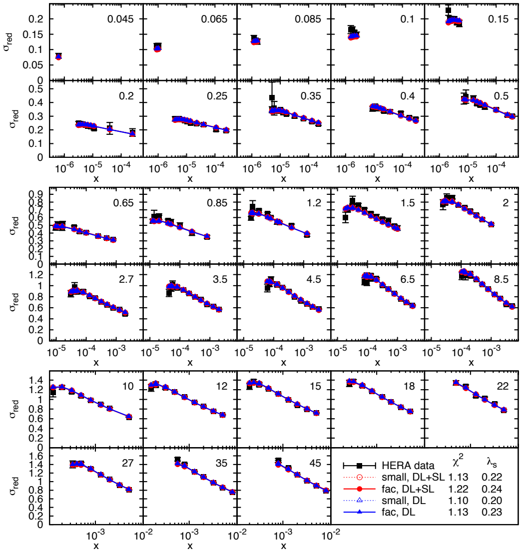
- Fixed coupling
  - LO: $\lambda_s \simeq 4.88\bar{\alpha}_s \simeq 1$
  - resummed DL: $\lambda_s \simeq 0.5$
  - DL + SL: $\lambda_s \simeq 0.4$
- Running coupling
  - LO: $\lambda_s = 0.25 \div 0.30$
  - DL + SL: $\lambda_s \simeq 0.2$
  - better convergence

Fitting the HERA data

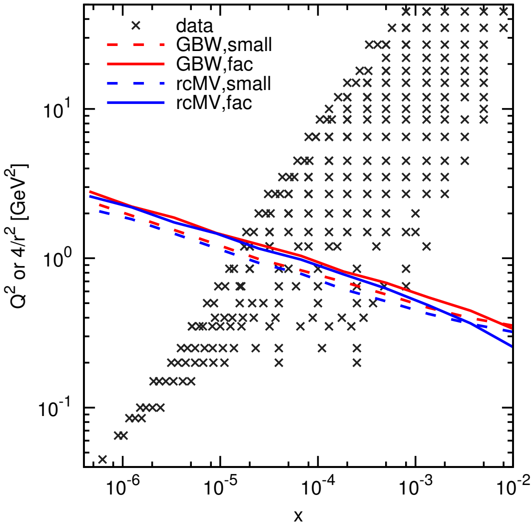
(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1507.03651)

- The most recent analysis of HERA data: very small error bars
  - Bjoerken' $x \leq 0.01$
  - $Q^2 < Q^2_{\max}$ with $Q^2_{\max} = 50 \div 400$ GeV^2
- Numerical solutions to the collinearly-improved BK equation with initial conditions (at $x_0 = 0.01$) which involve 4 free parameters
- 3 light quarks + charm quark, all treated on the same footing
  - good quality fits for $m_{u,d,s} = 0 \div 140$ MeV and $m_c = 1.3$ or 1.4 GeV
- Good quality fits: χ^2 per point around 1.1-1.2
- Very discriminatory: the fits favor
  - initial condition: MV model with running coupling
  - smallest-dipole prescription for the running
  - physical values for the free parameters

The HERA fit: rcMV initial condition



The HERA fit: rcMV initial condition

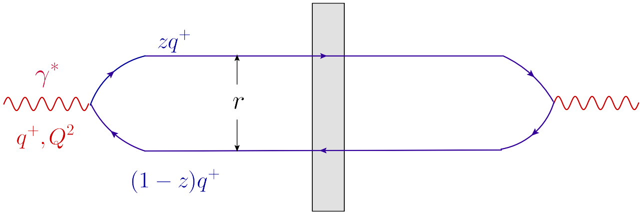


- Saturation line $Q_s^2(x)$ on top of the experimental data points
  - saturation exponent: $\lambda_s = 0.20 \div 0.24$

Conclusions & Perspectives

- The high energy evolution of a color dipole beyond leading order is by now under control
- The generalization to more complicated (still dilute) projectiles — proton, color quadrupole, ... — is straightforward
- The generalization to full JIMWLK is less obvious
  - essential for applications to AA collisions
- Similar 'NLO and beyond' calculations also needed for impact factors
- Previous attempts (single-inclusive particle production in pA) lead to unstable results (negative cross-section at $p_\perp > Q_s$)
  (Chirilli, Watanabe, Xiao, Yuan, Zaslavsky, 2012-2015)
- Still many open problems ... but we are likely close to having an accurate description of high-energy scattering in pQCD

