Statistical errors, efficiency and acceptance corrections in cumulants of measured net-charge (N^+-N^-) distributions, a theorem from Quantitative Finance and NBD fits to the PHENIX N^+ and N^- distributions

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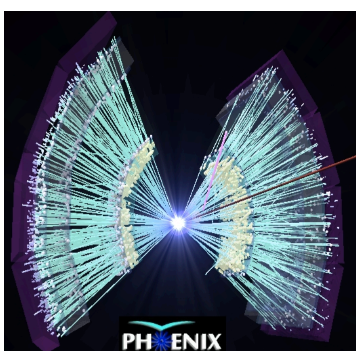
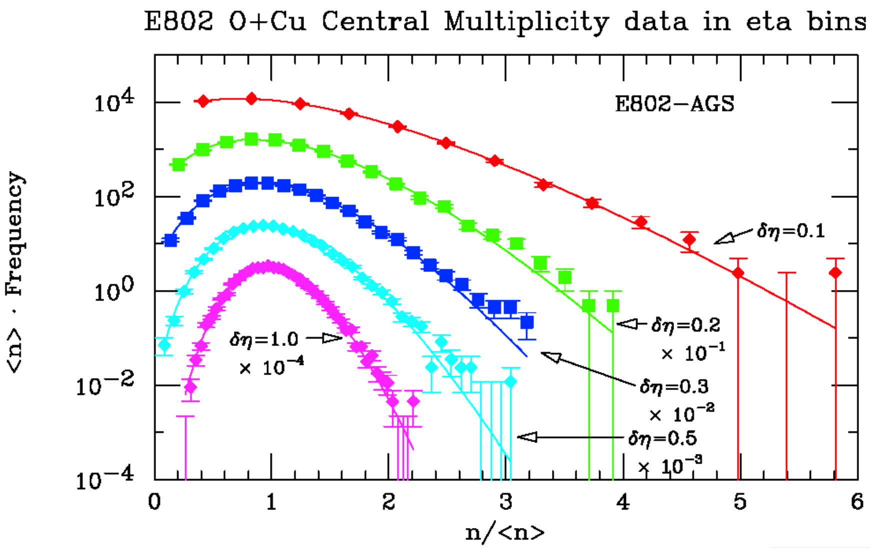


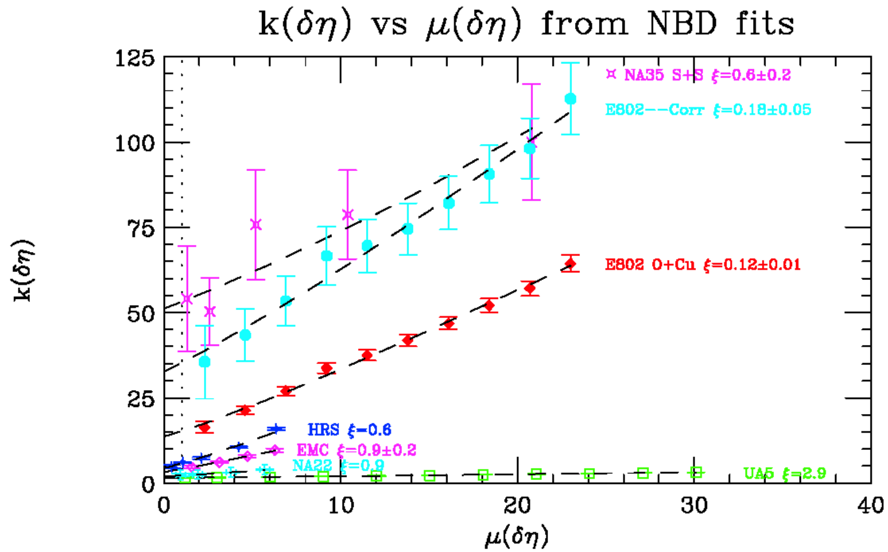
Quark Matter 2015
Kobe, Japan
September 29, 2015

NBD-p+p discoveryUA5 PLB **160**, 193,199 (1985); **167**, 476 (1986)

NBD in O+Cu central collisions at AGS vs $\Delta\eta$ central collisions defined by zero spectators (ZDC) Correlations due to to B-E don't vanish

E802 PRC **52**, 2663 (1995)





