Third order relativistic dissipative fluid dynamics for nucleus-nucleus collisions and astrophysics

Azwinndini Muronga

Department of Physics, University of Johannesburg, Auckland Park, Johannesburg, South Africa

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- Relativistic *second order* dissipative fluid dynamics is a very important scientific achievement of the last decade, and has inspired many authors to apply its methodology to the study of heavy ion collisions and astrophysics. In short it furnishes equations which are closed by imposing the entropy principle up to second order, with respect to equilibrium.

- We have so far refrained from exploiting subsequent orders because that requires long and cumbersome calculations

- However, the exploitation of subsequent orders with respect to equilibrium is desirable, for the following tthree reasons
  (i) a second order approach is necessary to link more closely the relativistic case with the classical one
  (ii) the higher order terms depends also on the lower order terms; so it might happen that the existing condition of the solution imposes further conditions on the lower order terms and it may also happen that the further conditions affect the equilibrium expressions which, however, are already known.
  (iii) the couplings between the three primary processes in dissipative fluids are only realized by going up to third order

Basics & Formalism

The objective of relativistic dissipative fluid dynamics for one component fluid is the determination of the 14 fields of

$N^\mu(x^\beta)$ net charge density — net charge flux vector

$T^{\mu\nu}(x^\beta)$ stress — energy — momentum tensor

$T^{\mu\nu}$ is assumed symmetric so that it has 10 independent components.

The 14 fields are determined from the field equations (fluid dynamical equations)

$\partial_\mu N^\mu = 0$ net charge (e.g., baryon, strangeness, etc) conservation

$\partial_\nu T^{\mu\nu} = 0$ energy – momentum conservation

$\partial_\lambda F^{\mu\nu\lambda} = P^{\mu\nu}$ balance law of fluxes

$F^{\mu\nu\lambda}$ is completely symmetric tensor of fluxes and $P^{\mu\nu}$ is its production density such that

$$F^{\mu\nu}_{\ \ \nu} = m^2 N^\mu \qquad \text{and} \qquad P^\nu_{\ \nu} = 0$$

Basics & Formulation

We then have a set of 14 independent equations (net charge conservation (1); energy-momentum conservation (4); balance of fluxes (9))

However, the dynamic equations cannot serve as the field equations for the thermodynamic fields N^μ and $T^{\mu\nu}$. Because the additional fields $F^{\mu\nu\lambda}$ and $P^{\mu\nu}$ have appeared.

Restriction on the general form of the constitutive functions $F^{\mu\nu\lambda}(N^\alpha, T^{\alpha\beta})$ and $P^{\mu\nu}(N^\alpha, T^{\alpha\beta})$ is imposed by

- entropy principle —the entropy density–entropy flux vector $S^\mu(N^\alpha, T^{\alpha\beta})$ is a constitutive quantity which obeys the inequality

$$\partial_\mu S^\mu \geq 0 \qquad \text{for all thermodynamic process}$$

- requirement of hyperbolicity — ensures that Cauchy problems of our field equations are well-posed and all wave speeds are finite $\Longrightarrow$ our set of field equations should be symmetric hyperbolic

Net charge 4-current $N^\mu = n u^\mu$

$n \equiv \sqrt{N^\mu N_\mu} = u_\mu N^\mu$ net charge density in fluid rest frame,

$u^\mu \equiv \dfrac{N^\mu}{\sqrt{N^\nu N_\nu}}$ the fluid 4-velocity,

$u^\nu u_\nu = 1 \implies u^\mu$ has 3 independent components

Stress–energy–momentum tensor $T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - (p+\Pi)\Delta^{\mu\nu} + 2q^{(\mu}u^{\nu)} + \pi^{\langle\mu\nu\rangle}$

$\varepsilon \equiv u_{\mu}u_{\nu}T^{\mu\nu}$ energy density in fluid rest frame,

$p \equiv p(\varepsilon, n)$ pressure in fluid rest frame,

Π bulk viscous pressure, $(p+\Pi) \equiv -\dfrac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$