Back-up Slides



$$\sigma_{\gamma^* p}(Q^2, x) = 2\pi R_p^2 \sum_f \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \left|\Psi_f(r, z; Q^2)\right|^2 T(r, x)$$

$$x = \frac{Q^2}{2P\cdot q} \simeq \frac{Q^2}{s} \ll 1 \quad \text{(Bjoerken' } x\text{)}$$

- $T(r, x)$: scattering amplitude for a $q\bar{q}$ color dipole with transverse size r
  - $r^2 \sim 1/Q^2$: the resolution of the dipole in the transverse plane
  - x : longitudinal fraction of a gluon from the target that scatters

Fitting the HERA data: initial conditions

- Use numerical solutions to collinearly-improved running-coupling BK equation using initial conditions which involve free parameters
  - a similar strategy as for the DGLAP fits
- Various choices for the initial condition at $x_0 = 0.01$:

$$\text{GBW :} \qquad T(Y_0, r) = \left\{1 - \exp\left[-\left(\frac{r^2 Q_0^2}{4}\right)^p\right]\right\}^{1/p}$$

$$\text{rcMV :} \quad T(Y_0, r) = \left\{1 - \exp\left[-\left(\frac{r^2 Q_0^2}{4}\,\bar{\alpha}_s(r)\left[1 + \ln\left(\frac{\bar{\alpha}_{\rm sat}}{\bar{\alpha}_s(r)}\right)\right]\right)^p\right]\right\}^{1/p}$$

- One loop running coupling with scale $\mu = 2C_\alpha/r$:

$$\bar{\alpha}_s(r) = \frac{1}{b_0 \ln\left[4C_\alpha^2/(r^2\Lambda^2)\right]}, \quad \text{with} \quad r = \min\{|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{x}-\boldsymbol{z}|, |\boldsymbol{y}-\boldsymbol{z}|\}$$

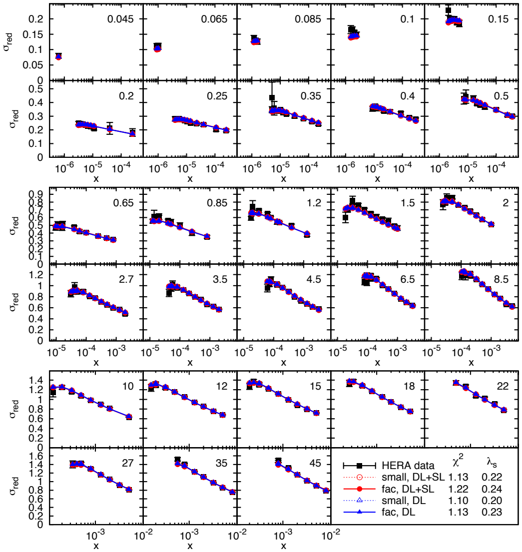
- 4 free parameters: R_p (proton radius), Q_0, p, C_α