Poisson, no correlation

$$\frac{\sigma^2}{\mu^2} - \frac{1}{\mu} = 0$$

NBD correlation= 1/k

$$\frac{\sigma^2}{\mu^2} - \frac{1}{\mu} = \frac{1}{k}$$

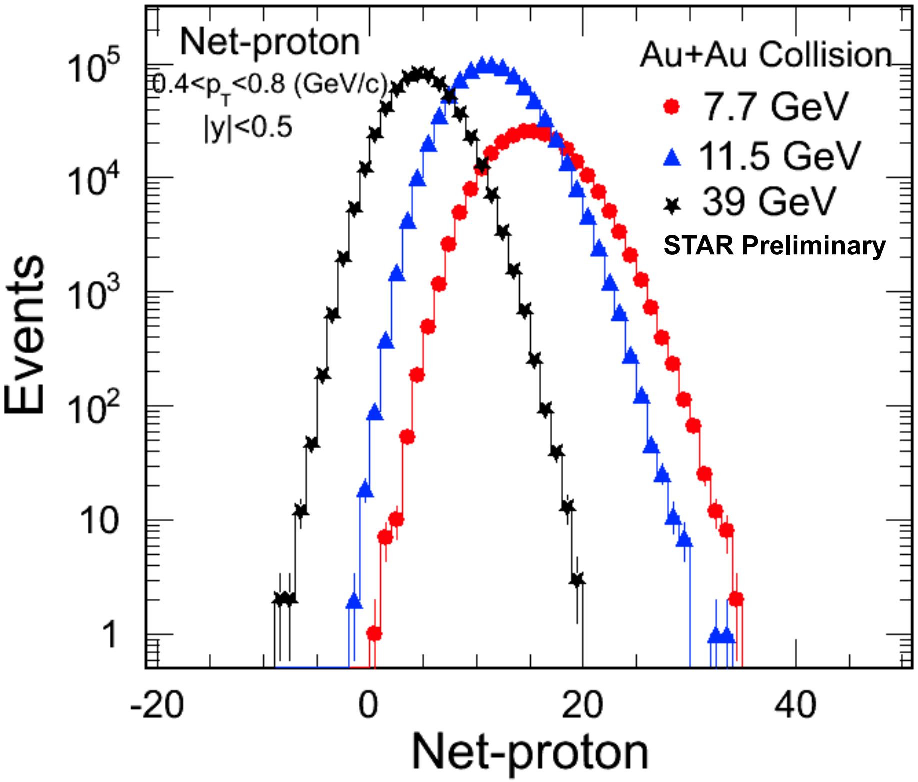
$$k(\delta\eta) \sim \frac{(\delta\eta/\xi)^2}{(\delta\eta/\xi - 1 + e^{-\delta\eta/\xi})}$$

The rapidity correlation length $\xi = 0.2$ for O+Cu is from B-E.

E802, PRC56(1977) 1544

Hot off the presses-LBL Press release June 24,2011 Higher Moments of Net-Proton Distributions

- 1^{st} moment: mean = $\mu=\langle x\rangle$
- 2^{nd} cumulant: variance $\kappa_2= \sigma^2=\langle(x-\mu)^2\rangle$
- 3^{rd} cumulant: $\kappa_3= \sigma^3=\langle(x-\mu)^3\rangle$
- 3^{rd} standardized cumulant: skewness = $S= \kappa_3/\kappa_2^{3/2}=\langle(x- \mu)^3\rangle/\sigma^3$
- 4^{th} cumulant: $\kappa_4= \langle(x-\mu)^4\rangle-3\kappa_2^2$
- 4^{th} standardized cumulant: kurtosis = $\kappa=\kappa_4 /\kappa_2^2=\{\langle(x- \mu)^4\rangle/\sigma^4\}$ -3
- Calculate moments from the event-by-event net proton distribution.
  - ✓ Have similar plots for net-charge and net-kaon distributions.



