Resumming large radiative corrections in the high-energy evolution of the Color Glass Condensate

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Collinearly-improved BK equation
Edmond Iancu
Introduction

- The CGC effective theory: the pQCD description for the wavefunction of an energetic hadron and its interactions
  - non-linear effects: gluon saturation, multiple scattering
- “Effective theory” : high-energy evolution described by pQCD
  - Balitsky-JIMWLK hierarchy \( \approx \) BK equation (at large \( N_c \))
  - non-linear generalizations of the BFKL equation
- The non-linear evolution has recently been promoted to NLO
  - essential for a realistic phenomenology
    \( (\text{Balitsky, Chirilli, 2008, 2013; Kovner, Lublinsky, Mulian, 2013}) \)
- Large corrections enhanced by double or single transverse logarithms
  - lack of convergence, unstable evolution at NLO
- Similar problems encountered & solved for NLO BFKL
  - collinear resummations, formulated in Mellin space
    \( (\text{Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03}) \)
Mellin representation is not suitable beyond the linear approximation.

Non-linear effects (multiple scattering) most naturally discussed in terms of transverse coordinates:

- Eikonal approximation, Wilson lines.

Alternative resummation, formulated in transverse coordinates

(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, 2015)

- Direct calculation of Feynman graphs (light-cone perturbation theory)

- Transparent physical interpretation

- Promising phenomenology (so far, only DIS)

- Extension to $pp$ and $pA$ (at least) is straightforward.
Dipole–hadron scattering ($\gamma^* p$, $\gamma^* A$, $pA$, ...)

- **Dipole**: large $q^+$, transverse size $r$, transverse resolution $Q^2 = 1/r^2$
- **Target**: large $P^-$, high gluon density (CGC), saturation momentum $Q_s^2$
- **Elastic $S$-matrix $S(r) \sim$ dipole survival probability**
  - scattering is weak ($S(r) \sim 1$) when the dipole is small: $Q^2 \gg Q_s^2$
  - scattering amplitude $T(r) \equiv 1 - S(r) : T(r) \propto r^2$ as $r \ll 1/Q_s$
Dipole–hadron scattering ($\gamma^* p$, $\gamma^* A$, $pA$, ...)

- **Dipole**: large $q^+$, transverse size $r$, transverse resolution $Q^2 = 1/r^2$
- **Target**: large $P^-$, high gluon density (CGC), saturation momentum $Q_s^2$
- **Elastic $S$-matrix $S(r) \sim$ dipole survival probability**
  - scattering is strong ($S(r) \ll 1$) when the dipole is large: $Q^2 \gtrsim Q_s^2$
  - unitarity bound (‘black disk limit’): $T(r) \leq 1$
High energy evolution

- Probability $\sim \alpha_s \ln \frac{1}{x}$ to radiate a soft gluon with $x \equiv \frac{k^+}{q^+} \ll 1$

$$r = x_\perp - y_\perp$$

$$Q^2 = 1/r^2$$

$$\frac{2xq^+}{Q^2} \sim \frac{1}{P^-} \implies x \sim \frac{Q^2}{s} \quad (s = 2q^+P^-)$$

- When $\alpha_s \ln \frac{1}{x} \sim 1$, need for resummation: $(\alpha_s Y)^n$ with $Y \equiv \ln \frac{1}{x}$
  - BFKL evolution of the dipole in the background of the dense target
  - multiple scattering $\implies$ non-linear evolution $\implies$ Balitsky-JIMWLK
The BK equation \((Balitsky, '96; Kovchegov, '99)\)

\[ \frac{\partial S_{xy}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int d^2z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \left[ S_{xz}S_{zy} - S_{xy} \right] \]

- **Large \(N_c\)**: the original dipole splits into two new dipoles

- ‘dipole kernel’ : probability for the dipole to split \((Al \, Mueller, 1990)\)
- ‘real term’ : the soft gluons exists at the time of scattering
- ‘virtual term’ : initial (or final) state evolution

Mean field approximation to the Balitsky-JIMWLK hierarchy
Double-logarithmic approximation: DLA 1.0