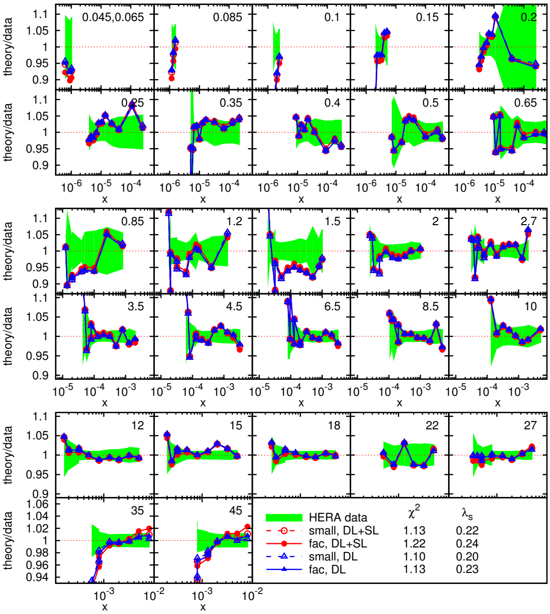
The HERA fit in tables

init cdt.	RC schm	sing. logs	χ^2 per data point $\sigma_{\rm red}$	$\sigma_{\rm red}^{cc}$	F_L	parameters R_p[fm]	Q_0[GeV]	C_α	p	$C_{\rm MV}$
GBW	small	yes	1.135	0.552	0.596	0.699	0.428	2.358	2.802	-
GBW	fac	yes	1.262	0.626	0.602	0.671	0.460	0.479	1.148	-
rcMV	small	yes	1.126	0.578	0.592	0.711	0.530	2.714	0.456	0.896
rcMV	fac	yes	1.222	0.658	0.595	0.681	0.566	0.517	0.535	1.550
GBW	small	no	1.121	0.597	0.597	0.716	0.414	6.428	4.000	-
GBW	fac	no	1.164	0.609	0.594	0.697	0.429	1.195	4.000	-
rcMV	small	no	1.097	0.557	0.593	0.723	0.497	7.393	0.477	0.816
rcMV	fac	no	1.128	0.573	0.591	0.703	0.526	1.386	0.502	1.015

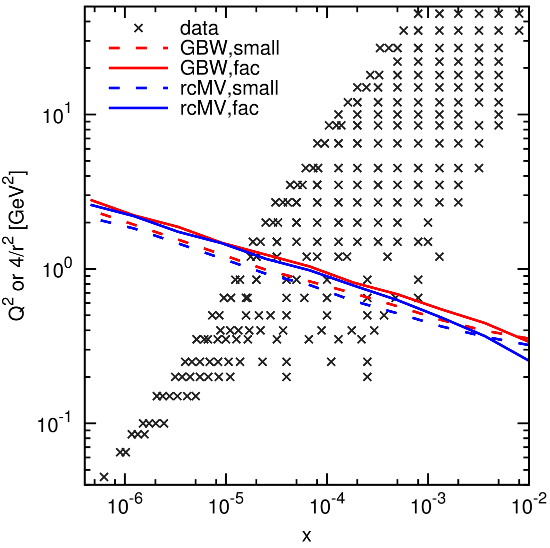
init cdt.	RC schm	sing. logs	χ^2/npts for $Q^2_{\rm max}$ 50	100	200	400
GBW	small	yes	1.135	1.172	1.355	1.537
GBW	fac	yes	1.262	1.360	1.654	1.899
rcMV	small	yes	1.126	1.172	1.167	1.158
rcMV	fac	yes	1.222	1.299	1.321	1.317
GBW	small	no	1.121	1.131	1.317	1.487
GBW	fac	no	1.164	1.203	1.421	1.622
rcMV	small	no	1.097	1.128	1.095	1.078
rcMV	fac	no	1.128	1.177	1.150	1.131

The Fit in plots

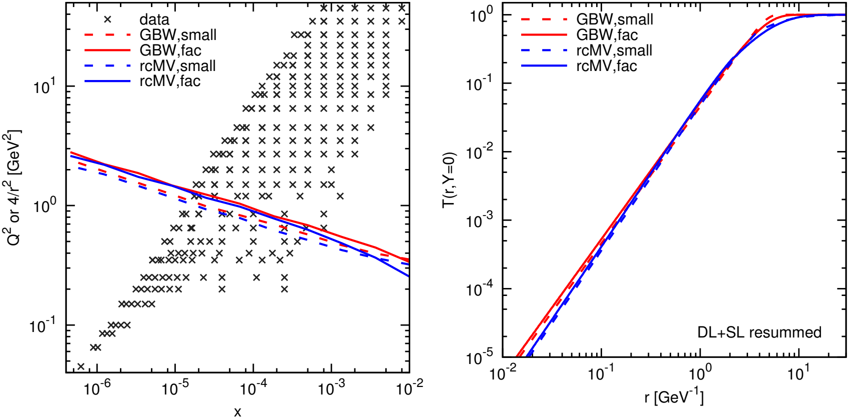



theory/data
x
0.045,0.065
0.085
0.1
0.15
0.2
0.25
0.35
0.4
0.5
0.65
0.85
1.2
1.5
2
2.7
3.5
4.5
6.5
8.5
10
12
15
18
22
27
35
45
HERA data
small, DL+SL
fac, DL+SL
small, DL
fac, DL
χ^2
λ_s
1.13 0.22
1.22 0.24
1.10 0.20
1.13 0.23