MJT-If you know the distribution, you know all the moments, but statistical mechanics and Lattice Gauge use Taylor expansions, hence moments/cumulants

Statistical Mechanics uses derivatives of the free energy to find susceptibilities

- Theoretical analyses tend to be made in terms of a Taylor expansion of the free energy $F=-T \ln Z$ around the critical temperature T_c where Z is the partition function or sum over states, $Z \approx \exp -[(E-\Sigma_i \mu_i Q_i)/kT]$ and μ_i chemical potentials associated with conserved charges Q_i

- The terms of the Taylor expansion are called susceptibilities or χ

- The only connection of this method to mathematical statistics is that the Cumulant generating function is also a Taylor expansion of the ln of an exponential:

$$\boxed{g_x(t) = \ln \langle e^{tx} \rangle = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} \qquad \kappa_m = \left. \frac{d^m g_x(t)}{dt^m} \right|_{t=0}}$$

If you measure the distribution, then you know all the cumulants

Cumulants for Poisson, Binomial and Negative Binomial Distributions

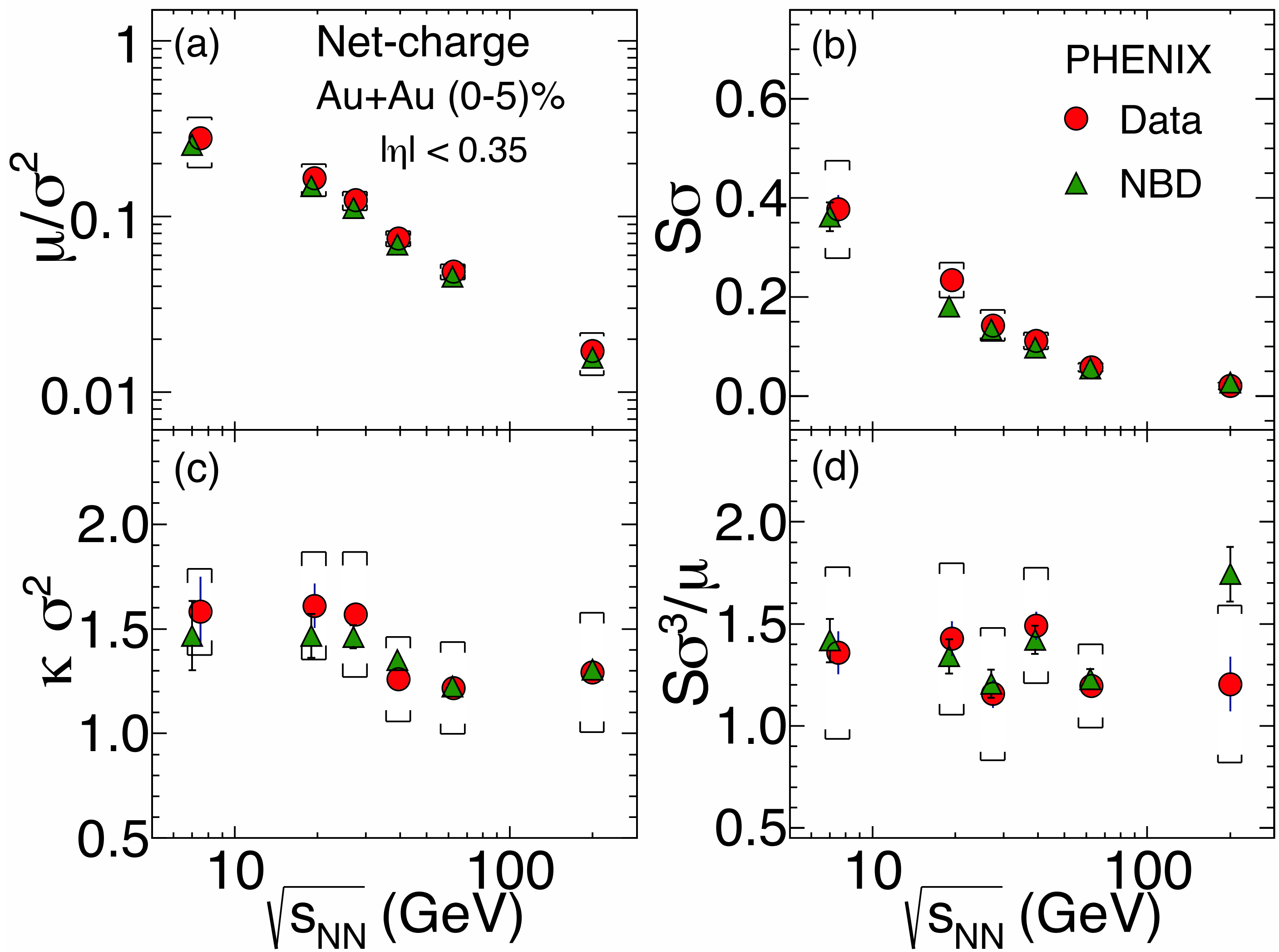
Cumulant	Poisson	Binomial	Negative Binomial
$\kappa_1 = \mu$	μ	np	μ
$\kappa_2 = \mu_2 = \sigma^2$	μ	$\mu(1-p)$	$\mu(1+\mu/k)$
$\kappa_3 = \mu_3$	μ	$\sigma^2(1-2p)$	$\sigma^2(1+2\mu/k)$
$\kappa_4 = \mu_4 - 3\kappa_2^2$	μ	$\sigma^2(1-6p+6p^2)$	$\sigma^2(1+6\mu/k+6\mu^2/k^2)$
$S \equiv \kappa_3/\sigma^3$	$1/\sqrt{\mu}$	$(1-2p)/\sigma$	$(1+2\mu/k)/\sigma$
$\kappa \equiv \kappa_4/\kappa_2^2$	$1/\mu$	$(1-6p+6p^2)/\sigma^2$	$(1+6\mu/k+6\mu^2/k^2)/\sigma^2$
$S\sigma = \kappa_3/\kappa_2$	1	$(1-2p)$	$(1+2\mu/k)$
$\kappa\sigma^2 = \kappa_4/\kappa_2$	1	$(1-6p+6p^2)$	$(1+6\mu/k+6\mu^2/k^2)$