- Large transverse separation between projectile and target: $Q^2 \gg Q_s^2$

- Large transverse phase-space for gluon emission at $r \ll z \ll 1/Q_s$
  - scattering is still weak: $T \equiv 1 - S \ll 1$ for all dipoles
  - the daughter dipoles scatter stronger (since larger): $T(z) \gg T(r)$

\[
\frac{\partial}{\partial Y} \frac{T(r^2)}{r^2} \simeq \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{dz^2}{z^2} \frac{T(z^2)}{z^2} \Rightarrow \Delta T \sim \bar{\alpha}_s Y \ln \frac{Q^2}{Q_s^2} T
\]
Any effect of $O(\bar{\alpha}_s^2 Y) \Rightarrow O(\bar{\alpha}_s)$ correction to the BFKL kernel

The prototype: two successive emissions, one soft and one non-soft

The maximal correction thus expected: $O(\bar{\alpha}_s \rho)$ with $\rho \equiv \ln(Q^2/Q_s^2)$

But one finds an even larger effect: $O(\bar{\alpha}_s \rho^2)$ ('double collinear log')

Originally found as a NLO correction to the BFKL kernel

(Fadin, Lipatov, Camici, Ciafaloni ... 95-98; Balitsky, Chirilli, 07)
BK equation at NLO  Balitsky, Chirilli (arXiv:0710.4330)

- Very complicated in full generality
- Here: $N_f = 0$, large $N_c$, tiny fonts

\[
\frac{dS_{xy}}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2} (S_{xz}S_{zy} - S_{xy}) \left\{ 1 + \right.
\]
\[+ \bar{\alpha}_s \left[ b \ln(x-y)^2 \mu^2 - \bar{b} \frac{(x-z)^2 - (y-z)^2}{(x-y)^2} \ln \frac{(x-z)^2}{(y-z)^2} \right.
\]
\[+ \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(x-z)^2}{(x-y)^2} \ln \frac{(y-z)^2}{(x-y)^2} \right]\}
\]
\[+ \frac{\bar{\alpha}_s^2}{8\pi^2} \int d^2u d^2z \left( S_{xz}S_{uz}S_{zy} - S_{xz}S_{uy} \right)
\]
\[
\left\{ -2 + \frac{(x-u)^2(y-z)^2 + (x-z)^2(y-u)^2 - 4(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2 - (x-z)^2(y-u)^2} \ln \frac{(x-u)^2(y-z)^2}{(x-z)^2(y-u)^2} \right.
\]
\[+ \frac{(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2} \left[ 1 + \frac{(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2 - (x-z)^2(y-u)^2} \right] \ln \frac{(x-u)^2(y-z)^2}{(x-z)^2(y-u)^2} \right\}
\]
\[
\frac{dS_{xy}}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x-y)^2}{(x-z)^2(y-z)^2} (S_{xz}S_{zy} - S_{xy}) \left\{ 1 + \right.
\]
\[
+ \bar{\alpha}_s \left[ \bar{b} \ln(x-y)^2 \mu^2 - \bar{b} \frac{(x-z)^2 - (y-z)^2}{(x-y)^2} \ln \frac{(x-z)^2}{(y-z)^2}
\]
\[
+ \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(x-z)^2}{(x-y)^2} \ln \frac{(y-z)^2}{(x-y)^2} \right]\}
\]
\[
+ \frac{\bar{\alpha}_s^2}{8\pi^2} \int d^2 u d^2 z \frac{(S_{xu}S_{uz}S_{zy} - S_{xu}S_{uy})}{(u-z)^4} \left\{ -2 + \frac{(x-u)^2(y-z)^2 + (x-z)^2(y-u)^2 - 4(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2 - (x-z)^2(y-u)^2} \ln \frac{(x-u)^2(y-z)^2}{(x-z)^2(y-u)^2}
\]
\[
+ \frac{(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2} \left[ 1 + \frac{(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2 - (x-z)^2(y-u)^2} \right] \ln \frac{(x-u)^2(y-z)^2}{(x-z)^2(y-u)^2} \right\}
\]