The Fit in plots

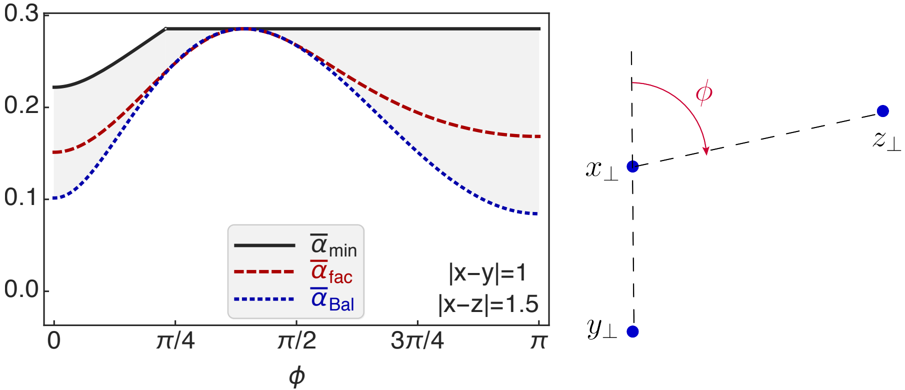


The Fit in plots



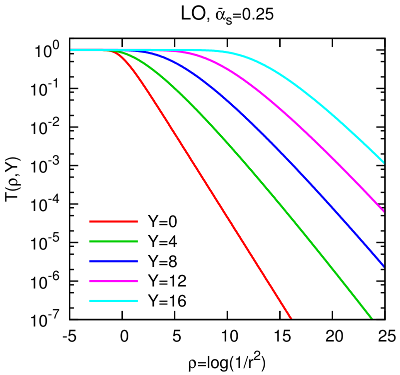
- Rather stable predictions for the saturation line and the shape of the initial amplitude

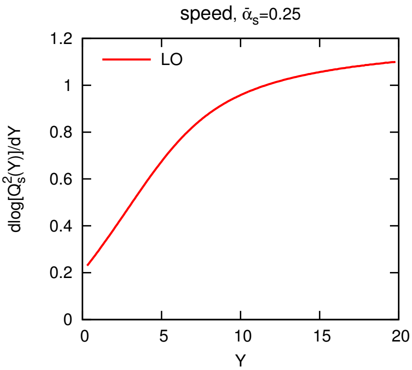
Prescriptions for running coupling



Numerical solutions: LO BK

- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2)$ with increasing Y



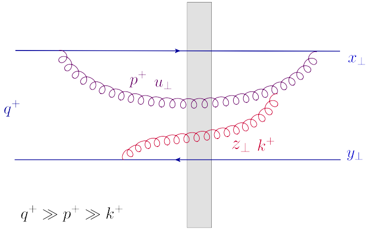


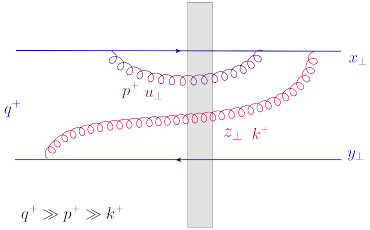
- color transparency at large ρ (small r) : $T \propto r^2 = \mathrm{e}^{-\rho}$
- unitarity limit at small ρ (large r) : $T = 1$
- saturation exponent (speed): $\lambda_s \equiv \frac{\mathrm{d}\ln Q_s^2}{\mathrm{d}Y} \simeq 1$ for $Y \gtrsim 10$

Light-cone perturbation theory

- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions

- time ordered graphs
- anti time-ordered graphs





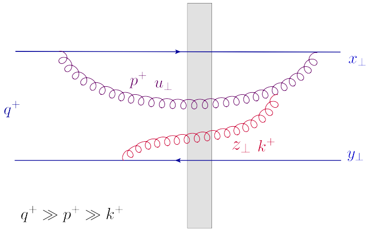
$$\frac{\tau_p}{\tau_p + \tau_k} \simeq \Theta(\tau_p - \tau_k)$$

$$\frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)$$

- The time (x^+) integrals yield energy denominators
- Integrate out the harder gluon $(p^+, u_\perp)$ to DLA : $r_\perp \ll u_\perp \ll z_\perp$