$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu}u^{\nu}$ projection tensor onto 3-space, $\Delta^{\mu\nu}u_{\nu} = \Delta^{\mu\nu}u_{\mu} = 0$

$g^{\mu\nu} \equiv \text{diag}(+1,-1,-1,-1)$ metric tensor

$q^{\mu} \equiv \Delta_{\alpha}^{\mu}u_{\beta}T^{\alpha\beta}$ heat flux 4-current,

$q^{\mu}u_{\mu} = 0 \implies q^{\mu}$ has 3 independent components

$\pi^{\langle\mu\nu\rangle} \equiv T^{\langle\mu\nu\rangle}$ shear stress tensor

$\pi^{\langle\mu\nu\rangle}u_{\mu} = \pi^{\langle\mu\nu\rangle}u_{\nu} = 0, \quad \pi^{\langle\nu}_{\nu\rangle} = 0 \implies \pi^{\langle\mu\nu\rangle}$ has 5 independent components

Production densities tensor $P^{\mu\nu} = \mathcal{P}_\Pi \Pi \Big(\Delta^{\mu\nu} - 3u^\mu u^\nu \Big) + 2\mathcal{P}_q q^{(\mu} u^{\nu)} + \mathcal{P}_\pi \pi^{\langle\mu\nu\rangle}$

The functions $\mathcal{P}_\Pi$, $\mathcal{P}_q$, $\mathcal{P}_\pi$ are related to the bulk viscosity, heat conductivity and shear viscosity and thus may be determined from measurements of these coefficients

Tensor of fluxes (up to 2nd order)

$$
\begin{aligned}
F^{\mu\nu\lambda} &= \frac{1}{2}\mathcal{F}_1^0 g^{(\mu\nu} u^{\lambda)} + \frac{1}{2}\mathcal{F}_2^0\Big(g^{(\mu\nu} u^{\lambda)} - 2u^\mu u^\nu u^\lambda\Big) \\
&\quad + \mathcal{F}_1^1\Pi\Big(\Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda\Big) + \mathcal{F}_2^1\Big(\Delta^{(\mu\nu} q^{\lambda)} - 5u^{(\mu} u^\nu q^{\lambda)}\Big) \\
&\quad + \mathcal{F}_3^1\pi^{(\langle\mu\nu\rangle} u^{\lambda)} \\
&\quad + \mathcal{F}_1^2\Pi^2\Big(\Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda\Big) + \mathcal{F}_2^2\Big(-q^\nu q_\nu\Delta^{(\mu\nu} u^{\lambda)} - 3u^{(\mu} q^\nu q^{\lambda)}\Big) \\
&\quad - \mathcal{F}_3^2 q^\alpha q_\alpha\Big(\Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda\Big) + \mathcal{F}_4^2\Big(3u^\mu\pi^{2\langle\nu\lambda\rangle} - \pi^{2\langle\alpha\alpha\rangle} u^\mu u^\nu u^\lambda\Big) \\
&\quad + \mathcal{F}_5^2\pi^{2\langle\alpha\alpha\rangle}\Big(\Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda\Big) + \mathcal{F}_6^2\Big(q^{(\mu}\pi^{\langle\nu\lambda\rangle)} - 2u^{(\mu} u^\nu\pi^{\langle\lambda)\nu} q_\nu\Big) \\
&\quad + \mathcal{F}_7^2\Big(\Delta^{(\mu\nu}\pi^{\langle\lambda)\alpha\rangle)} q_\alpha - 5u^\mu u^\nu\pi^{\langle\lambda)\alpha} q_\alpha\Big) + \mathcal{F}_8^2\Pi u^{(\mu}\pi^{\langle\nu\lambda\rangle)} \\
&\quad + \mathcal{F}_9^2\Pi\Big(\Delta^{(\mu\nu} q^{\lambda)} - 5q^{(\mu} u^\nu u^{\lambda)}\Big)
\end{aligned}
$$