- blue: leading-order (LO) terms
- red: NLO terms enhanced by (double or single) transverse logarithms
- black: pure \(\bar{\alpha}_s\) corrections (no logarithms)
\[
\frac{dS_{xy}}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x-y)^2}{(x-z)^2(y-z)^2} \left( S_{xz} S_{zy} - S_{xy} \right) \left\{ 1 + \right.
\]
\[
\bar{\alpha}_s \left[ \bar{b} \ln(x-y)^2 \mu^2 - \bar{b} \frac{(x-z)^2 - (y-z)^2}{(x-y)^2} \ln \frac{(x-z)^2}{(y-z)^2} \right.
\]
\[
+ \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(x-z)^2}{(y-z)^2} \ln \frac{(y-z)^2}{(x-y)^2} \left\} \right.
\]
\[
+ \frac{\bar{\alpha}_s^2}{8\pi^2} \int \frac{d^2 u d^2 z}{(u-z)^4} \left( S_{xu} S_{uz} S_{zy} - S_{xu} S_{uy} \right)
\]
\[
\left\{ -2 + \frac{(x-u)^2(y-z)^2 + (x-z)^2(y-u)^2 - 4(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2 - (x-z)^2(y-u)^2} \ln \frac{(x-u)^2(y-z)^2}{(x-z)^2(y-u)^2} \right.
\]
\[
+ \frac{(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2} \left[ 1 + \frac{(x-y)^2(u-z)^2}{(x-u)^2(y-z)^2 - (x-z)^2(y-u)^2} \right] \ln \frac{(x-u)^2(y-z)^2}{(x-z)^2(y-u)^2} \left\} \right.
\]
\]

- Keeping just the logarithmically enhanced terms \((z \gg r, \text{weak scattering})\)

\[
\frac{dT(r)}{dY} \simeq \bar{\alpha}_s \int dz^2 \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left( \frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} - \bar{b} \ln r^2 \mu^2 \right) \right\} T(z)
\]
NLO: unstable numerical solutions

- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2 Q_s^2)$ with increasing $Y$

Left: leading-order BK: the saturation front
- weak scattering at large $\rho$ (small $r$): $T \propto r^2 = e^{-\rho}$
- unitarity limit at small $\rho$ (large $r$): $T = 1$
- transition at the saturation scale: $T(Y, r) \sim 1$ when $r = 1/Q_s(Y)$
NLO : unstable numerical solutions

- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2 Q_0^2)$ with increasing $Y$

- **Left**: leading-order BK : the saturation front

- **Right**: LO BK + the double collinear logarithm at NLO

*(our calculation, arXiv:1502.05642)*
\( T(\rho, Y) \) as a function of \( \rho = \ln(1/r^2 Q_s^2) \) with increasing \( Y \)

Left: leading-order BK : the saturation front

Right: full NLO BK : evolution speed \( (\partial_Y Y T)/T \)

(Lappi, Mäntysaari, arXiv:1502.02400)

The main source of instability: the double collinear logarithm
The double collinear logarithm

- Two successive emissions which are **strongly ordered** in both ...

\[ Q^2 = 1/r^2 \]

- **longitudinal momenta**: \( q^+ \gg p^+ \gg k^+ \)
- ... and transverse sizes (or momenta): \( r_\perp^2 \ll u_\perp^2 \ll z_\perp^2 \ll 1/Q_s^2 \)

- "Two iterations of DLA 1.0 \( \Rightarrow \mathcal{O}((\bar{\alpha}_s Y \rho)^2)" \ ... not exactly!