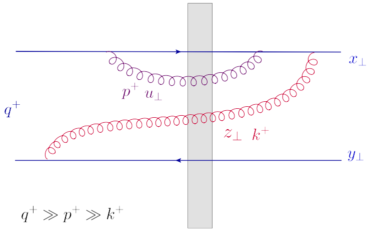
- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions

- time ordered graphs



$$\frac{\tau_p}{\tau_p+\tau_k} \simeq \Theta(\tau_p-\tau_k)$$

- anti time-ordered graphs



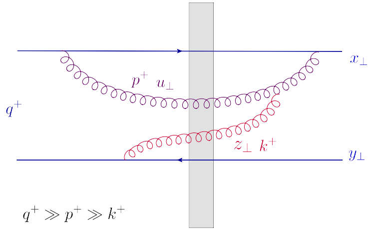
$$\frac{\tau_k}{\tau_p+\tau_k} \simeq \Theta(\tau_k-\tau_p)$$

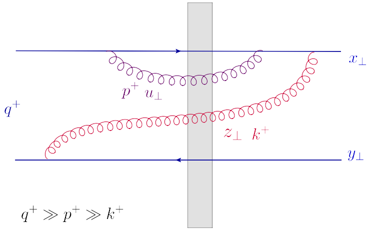
$$\text{TO}\ :\quad \bar{\alpha}_s\int_{r^2}^{z^2}\frac{\mathrm{d}u^2}{u^2}\int_{k^+}^{q^+}\frac{\mathrm{d}p^+}{p^+}\,\Theta(p^+u^2-k^+z^2) = \bar{\alpha}_s Y\rho - \frac{\bar{\alpha}_s\rho^2}{2}$$

Time-ordered (light-cone) perturbation theory

- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions

- time ordered graphs
- anti time-ordered graphs





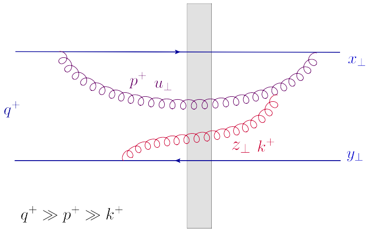
$$\frac{\tau_p}{\tau_p + \tau_k} \simeq \Theta(\tau_p - \tau_k) \qquad\qquad \frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)$$

$$\text{ATO}: \quad \bar{\alpha}_s \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+} \Theta(k^+ z^2 - p^+ u^2) = \frac{\bar{\alpha}_s \rho^2}{2}$$

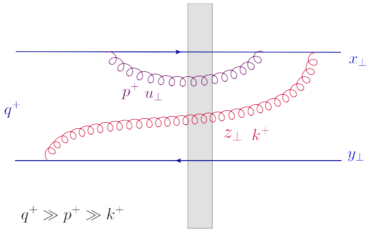
Time-ordered (light-cone) perturbation theory

- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$

- time ordered graphs



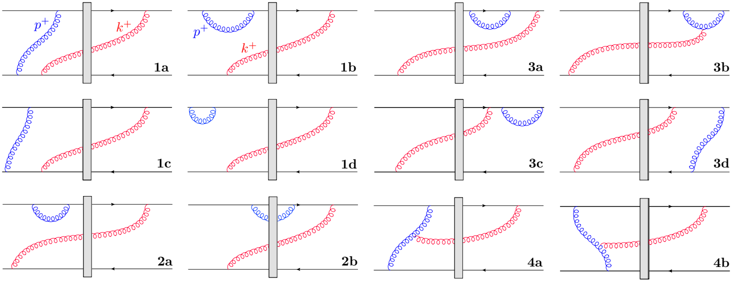
- anti time-ordered graphs



- TO graphs generate the expected LLA contributions: $\bar{\alpha}_s Y \rho$
- both TO and ATO graphs generate double collinear logs $\bar{\alpha}_s \rho^2$
- the latter precisely cancel in the sum of all the ATO graphs
- net result: the double-collinear logs come from TO graphs alone

(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1502.05642)

The Anti-Time-Ordered graphs



$$\frac{\tau_p}{\tau_p + \tau_k} \longrightarrow \frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)$$

- The softer gluon k^+ lives longer than the harder one p^+
- The DLA terms exactly cancel in the sum of all the ATO graphs
  - IR logs cancel between vertex (1a) and self-energy (1b) corrections
  - 'virtual' (2a) cancel against 'real' (2b) since hard gluon is not measured