The QCD Critical Point and Related Observables

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DAAD
Deutscher Akademischer Austauschdiensst
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Ideas about the QCD phase diagram


Critical End Point, “... which can be found in heavy-ion collision experiments”
Ideas about the QCD phase diagram


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“Most importantly one has to extrapolate to the continuum limit.”
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"...predictions for the location of the QCD critical point..."
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“Most importantly one has to extrapolate to the continuum limit.”


“...predictions for the location of the QCD critical point...”

C. Fischer, J. Luecker, PLB718 (2013)

“We ... find a potential critical endpoint...”
Ideas about the QCD phase diagram


C. Fischer, J. Luecker, PLB718 (2013)

B. Jacak, B. Müller Science 337 (2012)

"There may be a critical point in the phase diagram..."
The QCD phase diagram and heavy-ion collisions

highly dynamical
short times
small volume
inhomogeneous
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Heavy-ion collisions

QCD thermodynamics & critical point

homogeneous
infinite
long lived
static
The QCD phase diagram and heavy-ion collisions

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Heavy-ion collisions

Dynamical modeling

QCD thermodynamics & critical point

- Homogeneous
- Infinite
- Long lived
- Static
• Thermodynamic quantities change characteristically at the phase transition.
• Speed of sound $c_s^2 = (\partial p/\partial e)_S \rightarrow$ minimum around a crossover
  $\Rightarrow$ vanishes at the first-order PT
• Compressibility $\kappa_S = -1/V(\partial V/\partial p)_S \rightarrow$ maximum around a crossover
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Phase transitions: the equation of state

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"softest point" anomaly in the pressure
Phase transitions: the equation of state

- Modeling a phase transition dynamically is simple!
- Need to know the equation of state and transport coefficients ⇒ fluid dynamics!

A pronounced minimum in the slope of the directed flow $v_1$ is observed in a first-order phase transition.
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![Graph showing directed flow vs. center of mass energy](image)

- A pronounced minimum in the slope of the directed flow \(v_1\) is **not** observed in a first-order phase transition?
- In dynamical simulations: no clear sensitivity on a phase transition in the **equation of state** yet...

*J. Steinheimer, J. Auvinen, H. Petersen, M. Bleicher, H. Stöcker, PRC89 (2014)*
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\[ \text{Fluctuations matter at the phase transition!} \]
An order parameter changes characteristically at the phase transition -
discontinuously or continuously.
Phase transitions: order parameter & derivatives

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- Derivatives reveal more details!
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![Graph showing order parameter and 4th derivative vs. temperature (Tc)]
Phase transitions: order parameter & derivatives

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**Derivatives of thermodynamic quantities are related to fluctuations!**
What are fluctuation observables?

- Susceptibilities $\chi_n = \frac{\partial^n(P/T^4)}{\partial(\mu/T)^n} \bigg|_T$ relate to fluctuations in multiplicity

$$\chi_1 = \frac{1}{VT^3} \langle N \rangle, \quad \chi_2 = \frac{1}{VT^3} \langle (\Delta N)^2 \rangle, \quad \chi_3 = \frac{1}{VT^3} \langle (\Delta N)^3 \rangle, \quad \chi_4 = \frac{1}{VT^3} \langle (\Delta N)^4 \rangle_c \equiv \frac{1}{VT^3} \left( \langle (\Delta N)^4 \rangle - 3 \langle (\Delta N)^2 \rangle^2 \right).$$

- To zeroth-order in volume fluctuations:

$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M}, \quad \frac{\chi_3}{\chi_2} = S\sigma, \quad \frac{\chi_4}{\chi_2} = \kappa \sigma^2$$

variance Skewness Kurtosis

- $M, \sigma^2, S$ and $\kappa$ are obtained from measured event-by-event multiplicity distributions.