Zeroth order (Equilibrium) + First order + Second order

Entropy 4-current (up to 3rd order)

$$
\begin{aligned}
S^\mu \;=\;& \mathcal{S}_1^0 u^\mu \\
&+ \mathcal{S}_1^1 \Pi u^\mu + \mathcal{S}_2^1 q^\mu \\
&+ \Big(\mathcal{S}_1^2 \Pi^2 - \mathcal{S}_2^2 q^\alpha q_\alpha + \mathcal{S}_3^2 \pi^{2\langle\alpha\alpha\rangle} \Big) u^\mu \\
&+ \mathcal{S}_4^2 \Pi q^\mu + \mathcal{S}_5^2 \pi^{\langle\mu\alpha\rangle} q_\alpha \\
&+ \Big(\mathcal{S}_1^3 \Pi^3 - \mathcal{S}_2^3 \Pi q_\alpha q^\alpha + \mathcal{S}_3^3 \Pi \pi^{2\langle\alpha\alpha\rangle} + \mathcal{S}_4^3 q_\alpha q_\beta \pi^{\langle\alpha\beta\rangle} + \mathcal{S}_5^3 \pi^{3\langle\alpha\alpha\rangle} \Big) u^\mu \\
&+ \Big(\mathcal{S}_6^3 \Pi^2 - \mathcal{S}_7^3 q_\alpha q^\alpha + \mathcal{S}_8^3 \pi^{2\langle\alpha\alpha\rangle} \Big) q^\mu + \mathcal{S}_9^3 \Pi \pi^{\langle\mu\alpha\rangle} q_\alpha + \mathcal{S}_{10}^3 \pi^{2\langle\mu\alpha\rangle} q_\alpha
\end{aligned}
$$

Zeroth order (Equilibrium) + First order + Second order + Third order

Notation

Parentheses around some indices denote symmetrization, while angular brackets around two indices denote skew-symmetrization

$$a^{(\mu\nu)} \equiv \frac{1}{2}\left(a^{\mu\nu} + a^{\nu\mu}\right)$$

$$a^{\langle\mu\nu\rangle} \equiv \left(\Delta_\alpha^{(\mu}\Delta_\beta^{\nu)} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)a^{\alpha\beta}$$

The space-time derivative will be split into time and spatial components as follows

$$\partial_\mu \equiv u_\mu D + \nabla_\mu$$

with $D \equiv u^\alpha\partial_\alpha$ convective (comoving) time derivative
and $\nabla_\mu \equiv \Delta^{\mu\nu}\partial_\nu$ spatial gradient

$$\dot{a}^{\cdots} \equiv Da^{\cdots} = u^\mu\partial_\mu a^{\cdots} \qquad \text{convective (comoving) time derivative of } a^{\cdots}$$

$$\theta = \nabla_\mu u^\mu = \partial_\mu u^\mu \qquad \text{expansion scalar (divergence of 4-velocity)}$$

Equilibrium

Equilibrium is defined as a process in which production densities vanish and/or the entropy production vanishes

$$
\left.\begin{array}{rcl}
P_{Eq.}^{\mu\nu} &=& 0 \\
\Xi_{Eq.} &=& 0
\end{array}\right\} \implies \Pi_{Eq.} = 0\,, \quad q_{Eq}^{\mu} = 0\,, \quad \pi_{Eq}^{\langle\mu\nu\rangle} = 0
$$

$$
\begin{array}{rcl}
F_{Eq}^{\mu\nu\lambda} &=& \dfrac{1}{2}\mathcal{F}_1^0 g^{(\mu\nu} u^{\lambda)} + \dfrac{1}{2}\mathcal{F}_2^0\Big(g^{(\mu\nu} u^{\lambda)} - 2u^\mu u^\nu u^\lambda\Big) \\
S_{Eq.}^{\mu} &=& s(\varepsilon, n)u^\mu
\end{array}
$$