Thanks to Gary Westfall of STAR in a paper presented at Erice-International School of Nuclear Physics 2012, I found out that the cumulants of the difference of samples from two such distributions P(n-m) where $P^+(n)$ and $P^-(m)$ are both Poisson, Binomial or NBD with Cumulants κ_j^+ and κ_j^- respectively is the same as if they were statistically independent, so long as they are not 100% correlated. I call this the NBD Cumulant Theorem

$$\kappa_j = \kappa_j^+ + (-1)^j \kappa_j^-$$

PHENIX central cumulant ratios vs $\sqrt{s_{NN}}$

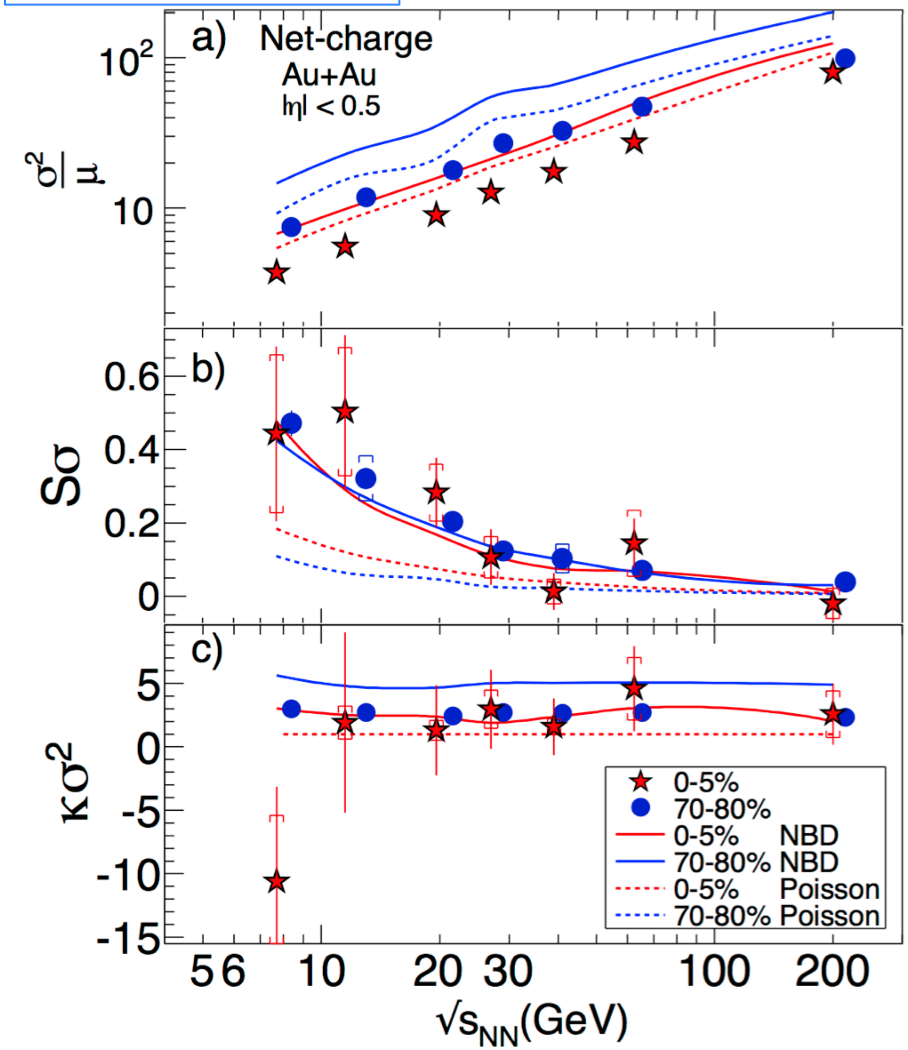
PHENIX arXiv:1506.07834



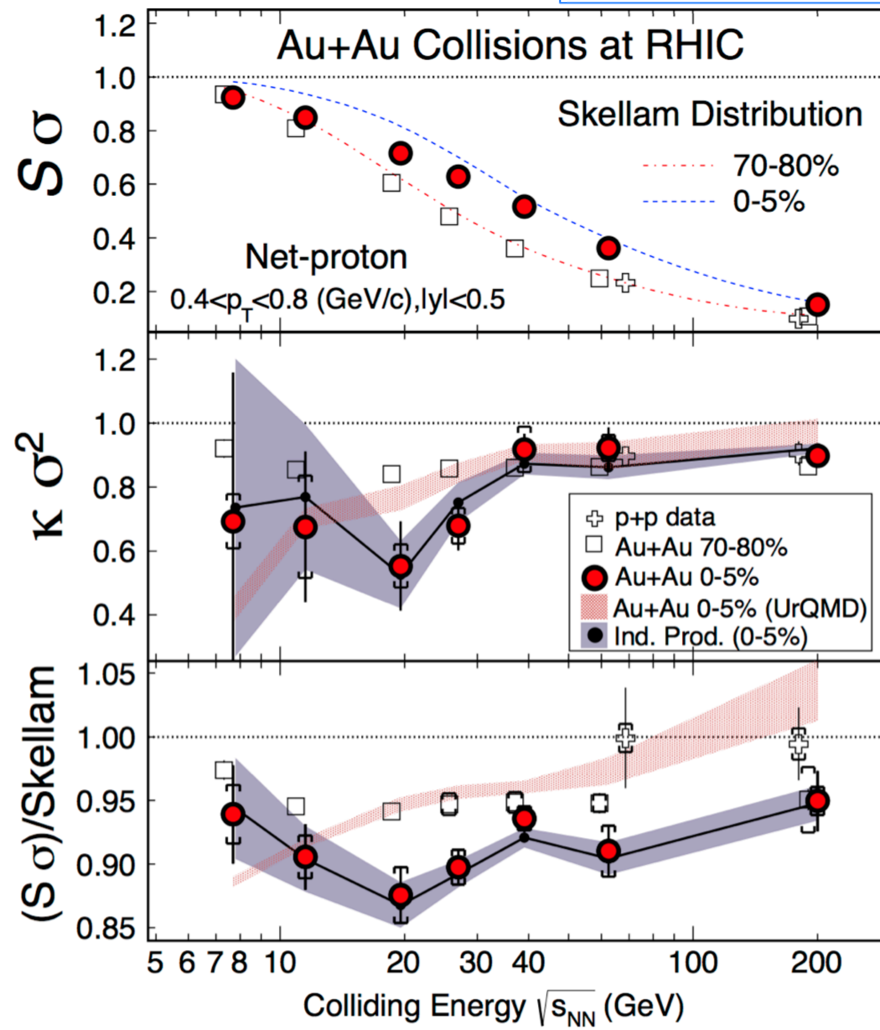
Note that the ``data'' • calculations from the $\Delta N_{ch}=N^{+} - N^{-}$ distributions agree with the NBD fits to the N^{+} and N^{-} distribution and the NBD Cumulant Theorem.

