Analytical solution of the Boltzmann equation in an expanding universe

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Abstract

Exact moments of distribution function are used to find the first analytical solution [1] of the nonlinear relativistic Boltzmann equation for a massless gas in a Friedmann-Robertson-Walker (FRW) spacetime.

Boltzmann equation in an FRW spacetime

- The FRW metric is $ds^2 = dt^2 - a^2(t)\left(dx^2+dy^2+dz^2\right)$ and the cosmological scale factor is $a(t)\sim t^{1/2}$ (radiation dominated universe).
- $H(t) \equiv \dot{a}(t)/a(t)$ is the Hubble parameter, the fluid flow is $u^\mu = (1,0,0,0)$ but $\theta(t) \equiv \partial_\mu(\sqrt{-g}\,u^\mu)/\sqrt{-g} = 3H(t)$. The fluid is expanding in a locally isotropic and homogeneous manner.
- Symmetries imply that the single-particle distribution function f_k depends only on t and $u\cdot k = k^0$ where for massless particles $k^0 = k/a(t)$ with $k = |\boldsymbol{k}|$.
- Even when $f_k \neq f_k^{eq}$ (local equilibrium), the energy and number density always take their equilibrium form determined by

$$\partial_t n + 3nH(t) = 0, \qquad \partial_t \varepsilon + 4\varepsilon H(t) = 0. \tag{1}$$

With initial condition $a(t_0)=1$, they are solved by $n(t)=n(t_0)/a^3(t)$ and $\varepsilon(t)=\varepsilon(t_0)/a^4(t)$.

- Assuming classical statistics and a constant cross section σ, the Boltzmann equation in FRW spacetime reduces to [1]

$$k^0\partial_t f_k = \frac{(2\pi)^5}{2}\sqrt{-g}\,\sigma\int_{k'pp'} s\,\delta^4(k+k'-p-p')(f_p f_{p'} - f_k f_{k'}), \tag{2}$$

where $\int_k \equiv \int d^3k/\left[(2\pi)^3\sqrt{-g}\,k^0\right]$ and $s=(k^\mu+k'^\mu)(k_\mu+k'_\mu)$.

Scalar moments of the distribution function

- The distribution function f_k can be fully described by the scalar moments [2]

$$\rho_m(t) = \int_{\boldsymbol{k}} (u\cdot k)^{m+1} f_k(t) \tag{3}$$

- $T(t)$ and λ (fugacity) define the local equilibrium distribution function $f_k^{\rm eq}(t) = \lambda \exp(-u\cdot k/T(t))$ and in equilibrium the scalar moments are

$$\rho_m^{\rm eq}(t) = \frac{(m+2)!}{2\pi^2}\lambda T^{m+3}(t). \tag{4}$$

Exact equations for the scalar moments in closed form

- To express the Boltzmann equation in terms of the scalar moments ρ_m we multiply (2) by $(u\cdot k)^m$ and integrate over k. This results in (see [1] for the many details)

$$\partial_t\rho_m(t) + [(3+m)H(t)+\sigma n(t)]\rho_m(t) = 2\,(m+2)\,m!\,\sigma\sum_{j=0}^{m}\frac{\rho_j(t)}{(j+2)!}\frac{\rho_{m-j}(t)}{(m-j+2)!}. \tag{5}$$

- We can further simplify the moment equations by introducing the scaled moments $M_m(t) \equiv \rho_m(t)/\rho_m^{\rm eq}(t)$ and the scaled time $\hat{t} = t/\ell_0$ where $\ell_0 = 1/(\sigma\, n(t_0))$ is the (constant) mean free path at time t_0. This yields the surprisingly simple evolution equations

$$a^3(\hat{t})\,\frac{\partial}{\partial\hat{t}}M_m(\hat{t}) + M_m(\hat{t}) = \frac{1}{m+1}\sum_{j=0}^{m} M_j(\hat{t})M_{m-j}(\hat{t}). \tag{6}$$

- This infinite set of coupled nonlinear differential equations for the scalar moments $\rho_m(t)$ is equivalent to the original relativistic integro-differential Boltzmann equation.
- Eq. (6) can be solved recursively, i.e. the solution of the evolution equation for M_n requires only previously solved moments $M_k(t)$ of lower order $k<n$.

Analytical solution of the Boltzmann equation

- It can be shown [1] that the moment equations (6) possess the following analytical solutions