- Additional constraint due to time ordering: \( \tau_p > \tau_k \)
Heisenberg: fluctuations have a finite lifetime $\tau_k \sim \frac{k^+}{k_\perp^2} \sim k^+ z_\perp^2$

Time ordering enters perturbation theory via energy denominators

Additional restriction on the DLA phase-space $\Rightarrow$ double collinear logs

\[
p^+ u_\perp^2 > k^+ z_\perp^2 \quad \Rightarrow \quad \Delta Y \equiv \ln \frac{p^+}{k^+} > \Delta \rho \equiv \ln \frac{z_\perp^2}{u_\perp^2}
\]

$Y > \rho \Rightarrow \tilde{\alpha}_s Y \rho \Rightarrow \tilde{\alpha}_s (Y - \rho) \rho = \tilde{\alpha}_s Y \rho - \tilde{\alpha}_s \rho^2$

Time ordering enters perturbation theory via energy denominators
Mixed Fourier representation: $p^+$ and $x^+$, with $p^- = p_{\perp}^2 / 2p^+ = 1/\tau_p$

All possible time orderings for successive, soft, emissions

The time ($x^+$) integrals yield energy denominators

Integrate out the harder gluon ($p^+, u_{\perp}$) to DLA:

$$\bar{\alpha}_s \int_{r^2} z^2 \frac{du^2}{u^2} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \Theta(p^+ u^2 - k^+ z^2) = \bar{\alpha}_s Y \rho - \frac{\bar{\alpha}_s \rho^2}{2}$$
The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within DLA 1.0

\[ \frac{\partial T(Y, r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{dz^2}{z^2} \frac{r^2}{z^2} T(Y, z^2) \]
The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within DLA 1.0


\[
\frac{\partial T(Y, r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{d\bar{z}^2}{\bar{z}^2} \frac{r^2}{\bar{z}^2} T \left( Y - \ln \frac{\bar{z}^2}{r^2}, \bar{z}^2 \right)
\]

- Non-local in \( Y \)
- DLA 2.0: Resums all the powers of \( \bar{\alpha}_s Y \rho \) and \( \bar{\alpha}_s \rho^2 \)
The double-collinear logs can be \textit{systematically resummed to all orders} by enforcing time-ordering within DLA 1.0

\begin{equation}
\frac{\partial T(Y, r^2)}{\partial Y} = \bar{\alpha}_s \int^{1/Q_s^2}_{r^2} \frac{d\bar{z}^2}{\bar{z}^2} \frac{r^2}{\bar{z}^2} T\left(Y - \ln \frac{\bar{z}^2}{r^2}, \bar{z}^2\right)
\end{equation}

- Non-local in $Y$
- DLA 2.0: Resums all the powers of $\bar{\alpha}_s Y \rho$ and $\bar{\alpha}_s \rho^2$
- The importance of time-ordering had already been recognized
  \begin{itemize}
  \item Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96),
  \item Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)
  \end{itemize}
- The diagrammatic foundation in pQCD was not properly appreciated
The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within DLA 1.0

\[ \frac{\partial T(Y, r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{dz^2}{z^2} \frac{r^2}{z^2} T \left( Y - \ln \frac{z^2}{r^2}, z^2 \right) \]

- Non-local in \( Y \)
- DLA 2.0: Resums all the powers of \( \bar{\alpha}_s Y \rho \) and \( \bar{\alpha}_s \rho^2 \)
- The importance of time-ordering had already been recognized
  
  \[ \text{Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)} \]
- The diagrammatic foundation in pQCD was not properly appreciated
- So far, no change in the kernel: double-logs come from non-locality
The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within DLA 1.0

\[ \frac{\partial T(Y, r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{dz^2}{z^2} \frac{r^2}{z^2} T \left( Y - \ln \frac{z^2}{r^2}, z^2 \right) \]

Non-local in \( Y \)

DLA 2.0: Resums all the powers of \( \bar{\alpha}_s Y \rho \) and \( \bar{\alpha}_s \rho^2 \)

The importance of time-ordering had already been recognized

Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)

The diagrammatic foundation in pQCD was not properly appreciated

Equivalently: a local equation but with an all-order resummed kernel
The collinearly improved BK equation

- The argument extends beyond DLA, that is, to BFKL/BK equations
- The generalized, non-local, BK equation (with the usual kernel, but with time-ordering) is equivalent to the following, local, equation:

\[
\frac{d\tilde{S}_{xy}}{dY} = \tilde{\alpha}_s \int \frac{d^2 z}{2\pi} \frac{(x - y)^2}{(x - z)^2 (z - y)^2} K_{DLA}(\tilde{\rho}^2(x, y, z)) (\tilde{S}_{xz} \tilde{S}_{zy} - \tilde{S}_{xy})
\]