STAR publications 2014

PRL 113(2014) 092301



PRL 112(2014) 032302

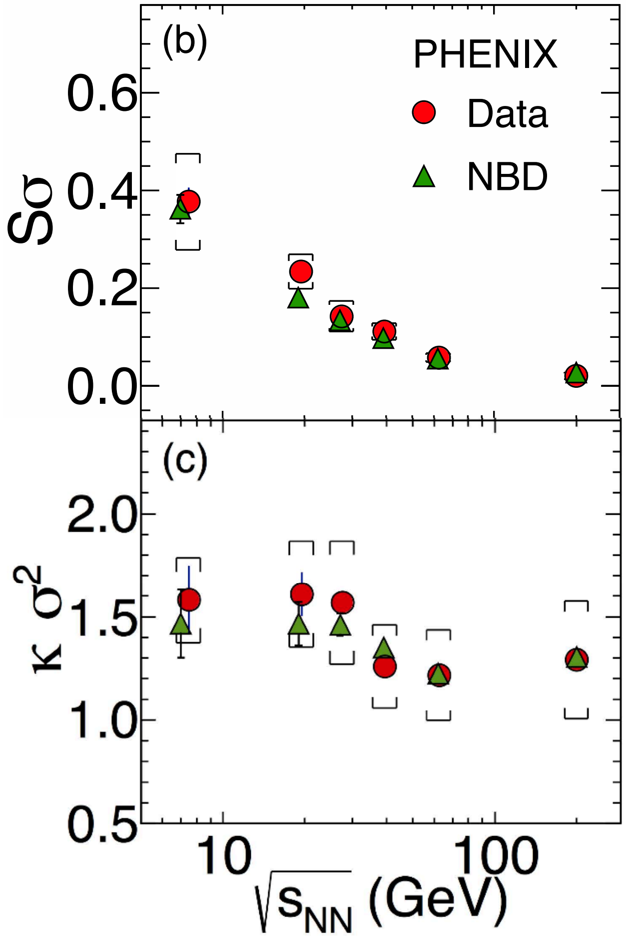


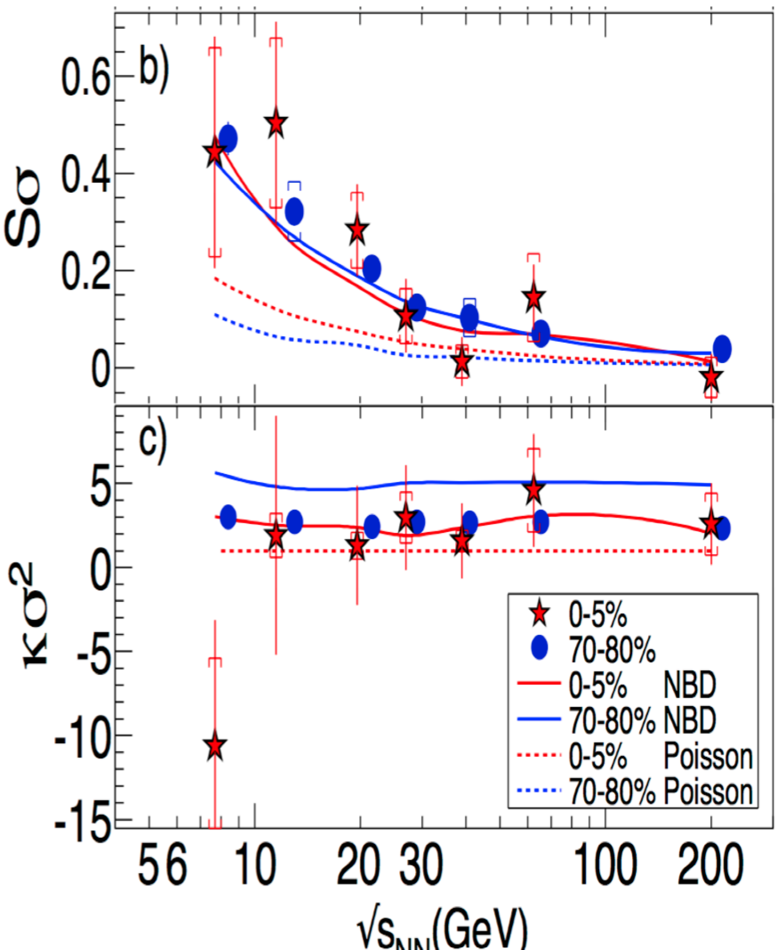
Sσ clearly favors NBD, not Poisson (!).
No non- monotonic behavior in Sσ or $\kappa\sigma^2$
but $\kappa\sigma^2$=-1.5 at $\sqrt{s_{NN}}$ =20 can't be ruled out

$\kappa\sigma^2$=-1.5 at $\sqrt{s_{NN}}$ =20 **can** be ruled out
$\kappa\sigma^2$ changes for $\sqrt{s_{NN}}$ ≤20 GeV but
antiprotons become negligible <0.1 p

PHENIX and STAR comparison!!!