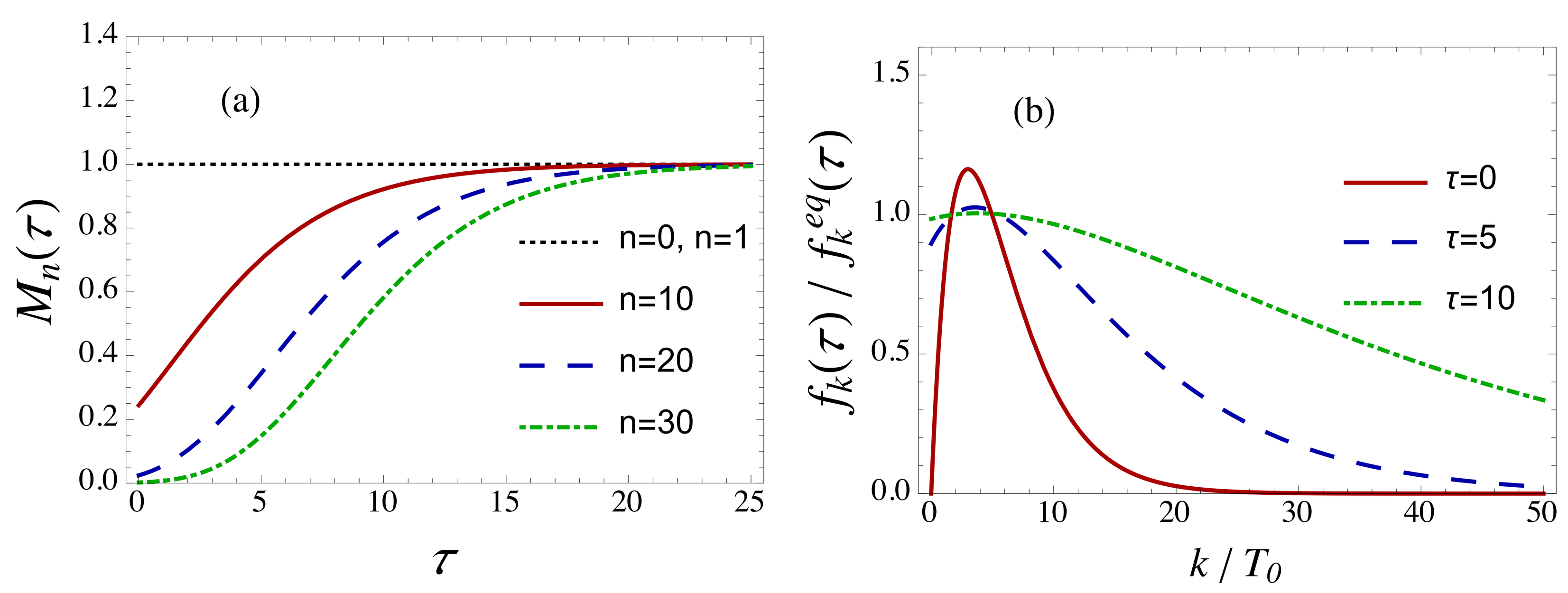
$$M_m(\tau) = \mathcal{K}(\tau)^{m-1}\left[m-(m-1)\mathcal{K}(\tau)\right] \quad (m\geq 0), \tag{7}$$

where $\tau = \int_{\hat{t}_0}^{\hat{t}} dt'/a^3(t')$ to take into account the expansion of the universe, and $\mathcal{K}(\tau) = 1-\frac{1}{4}\exp(-\tau/6)$.

- The time evolution of the moments is shown in Fig. 1 (a). Low-order moments equilibrate more quickly than the higher-order ones that are needed to describe the high-momentum non-equilibrium tail of the distribution function.
- Given the exact form (7) of the moments $M_m(\tau)$, the distribution function can be found analytically [1]

$$f_k(\tau) = \lambda\,\exp\left(-\frac{u\cdot k}{\mathcal{K}(\tau)T(\tau)}\right)\left[\frac{4\mathcal{K}(\tau)-3}{\mathcal{K}^4(\tau)} + \frac{u\cdot k}{T(\tau)}\left(\frac{1-\mathcal{K}(\tau)}{\mathcal{K}^5(\tau)}\right)\right]. \tag{8}$$

- At the initial time $\hat{t}_0$ (corresponding to $\tau=0$ and $T(\tau)=T_0$) one finds the initial condition $f_k(0) = \frac{256}{243}(k/T_0)\,\lambda\,\exp[-4k/(3T_0)] > 0$.
- In Fig. 1 (b) we show how the shape of the non-equilibrium distribution function (8) as a function of $u\cdot k/T(\tau) = k/T_0$ evolves with τ.
- Initially, at $\tau=0$, the high momentum tail is largely underpopulated whereas momentum modes in the range $k/T_0\sim 1.6-5$ are over-occupied relative to local equilibrium. As time evolves, high momentum modes are populated at the expense of the over-populated moderate momentum region, in a process resembling an energy cascade.
- For a radiation dominated universe (8) never assumes the equilibrium form, as expected.



Conclusions and Outlook

- We computed, in exact form, the nonlinear collision term of the relativistic Boltzmann equation for a massless relativistic gas in an expanding universe.
- For a constant cross section the non-linear Boltzmann equation could be transformed into a simple set of non-linearly coupled moment equations which, for a particular initial condition, could be solved exactly. For other initial conditions the moment equations can be simply solved numerically in a recursive fashion.
- As far as we know, (8) is the first nontrivial analytic solution of the non-linear Boltzmann equation that includes general relativity effects. It may be used to study the non-equilibrium behavior of matter in the early universe.
- One may consider different types of cross sections and include nonzero particle masses to investigate bulk viscous effects.
- A comparison between the solution of the nonlinear Boltzmann equation presented here with a simple linear relaxation-time collision term could shed light on the importance of non-linear effects for the rate of thermalization.
- The generalization of the method presented here to expanding systems with different symmetries (i.e., Bjorken or Gubser flow) that are relevant for the study of the quark-gluon plasma formed in heavy ion collisions is in progress.
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References

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