The energy-momentum tensor reduces to

$$
T_{Eq.}^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu}
$$

In the ideal "perfect" fluid limit one has 5 independent fields
($p(n, e)(2)$, $u^\mu(3)$) and 5 field equations

In dissipative(non-ideal) fluid dynamics one needs 9 additional equations for the dissipative fluxes. The 14 fields $p(n, \varepsilon)$, Π, u^α, q^α, $\pi^{\langle\alpha\beta\rangle}$ are governed by the following fields equations

$$
\begin{aligned}
\partial_\mu N^\mu &= 0 \\
\Delta_{\alpha\mu}\partial_\nu T^{\mu\nu} &= 0 \\
u_\mu \partial_\nu T^{\mu\nu} &= 0 \\
u_\mu u_\nu \partial_\lambda F^{\mu\nu\lambda} &= -\mathcal{P}_\Pi \Pi \\
\Delta_\alpha^\mu u_\nu \partial_\lambda F^{\alpha\nu\lambda} &= \mathcal{P}_q q^\mu \\
\left(\Delta_\alpha^{(\mu}\Delta_\beta^{\nu)} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)\partial_\lambda F^{\alpha\beta\lambda} &= \mathcal{P}_\pi \pi^{\langle\mu\nu\rangle}
\end{aligned}
$$

For all thermodynamic processes the entropy principle holds

$$
\partial_\mu S^\mu \geq 0
$$

$$
\begin{aligned}
\Pi &= \Pi_{Eq.} = 0 \\
q^{\alpha} &= q^{\alpha}_{Eq.} = 0 \\
\pi^{\langle \alpha \beta \rangle} &= \pi^{\langle \alpha \beta \rangle}_{Eq.} = 0
\end{aligned}
$$

$$\Pi^{(1)} = \Pi_E = -\zeta \nabla_\alpha u^\alpha$$

$$q^{\alpha\,(1)} = q_E^\alpha = \kappa T \Delta^{\alpha\mu}\Big(\frac{\nabla_\alpha T}{T} - \dot{u}_\alpha\Big)$$

$$\pi^{\langle\alpha\beta\rangle\,(1)} = \pi_E^{\langle\alpha\beta\rangle} = 2\eta \Delta^{\alpha\mu}\Delta^{\beta\nu}\nabla_{\langle\alpha} u_{\beta\rangle}$$

Relativistic versions of the laws of Navier-Stokes and Fourier
first derived by Eckart, Landau-Lifshitz.
ζ is the bulk viscosity, κ is the thermal conductivity, η is the shear viscosity

- simple algebraic expressions of dissipative fluxes
- may lead to acausal and unstable equations of motion

Müller-Israel-Stewart (MIS) equations: $F^{\mu\nu\lambda}$ linear (first-order) in dissipative fluxes and S^μ quadratic (second-order) in dissipative fluxes
Resulting equations causal and hyperbolic

$$
\begin{aligned}
\Pi^{(2)} &= \Pi_{MIS} = -\zeta\Big[2\mathcal{S}_1^2\dot{\Pi} + \mathcal{S}_4^2\nabla_\alpha q^\alpha\Big] \\
&\quad -\zeta\Big[\Pi(\dot{\mathcal{S}}_1^2 + \mathcal{S}_1^2\nabla_\alpha u^\alpha) + q^\alpha(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2\dot{u}_\alpha)\Big] \\
q^{\mu\,(2)} &= q_{MIS}^\mu = \kappa T\Delta^{\alpha\mu}\Big[2\mathcal{S}_2^2\dot{q}_\alpha + \mathcal{S}_4^2\nabla_\alpha\Pi + \mathcal{S}_5^2\nabla^\beta\pi_{\langle\alpha\beta\rangle}\Big] \\
&\quad +\kappa T\Delta^{\alpha\mu}\Big[q_\alpha(\dot{\mathcal{S}}_2^2 + \mathcal{S}_2^2\nabla_\nu u^\nu) + \Pi(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2\dot{u}_\alpha) \\
&\quad +\pi_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_5^2 - \mathcal{S}_5^2\dot{u}^\beta)\Big] \\
\pi^{\langle\mu\nu\rangle\,(2)} &= \pi_{MIS}^{\langle\mu\nu\rangle} = 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[2\mathcal{S}_3^2\dot{\pi}_{\langle\alpha\beta\rangle} + \mathcal{S}_5^2\nabla_{\langle\alpha}q_{\beta\rangle}\Big] \\
&\quad +2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^2 + \mathcal{S}_3^2\nabla_\lambda u^\lambda) \\
&\quad +q_{\langle\alpha}(\nabla_{\beta\rangle}\mathcal{S}_5^2 - \mathcal{S}_5^2\dot{u}_{\beta\rangle})\Big]
\end{aligned}
$$