... with the all-order resummed kernel: (see also Sabio-Vera, 2005)

\[
K_{DLA}(\rho^2) \equiv \frac{J_1(2\sqrt{\tilde{\alpha}_s \rho^2})}{\sqrt{\tilde{\alpha}_s \rho^2}} = 1 - \frac{\tilde{\alpha}_s \rho^2}{2} + \frac{(\tilde{\alpha}_s \rho^2)^2}{12} + \cdots
\]

... and the symmetrized version of the collinear double-log:

\[
\tilde{\rho}^2(x, y, z) \equiv \ln \left( \frac{(x - z)^2}{(x - y)^2} \ln \frac{(y - z)^2}{(x - y)^2} \right)
\]

- The first correction, of $O(\tilde{\alpha}_s \tilde{\rho}^2)$, coincides with the NLO double-log
Extending to single-logs & running coupling

- Recall the NLO equation with all the transverse logs

\[
\frac{dT(r)}{dY} = \bar{\alpha}_s \int d\bar{z}^2 \frac{r^2}{\bar{z}^4} \left\{ 1 - \bar{\alpha}_s \left( \frac{1}{2} \ln^2 \frac{\bar{z}^2}{r^2} + \frac{11}{12} \ln \frac{\bar{z}^2}{r^2} - \bar{b} \ln r^2 \mu^2 \right) \right\} T(\bar{z})
\]

- the double-logarithm is already included within \( K_{DLA}(\rho) \) ✔
- the collinear single-log is part of the DGLAP anomalous dimension ✔
- the running coupling log is resummed by replacing \( \bar{\alpha}_s \to \bar{\alpha}_s(r_{\text{min}}) \) ✔

\[
\frac{d\tilde{S}_{xy}}{dY} = \int \frac{d^2\bar{z}}{2\pi} \bar{\alpha}_s(r_{\text{min}}) \frac{(x-y)^2}{(x-z)^2(z-y)^2} \left[ \frac{r^2}{\bar{z}^2} \right]^{\pm A_1 \bar{\alpha}_s} K_{DLA}(\bar{\rho}^2(x, y, z)) \times (\tilde{S}_{xz} \tilde{S}_{zy} - \tilde{S}_{xy})
\]

\[ A_1 \equiv \frac{11}{12}, \quad z_\prec \equiv \min\{(x-z)^2, (y-z)^2\} \]
Numerical solutions: saturation front

- Fixed coupling $\bar{\alpha}_s = 0.25$, double collinear logs alone
  - left: expanded to NLO
  - right: resummed to all orders
- The resummation stabilizes & slows down the evolution
Saturation exponent $\lambda_s \equiv \frac{d \ln Q_s^2}{dY}$

- **Fixed coupling**
  - LO: $\lambda_s \approx 4.88\bar{\alpha}_s \approx 1$
  - resummed DL: $\lambda_s \approx 0.5$
  - DL + SL: $\lambda_s \approx 0.4$

- **Running coupling**
  - LO: $\lambda_s = 0.25 \div 0.30$
  - DL + SL: $\lambda_s \approx 0.2$
  - better convergence
Fitting the HERA data

(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1507.03651)

- The most recent analysis of HERA data: very small error bars
  - Bjoerken’ $x \leq 0.01$
  - $Q^2 < Q^2_{\text{max}}$ with $Q^2_{\text{max}} = 50 \div 400$ GeV$^2$

- Numerical solutions to the collinearly-improved BK equation with initial conditions (at $x_0 = 0.01$) which involve 4 free parameters
  - 3 light quarks + charm quark, all treated on the same footing
    - good quality fits for $m_{u,d,s} = 0 \div 140$ MeV and $m_c = 1.3$ or 1.4 GeV
  - Good quality fits: $\chi^2$ per point around 1.1-1.2

- Very discriminatory: the fits favor
  - initial condition: MV model with running coupling
  - smallest-dipole prescription for the running
  - physical values for the free parameters
The HERA fit: rcMV initial condition
Saturation line $Q_s^2(x)$ on top of the experimental data points

- saturation exponent: $\lambda_s = 0.20 \div 0.24$
Conclusions & Perspectives

- The high energy evolution of a color dipole beyond leading order is by now under control.