STAR Coll. PRL112 (2014), PRL113 (2014); PHENIX Coll. arxiv:1506.07834
Non-critical effects on fluctuation observables

- Limited acceptance & detector efficiency.  
  A. Bzdak, V. Koch, PRC86 (2012); PRC91 (2015)

- Isospin randomization.  
  M. Kitazawa, M. Asakawa, PRC85, PRC86 (2012)

- Volume fluctuations  
  V. Skokov, B. Friman, K. Redlich, PRC88 (2013)
  \(\rightarrow\) strongly intensive measures.
  E. Sangaline, arxiv:1505.00261; M. Gorenstein, M. Gazdzicki, PRC84 (2011)

- Global net-baryon number conservation.
  MN, T. Schuster, M. Mitrovski, R. Stock, M. Bleicher, EPJC72 (2012); A. Bzdak, V. Koch, V. Skokov, PRC87 (2013)

\(\Rightarrow\) These effects are or can be included in microscopic transport models, e.g. UrQMD, (P)HSD, or hybrid models = valuable baseline studies!

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- Initial fluctuations due to baryon stopping.

Need to be well understood!
Phase transitions: fluctuations

**Critical point**
- Universal behavior of the long-wavelength modes.
- Correlation length diverges $\xi \to \infty$.
- Fluctuations of the critical mode $\sigma$ diverge.
- Higher moments more sensitive to $\xi$:
  \[ \langle \Delta \sigma^2 \rangle \propto \xi^2, \quad \langle \Delta \sigma^3 \rangle \propto \xi^{9/2}, \quad \langle \Delta \sigma^4 \rangle_c \propto \xi^7. \]
- For QCD: parameters from 3d Ising universality class.
- Relaxation time $\tau_{\text{rel}} \propto \xi^z$ diverges $\Rightarrow$ critical slowing down!

**First-order phase transition**
- Coexistence of two stable thermodynamic phases.
- Metastable states above and below $T_c \Rightarrow$ supercooling and -heating.

\[ P_{\text{crit}}(\sigma) = \exp\left(\frac{\sigma^2}{\xi^2} - \frac{\sigma^4}{c_2 \xi^4} \right) \]

\[ \text{Distribution } P_{\text{crit}}(\sigma), \quad \text{Energy density } \epsilon/\kappa_0 \]

- Nucleation & spinodal decomposition $\Rightarrow$ domain formation.

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LARGE fluctuations in equilibrium

LARGE fluctuations in nonequilibrium

References:
• Excellent opportunity to study critical fluctuations in conserved-charge densities at finite $\mu_B$.

Critical fluctuations in QCD effective models

Strong $T$-$\mu_B$-dependence of $R_{4,2} = \chi_4/\chi_2$ toward critical point in FRG approach.

Divergence of fluctuations along the spinodal lines.

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![Graph showing critical fluctuations](image)

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Clear signals for the phase transition in effective models!

**Critical net-proton fluctuations - phenomenology**

IDEA: couple order parameter to measurable particles: $g_p\bar{p}\sigma p$

- Mass change in a Hadron Resonance Gas: $m_h \rightarrow m_h + g_h\Delta\sigma$.
- Equilibrium 3d Ising model assumptions for $\Delta\sigma$.
- Fluctuations in net-protons at chemical freeze-out.
- Critical fluctuations are reduced but survive when resonance decays are included!

- Particle emission during Cooper-Frye freeze-out over a hypersurface from fluid dynamical evolution.


M. Bluhm, MN, work in progress

H. Song, L. Jiang, work in progress
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Still no dynamical fluctuations...
Toward dynamics: memory effects

IDEA: real-time evolution of non-Gaussian cumulants in the scaling regime, where

\[ L_{\text{micro}} \ll \xi \ll L_{\text{sys}} \]

- Memory effects are important!
- Magnitude and sign can be different in non-equilibrium compared to equilibrium expectations!
- Different trajectories, chemical freeze-out conditions and \( \tau_{\text{rel}} \) can give similar results.
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Needs dynamical space-time evolution!
IDEA: explicit propagation of order parameters coupled to QGP evolution.

- Relaxation equation for order parameter:
  \[ \partial_\mu \partial^\mu \sigma + \frac{\delta U}{\delta \sigma} + g \rho_s + \eta \partial_t \sigma = \xi \]

- Interaction from effective (P)QM model.

- Fluctuations due to noise \( \xi \).