- The terms in red are neglected in the original MIS formulation. Terms of the general form $\Pi\partial_\nu u^\mu$, $\Pi\partial_\lambda n$, $\Pi\partial_\lambda \varepsilon$, $q_\alpha\partial_\nu u^\mu$, $q_\alpha\partial_\lambda n$, $q_\alpha\partial_\lambda \varepsilon$, $\pi_{\langle\alpha\beta\rangle}\partial_\nu u^\mu$, $\pi_{\langle\alpha\beta\rangle}\partial_\lambda n$, $\pi_{\langle\alpha\beta\rangle}\partial_\lambda \varepsilon$ have been considered non-linear and thus ignored. These terms have been shown to be important in heavy ion collisions. They will be even more important at low energies and high densities.
- Derivations of the equations from kinetic theory reveals terms that are not explicit from phenomenological considerations (e.g., vorticity terms)

$$
\begin{aligned}
\Pi^{(3)} &= -\zeta\Big[3\mathcal{S}_1^3\dot{\Pi} + 2\mathcal{S}_2^3\dot{q}_\lambda q^\lambda + 2\mathcal{S}_3^3\dot{\pi}_{\langle\alpha\beta\rangle}\pi^{\langle\alpha\beta\rangle} \\
&\quad + \mathcal{S}_6^3(\Pi\nabla_\alpha q^\alpha + q^\alpha\nabla_\alpha\Pi) + \mathcal{S}_9^3(\pi^{\langle\alpha\beta\rangle}\nabla_\alpha q_\beta + q_\beta\nabla_\alpha\pi^{\langle\alpha\beta\rangle})\Big] \\
&\quad -\zeta\Big[\Pi^2(\dot{\mathcal{S}}_1^3 + \mathcal{S}_1^3\nabla_\alpha u^\alpha) - q^\alpha q_\alpha(\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3\nabla_\alpha u^\alpha) \\
&\quad + \pi^{2\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_\alpha u^\alpha) \\
&\quad + \Pi q^\alpha(\nabla_\alpha\mathcal{S}_6^3 - \mathcal{S}_6^3 a_\alpha) + \pi^{\langle\alpha\beta\rangle}q_\beta(\nabla_\alpha\mathcal{S}_9^3 - \mathcal{S}_9^3 a_\alpha)\Big]
\end{aligned}
$$