- The generalization to more complicated (still dilute) projectiles — proton, color quadrupole, ... — is straightforward.

- The generalization to full JIMWLK is less obvious.
  - Essential for applications to AA collisions.

- Similar ‘NLO and beyond’ calculations also needed for impact factors.

- Previous attempts (single-inclusive particle production in $pA$) lead to unstable results (negative cross-section at $p_\perp > Q_s$).

  *(Chirilli, Watanabe, Xiao, Yuan, Zaslavsky, 2012-2015)*

- Still many open problems ... but we are likely close to having an accurate description of high-energy scattering in pQCD.
Dipole factorization for DIS at small $x$

\[ \sigma_{\gamma^*p}(Q^2, x) = 2\pi R_p^2 \sum_f \int d^2 r \int_0^1 dz |\Psi_f(r, z; Q^2)|^2 T(r, x) \]

\[ x = \frac{Q^2}{2P \cdot q} \simeq \frac{Q^2}{s} \ll 1 \quad \text{(Bjorken' x)} \]

- $T(r, x)$: scattering amplitude for a $q\bar{q}$ color dipole with transverse size $r$
- $r^2 \sim 1/Q^2$: the resolution of the dipole in the transverse plane
- $x$: longitudinal fraction of a gluon from the target that scatters
Fitting the HERA data: initial conditions

- Use numerical solutions to collinearly-improved running-coupling BK equation using initial conditions which involve free parameters
  - a similar strategy as for the DGLAP fits
- Various choices for the initial condition at $x_0 = 0.01$:
  - GBW: $T(Y_0, r) = \left\{ 1 - \exp \left[ - \left( \frac{r^2 Q_0^2}{4} \right)^p \right] \right\}^{1/p}$
  - rcMV: $T(Y_0, r) = \left\{ 1 - \exp \left[ - \left( \frac{r^2 Q_0^2}{4} \tilde{\alpha}_s(r) \left[ 1 + \ln \left( \frac{\tilde{\alpha}_{\text{sat}}}{\tilde{\alpha}_s(r)} \right) \right] \right)^p \right\}^{1/p}$
- One loop running coupling with scale $\mu = 2C_\alpha / r$:
  $$\tilde{\alpha}_s(r) = \frac{1}{b_0 \ln \left[4C_\alpha^2/(r^2 \Lambda^2)\right]}, \quad \text{with} \quad r = \min\{ |x-y|, |x-z|, |y-z| \}$$
- 4 free parameters: $R_p$ (proton radius), $Q_0$, $p$, $C_\alpha$
### The HERA fit in tables

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<td>no</td>
<td>1.128</td>
<td>0.573</td>
</tr>
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</table>

### $\chi^2$/npts for $Q^2_{\text{max}}$

<table>
<thead>
<tr>
<th>init</th>
<th>RC schm</th>
<th>sing. logs</th>
<th>$\chi^2$/npts for $Q^2_{\text{max}}$</th>
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<td>fac</td>
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<tr>
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<td>1.097</td>
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<tr>
<td>rcMV</td>
<td>fac</td>
<td>no</td>
<td>1.128</td>
</tr>
</tbody>
</table>
The Fit in plots

Quark Matter 2015, Kobe, Japan  Collinearly-improved BK equation  Edmond Iancu
The Fit in plots

\[ Q^2 \text{ or } \frac{4}{r^2} \ \text{[GeV}^2] \]

- \( \times \) data
- \( \text{red dashed line} \) GBW,small
- \( \text{red line} \) GBW,fac
- \( \text{blue dashed line} \) rcMV,small
- \( \text{blue line} \) rcMV,fac