PHENIX arXiv:1506.07834





STAR arXiv: 1402.1558

The key difference of the PHENIX and STAR results is that the error on all corrected cumulant ratios is 20-30% for PHENIX while for STAR the error on e.g. Sσ is ~ 50%, on $\kappa\sigma^2$ is >100% but <1% for σ^2/μ!!! WHY?

Efficiency Corrected Cumulants

It must be that statistical errors and efficiency corrections are a BIG issue in these measurements even though the correction is simply Binomial; and analytical for NBD N^+ and N^- distributions (**k unchanged, $\mu_t=\mu/p$ where p is the efficiency**) thanks to the NBD "integer value Levy process" cumulant theorem:

Tarnowsky, Westfall PLB 724 (2013) 51

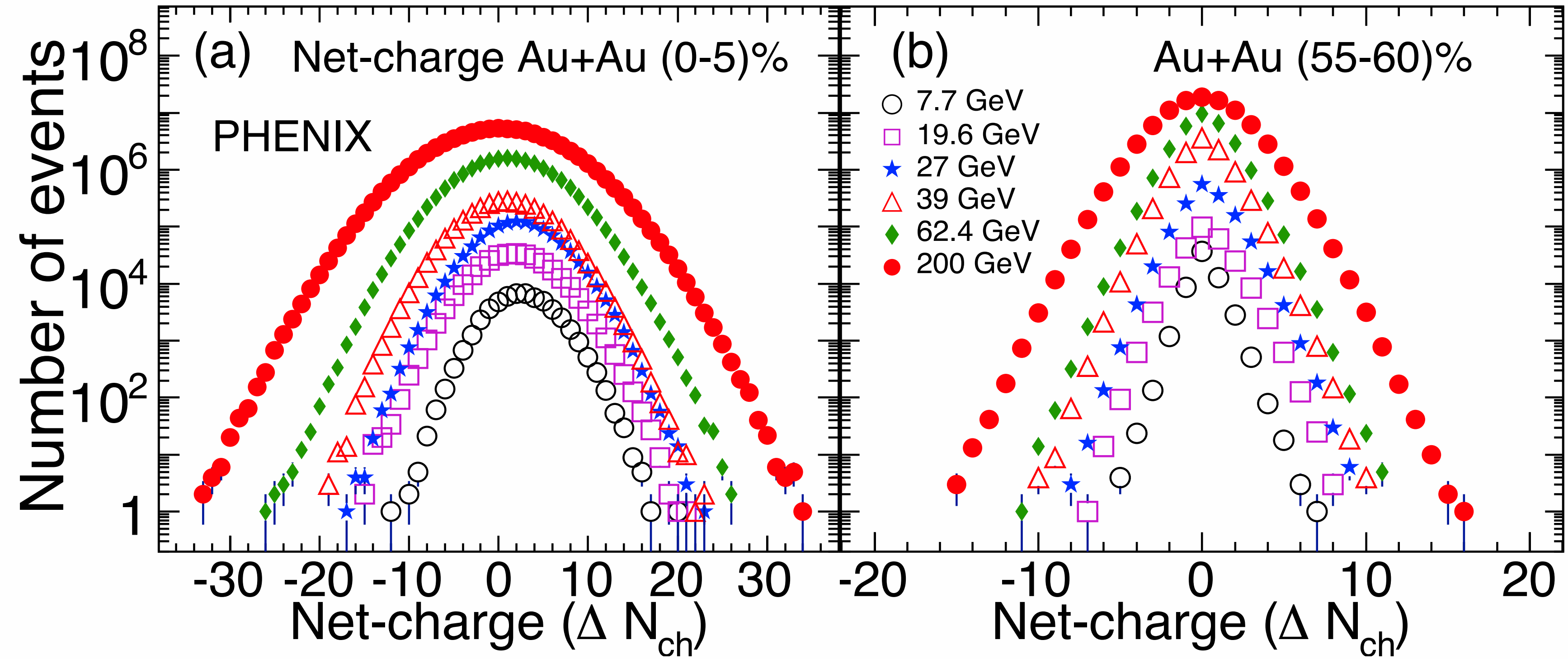
Barndorff-Nielsen,Pollard,Shephard:Quantative Finance 12(2012)587