$$
\begin{aligned}
q^{\mu\,(3)} &= \kappa T \Delta^{\alpha\mu} \Big[-\mathcal{S}_2^3 (2\Pi\dot{q}_\alpha + q_\alpha \dot{\Pi}) + \mathcal{S}_4^3 (2\dot{q}^\beta \pi_{\langle\alpha\beta\rangle} + q^\beta \dot{\pi}_{\langle\alpha\beta\rangle}) \\
&\quad + 2\mathcal{S}_6^3 \Pi \nabla_\alpha \Pi - 2\mathcal{S}_7^3 q^\beta \nabla_\alpha q_\beta + \mathcal{S}_9^3 (\Pi \nabla^\beta \pi_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle} \nabla^\beta \Pi) \\
&\quad + 2\mathcal{S}_{10}^3 \pi_{\langle\beta\nu\rangle} \nabla_\alpha \pi^{\langle\beta\nu\rangle} \Big] \\
&\quad + \kappa T \Delta^{\alpha\mu} \Big[\Pi q_\alpha (\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3 \nabla_\nu u^\nu) + q^\beta \pi_{\langle\alpha\beta\rangle} (\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3 \nabla_\nu u^\nu) \\
&\quad + (\Pi^2 \nabla_\alpha \mathcal{S}_6^3 - q^\lambda q_\lambda \nabla_\alpha \mathcal{S}_7^3 + \pi^{2\langle\lambda\lambda\rangle} \nabla_\alpha \mathcal{S}_8^3) \\
&\quad + \Pi \pi_{\langle\alpha\beta\rangle} (\nabla^\beta \mathcal{S}_9^3 - \mathcal{S}_9^3 a^\beta) + \pi_{\langle\alpha\beta\rangle}^2 (\nabla^\beta \mathcal{S}_{10}^3 - \mathcal{S}_{10}^3 a^\beta) \\
&\quad + \mathcal{S}_7^3 q_\alpha q^\lambda a_\lambda \Big]
\end{aligned}
$$

$$
\begin{aligned}
\pi^{\langle\mu\nu\rangle\,(3)} &= 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\mathcal{S}_3^3(2\Pi\dot{\pi}_{\langle\alpha\beta\rangle}+\pi_{\langle\alpha\beta\rangle}\dot{\Pi})+2\mathcal{S}_4^3\dot{q}_{\langle\alpha}q_{\beta\rangle} \\
&\quad +3\mathcal{S}_5^3\dot{\pi}_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle}+\mathcal{S}_8^3\pi_{\langle\alpha\beta\rangle}\nabla_\lambda q^\lambda+\mathcal{S}_9^3(\Pi\nabla_{\langle\alpha}q_{\beta\rangle}+q_{\langle\alpha}\nabla_{\beta\rangle}\Pi) \\
&\quad +\mathcal{S}_{10}^3(q_{\langle\beta}\nabla^\lambda\pi_{\langle\alpha\rangle\lambda\rangle}+\pi_{\langle\lambda(\alpha\rangle}\nabla^\lambda q_{\beta\rangle})\Big] \\
&\quad +2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\Pi\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3+\mathcal{S}_3^3\nabla_\lambda u^\lambda) \\
&\quad +q_{\langle\alpha}q_{\beta\rangle}(\dot{\mathcal{S}}_4^3+\mathcal{S}_4^3\nabla_\lambda u^\lambda)+\pi_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle}(\dot{\mathcal{S}}_5^3+\mathcal{S}_5^3\nabla_\lambda u^\lambda) \\
&\quad +\pi_{\langle\alpha\beta\rangle}q^\lambda(\nabla_\lambda\mathcal{S}_8^3-\mathcal{S}_8^3 a_\lambda)+\Pi q_{\langle\beta}(\nabla_{\alpha\rangle}\mathcal{S}_9^3-\mathcal{S}_9^3 a_{\alpha\rangle}) \\
&\quad +\pi_{\langle\alpha\lambda}q^\lambda(\nabla_{\beta\rangle}\mathcal{S}_{10}^3-\mathcal{S}_{10}^3 a_{\beta\rangle})\Big]
\end{aligned}
$$

Entropy from Kinetic Theory

We derive the third order entropy 4-current as well the non-classical coefficients by going beyond Israel-Stewart entropy 4-current expression in kinetic theory. The kinetic expression for entropy, can be written as

$$S^\mu = -\int dw p^\mu \psi[f(x,p)] \, ,$$

where

$$\psi[f(x,p)] = f(x,p) \left\{ \ln[A_0^{-1} f(x,p)] - 1 \right\} \, ,$$

and $f(x,p)$ is the out of the equilibrium distribution function. Expanding $\psi(f)$ around $\psi(f^{eq})$ up to third order we get,