\( x \)
Rather stable predictions for the saturation line and the shape of the initial amplitude
Prescriptions for running coupling

\[ \alpha(\pi \sim) \]

\[ \phi |x-y| = 1 \]

\[ |x-z| = 1.5 \]

Quark Matter 2015, Kobe, Japan

Collinearly-improved BK equation

Edmond Iancu
Numerical solutions: LO BK

- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2)$ with increasing $Y$

- color transparency at large $\rho$ (small $r$): $T \propto r^2 = e^{-\rho}$

- unitarity limit at small $\rho$ (large $r$): $T = 1$

- saturation exponent (speed): $\lambda_s \equiv \frac{d \ln Q^2_s}{dY} \simeq 1$ for $Y \gtrsim 10$
Mixed Fourier representation: $p^+$ and $x^+$, with $p^- = p_\perp^2 / 2p^+ = 1/\tau_p$

All possible time orderings for successive, soft, emissions

time ordered graphs

anti time-ordered graphs

$$\frac{\tau_p}{\tau_p + \tau_k} \simeq \Theta(\tau_p - \tau_k)$$

$$\frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)$$

The time ($x^+$) integrals yield energy denominators

Integrate out the harder gluon ($p^+, u_\perp$) to DLA: $r_\perp \ll u_\perp \ll z_\perp$
Mixed Fourier representation: $p^+$ and $x^+$, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$

All possible time orderings for successive, soft, emissions

time ordered graphs

\[
\tau_p \approx \Theta(\tau_p - \tau_k)
\]

\[
\frac{\tau_p}{\tau_p + \tau_k} \approx \Theta(\tau_p - \tau_k)
\]

anti time-ordered graphs

\[
\tau_k \approx \Theta(\tau_k - \tau_p)
\]

\[
\frac{\tau_k}{\tau_p + \tau_k} \approx \Theta(\tau_k - \tau_p)
\]

TO:

\[
\bar{\alpha}_s \int_{r^2} \frac{du^2}{u^2} \int_{k^+} \frac{dp^+}{p^+} \Theta(p^+ u^2 - k^+ z^2) = \bar{\alpha}_s Y \rho - \frac{\bar{\alpha}_s \rho^2}{2}
\]
Mixed Fourier representation: \( p^+ \) and \( x^+ \), with \( p^- = p^2_\perp / 2p^+ = 1 / \tau_p \)

All possible time orderings for successive, soft, emissions

time ordered graphs

\[
\frac{\tau_p}{\tau_p + \tau_k} \simeq \Theta(\tau_p - \tau_k)
\]

anti time-ordered graphs

\[
\frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)
\]

ATO:

\[
\bar{\alpha}_s \int_{r^2} \frac{d u^2}{u^2} \int_{k^+}^q \frac{d p^+}{p^+} \Theta(k^+ z^2 - p^+ u^2) = \frac{\bar{\alpha}_s \rho^2}{2}
\]
Mixed Fourier representation: \( p^+ \) and \( x^+ \), with \( p^− = p_⊥^2 / 2p^+ = 1/\tau_p \)

time ordered graphs

\[ q^- \gg p^+ \gg k^+ \]

\[ q^+ \gg p^+ \gg k^+ \]

anti time-ordered graphs

\[ p^+ u_⊥ \]

\[ z_⊥ k^+ \]

\[ y_⊥ \]

\[ x_⊥ \]

TO graphs generate the expected LLA contributions: \( \bar{\alpha}_s Y \rho \)

both TO and ATO graphs generate double collinear logs \( \bar{\alpha}_s \rho^2 \)

the latter precisely cancel in the sum of all the ATO graphs

net result: the double-collinear logs come from TO graphs alone

The Anti-Time-Ordered graphs

\[
\frac{\tau_p}{\tau_p + \tau_k} \rightarrow \frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)
\]

- The softer gluon \( k^+ \) lives longer than the harder one \( p^+ \)
- The DLA terms exactly cancel in the sum of all the ATO graphs
  - IR logs cancel between vertex (1a) and self-energy (1b) corrections
  - ‘virtual’ (2a) cancel against ‘real’ (2b) since hard gluon is not measured