- Coupling to fluid dynamical expansion:
  \[ \partial_\mu T^{\mu \nu}_q = S^\nu = -\partial_\mu T_{\sigma}^{\mu \nu}, \quad \partial_\mu N^\mu_q = 0 \]

- Stochastic source term \( \Rightarrow \) dynamical evolution of fluctuations!
Dynamical modeling of fluctuations

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Nonequilibrium chiral fluid dynamics ($N\chi FD$)

---

Conventional fluid dynamics propagates thermal averages of the energy density, pressure, velocities, charge densities, etc.

However, ...

- already in equilibrium there are thermal fluctuations
- the fast processes, which lead to local equilibration also lead to noise!

Conventional ideal fluid dynamics:

\[
T^{\mu\nu} = T_{\text{eq}}^{\mu\nu}
\]
\[
N^\mu = N_{\text{eq}}^\mu
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Fluid dynamical fluctuations

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**Conventional viscous** fluid dynamics:

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**Fluctuating viscous fluid dynamics:**

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T^{\mu\nu} = T^{\mu\nu}_{eq} + \Delta T^{\mu\nu}_{visc} + \Xi^{\mu\nu}
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- \(\langle T^{\mu\nu} T^{\nu\mu} \rangle\) give viscosities (Kubo-formula), consistently with dissipation-fluctuation theorem fluctuations need to be included as well!

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**Important at the critical point: true critical mode is a fluid dynamical density!**

Bjorken expansion example with a critical point:

- Near the CP the thermal conductivity is enhanced $\Rightarrow$ enhancement of the rapidity correlator of protons.
Fluid dynamical fluctuations

Bjorken expansion example with a critical point:

- Near the CP the thermal conductivity is enhanced ⇒ enhancement of the rapidity correlator of protons.

How to implement in a $3+1$d relativistic causal fluid dynamical evolution?
Fluid dynamical fluctuations

\[ \partial_\mu T^{\mu \nu} = \partial_\mu \left( T^{\mu \nu}_{eq} + \Delta T^{\mu \nu}_{visc} + \Xi^{\mu \nu} \right) = 0 \]

- Enhancement of flow due to additional fluctuations?
- Important check: equilibrium expectations for fluctuations and nonlinear effects.

- Implementing fluid dynamical fluctuations is important, but requires a sustained and systematic effort!
Fluid dynamical fluctuations

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Future: include net-baryon densities!

talk by K. Murase, T. Hirano; arxiv:1304.3243

MN, M. Bluhm, Y. Karpenko, T. Schäfer, S. Bass, work in progress
Toward the discovery of the critical point

Connect the QCD critical point to experimental observables via: realistic dynamical modeling!
Non-critical effects on fluctuation observables

- Global net-baryon number conservation.

\[ \text{MN, T. Schuster, M. Mitrovski, R. Stock, M. Bleicher, EPJC72 (2012); A. Bzdak, V. Koch, V. Skokov, PRC87 (2013)} \]

- In a microscopic transport model the microcanonical nature of individual scatterings is preserved.

- Strongly negative kurtosis of net-baryon number due to global conservation and volume fluctuations.

- Net-proton fluctuations follow this trend slightly.

\[ K_{\text{eff}} \]

\[ \sqrt{S_{\text{NN}}} \text{ (GeV)} \]

\[ E_{\text{lab}} = 158A \text{ GeV} \]

\[ \text{fixed } |y| \leq 0.5 \]

- net baryon
- net proton
- * net charge

\[ \text{Mean Net Baryon Number} \]
Nonequilibrium correlation length

Phenomenological equation: \[
\frac{d}{dt} m_\sigma(t) = -\Gamma[m_\sigma(t)] \left( m_\sigma(t) - \frac{1}{\xi_{eq}(t)} \right)
\]
with input from the dynamical universality class \( \Rightarrow \xi \sim 1.5 - 2.5 \text{ fm} \)


\[
G(r) = \int d^3x d^3y \langle \sigma(x) - \sigma_0 \rangle \langle \sigma(y) - \sigma_0 \rangle
\sim \exp(-r/\xi)
\]

Assume \( \sigma_0 \) is the volume averaged field.