$$\begin{aligned}
\psi(f) &= \psi(f^{eq}) + \psi'(f^{eq})(f - f^{eq}) + \frac{1}{2}\psi''(f^{eq})(f - f^{eq})^2 \\
&\quad + \frac{1}{6}\psi'''(f^{eq})(f - f^{eq})^3 + .. \, ,
\end{aligned}$$

$$S^{\mu(1)} = \frac{q^\mu}{T} \ ,$$

$$S^{\mu(2)} = \frac{1}{2}\beta u^\mu \left[\mathcal{S}_1^2 \Pi^2 - \mathcal{S}_2^2 q^\alpha q_\alpha + \mathcal{S}_3^2 \pi^{\nu\alpha}\pi_{\nu\alpha} \right] + \beta \left[\mathcal{S}_4^2 q^\mu \Pi + \mathcal{S}_5^2 q_\alpha \pi^{\mu\alpha} \right] \ ,$$

$$\begin{aligned} S^{\mu(3)} &= \frac{1}{6}\beta u^\mu \left\{ \mathcal{S}_1^3 \Pi^3 + \mathcal{S}_2^3 \Pi q^\alpha q_\alpha + \mathcal{S}_3^3 \Pi \pi^{\nu\alpha}\pi_{\nu\alpha} + \mathcal{S}_4^3 q_\nu q_\alpha \pi^{\nu\alpha} + \mathcal{S}_5^3 \pi_{\nu\alpha}\pi_\beta^\nu \pi^{\alpha\beta} \right\} \\ &\quad - \frac{1}{6}\beta q^\mu \left\{ \mathcal{S}_6^3 \Pi^2 + \mathcal{S}_7^3 q^\alpha q_\alpha - \mathcal{S}_8^3 \pi^{\nu\alpha}\pi_{\nu\alpha} \right\} - \beta \mathcal{S}_9^3 \Pi q_\alpha \pi^{\mu\alpha} \\ &\quad + \frac{1}{2}\beta \mathcal{S}_{10}^3 q_\alpha \pi^{\nu\alpha}\pi_\nu^\mu \ , \end{aligned}$$

As function of m/T in the large temperature limit

$$
\left.
\begin{array}{rcl}
\mathcal{S}_1^2 & = & \infty \\
\mathcal{S}_2^2 & = & \frac{5}{4p} \\
\mathcal{S}_3^2 & = & \frac{3}{4p} \\
\mathcal{S}_4^2 & = & \infty \\
\mathcal{S}_5^2 & = & \frac{1}{4p}
\end{array}
\right\} \implies \text{Second order coefficients known}
$$

Third order coefficients

$$
\begin{array}{rcl}
\mathcal{S}_1^3 & = & \infty, \quad \mathcal{S}_2^3 = \infty, \quad \mathcal{S}_3^3 = \infty \\
\mathcal{S}_4^3 & = & \frac{6}{p^2}, \quad \mathcal{S}_5^3 = \frac{3}{4p^2} = \frac{\mathcal{S}_3^2}{p}, \quad \mathcal{S}_6^3 = \infty \\
\mathcal{S}_7^3 & = & \frac{2}{p^2} = 2\frac{\mathcal{S}_2^2}{p}, \quad \mathcal{S}_8^3 = \frac{27}{32p^2}, \quad \mathcal{S}_9^3 = \infty \\
\mathcal{S}_{10}^3 & = & \frac{9}{32p^2}
\end{array}
$$

- For a one component fluid there are three main mechanisms for entropy production. One is related to the dynamic preassure, one due to heat flux and one due to shear stress.

- For the entropy production to be non-negative the coefficients related to bulk viscosity, shear viscosity and heat flux must satisfy some inequality relations.

- Third order relativistic fluid dynamics reveals the couplings and relaxation times that are not present up to second order

- The many identities one encounter in deriving the equations implies that perhaps it is possible to constrain the lower order known functions such as the equation of state and the transport, relaxation and coupling coefficients

- The equations presented here are the same whether derived via divergence theory or kinetic theory

- The relaxation and coupling coefficients are not *new* parameters - but may be determined from the equation of state