From the curvature of \( V_{\text{eff}} \):
\[
\langle \xi^2 \rangle = \langle 1/m_\sigma^2 \rangle = \left\langle \left( \frac{d^2 V_{\text{eff}}}{d\sigma^2} \right)^{-1} \right\rangle
\]

Definition of \( \xi \) in inhomogeneous systems involves averaging!

\( \Rightarrow \) Similar magnitude of \( \xi \sim 1.5 - 3 \text{ fm}! \)
Finite-size scaling

In the scaling regime $L_{\text{micro}} \ll \xi \ll L$:

- Finite-size scaling for any intensive thermodynamic quantity $X$ with an algebraic singularity at the critical point:

$$X_L(T) \propto L^{\gamma/\nu}$$

- Equilibrium critical exponents:

$$\xi_\infty(T) \propto t^{-\nu}, \quad X_\infty(T) \propto t^{-\gamma}$$

- Position of the peak is shifted:

$$\Delta t_L = (T_c - T_{c,L})/T_c \propto L^{-\lambda}$$

- Vary the system size via centrality, species of nuclei: Can finite-size scaling be seen in observables?

- Expanding system size, freeze-out of fluctuations, critical slowing down... $\Rightarrow$ need dynamical models!

E. Fraga, L. Palhares, P. Sorensen, PRC84 (2011); R. Lacey, PRL114 (2015)
Amplification of initial fluctuations at a FOPT

- Nonequilibrium construction of the EoS from QGP and hadronic matter:

- Significant amplification of initial density irregularities.
- But: no clear signals after final hadronic phase.

Amplification of initial fluctuations at a FOPT

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\[ T=0 \]

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- But: no clear signals after final hadronic phase.

\[ J. \ Steinheimer, \ J. \ Randrup, \ PRL \ 109 \ (2012), \ PRC \ 87 \ (2013) \]

Deterministic evolution \( \Rightarrow \) no dynamical fluctuations
Dynamical modeling at finite $\mu_B$

- Inclusion of net-baryon diffusion into fluid dynamical simulations:
  - Baryon dissipation.
  - Baryon-shear and baryon-bulk coupling terms.
  - Out-of-equilibrium $\delta f$ corrections.

- Initial state and initial baryon stopping $\Rightarrow$ explore net-baryon rapidity correlations and fluctuations!

- Is there a fluid dynamical phase at high-baryon densities.

- Importance of correct description of the hadronic phase.

- Bulk viscosity needs to be included.
Equation of state + transport coefficients at finite $\mu_B$

Equation of state:
- lattice QCD: cont.-extr. in Taylor expansion up to $O(\mu^2)$
  - Wuppertal-Budapest Coll., JHEP1208 (2012)
  - up to $O(\mu^4)$ not cont.-extr.
  - BNL-Bielefeld-CCNU-Coll., NPA931 (2014)
- effective models: e. g. from
- from 3d Ising model
  - C. Nonaka, M. Asakawa PRC71 (2005); M. Bluhm, B. Kampfer CPOD 2006

Transport coefficients:
- $\eta/s$ in DQPM, reproduces lattice at $\mu_B = 0$, crossover
  - H. Berrehrah et al., arxiv:1412.1017
- $\zeta$: universal properties in vicinity of critical point, $\zeta \to \infty$ in $Z(2)$
- from transport model calculations
  - talk by Y. Karpenko
HRG + critical fluctuations

\[ \xi(\mu_B) = \frac{\xi_{\text{max}}}{\left(1 + \left(\frac{\mu_B - \mu_B^c}{W(\mu_B)}\right)^2\right)^{1/3}} \]

- Model for correlation length from
  C. Athanasiou, K. Rajagopal, M. Stephanov, PRD82 (2010)

- Coupling of resonances to the \( \sigma \)-field: 
  \[ g_{R\sigma} = \frac{m_R}{m_p} (3 - |S_R|) \frac{g_p \sigma}{3} . \]

- Additional parameters \( \tilde{\lambda}_3, \tilde{\lambda}_4 \) should depend on \( \sqrt{s} \).