http://www.economics.ox.ac.uk/materials/papers/4382/paper490.pdf

$$\boxed{\kappa_j = \kappa_j^+ + (-1)^j \kappa_j^-}$$

From PHENIX net-charge fluctuations

PHENIX arXiv:1506.07834



$\Delta N_{ch}=N^{+} - N^{-}$ distribution in $|\eta|<0.35$, $\delta\phi=\pi$, $0.3<p_T<2.0$ GeV/c
Not corrected for detection efficiency $\varepsilon\approx0.70$ in acceptance

The raw moments of the uncorrected distributions can be easily calculated

$$\mu'_k \equiv \langle x^k \rangle \equiv \sum_{i=1}^{n} x_i^k E(x_i) / \sum_{i=1}^{n} E(x_i)$$

$\mu'_1 \equiv \mu = \langle x \rangle$ and x_i is a bin in the $\Delta N_{\rm ch}$ plot with $E(x_i)$ events.

Statistical errors--the complications begin

$$\mu'_k \equiv \langle x^k \rangle \equiv \sum_{i=1}^{n} x_i^k E(x_i) / \sum_{i=1}^{n} E(x_i)$$

The statistical errors for every μ'_k can be calculated from the statistical errors of each data point $E(x_i) \pm \sigma_{E(x_i)}$. Even though the $\sigma_{E(x_i)}$ on each point are independent, the errors on each μ'_k are not independent because the same $\sigma_{E(x_i)}$ appears in all the moments.

Next one computes the cumulants κ_i from the raw (aka) non-central moments:

$$\begin{aligned}
\mu = \kappa_1 &= \mu'_1 \\
\sigma^2 = \mu_2 \equiv \langle (x-\mu)^2 \rangle = \kappa_2 &= \mu'_2 - \mu_1'^2 \\
\mu_3 = \kappa_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3 \\
\mu_4 - 3\mu_2^2 = \kappa_4 &= \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4
\end{aligned}$$

Next correction---Efficiency

A certain random fraction of the tracks that fall on the acceptance are not detected because of inefficiency---a clearly random, thus binomial effect. This is further complicated if the N^+ and N^- measurements have different efficiencies.

Long Range Correlations: Binomial Split of NBD

Carruthers and Shih PLB 165 (1985)209

If a population n is distributed as NBD(μ_t, k) and then divided randomly into 2 subpopulations with probabilities p and $q=1-p$, then the distribution on p is NBD ($p\mu_t$, k) and on q is NBD ($q\mu_t$, k) BUT the two sub-intervals are not statistically independent. Also k does not change!!

So if you measure $\mu=p\mu_t$ with effieicncy p the true value is $\mu_t=\mu/p$

If you measure the distribution, then you know all the corrected cumulants

Cumulants for Poisson, Negative Binomial Distributions Measured with **efficiency** p corrected to true value, explicit in μ_t and k

Measured Cumulant	Corrected Poisson	Corrected Negative Binomial Expanded
$\kappa_1 = \mu$	$\mu_t \equiv \mu/p$	$\mu_t \equiv \mu/p$
$\kappa_2 = \mu_2 = \sigma^2$	μ_t	$\mu_t(1+\mu_t/k) \equiv \sigma_t^2$
$\kappa_3 = \mu_3$	μ_t	$\mu_t(1+3\mu_t/k+2\mu_t^2/k^2)$
$\kappa_4 = \mu_4 - 3\kappa_2^2$	μ_t	$\mu_t(1+7\mu_t/k+12\mu_t^2/k^2+6\mu_t^3/k^3)$
$S \equiv \kappa_3/\sigma^3$	$1/\sqrt{\mu_t}$	$(1+2\mu_t/k)/\sqrt{\mu_t(1+\mu_t/k)}$
$\kappa \equiv \kappa_4/\kappa_2^2$	$1/\mu_t$	$(1+6\mu_t/k+6\mu_t^2/k^2)/[\mu_t(1+\mu_t/k)]$
$S\sigma = \kappa_3/\kappa_2$	1	$(1+2\mu_t/k)$
$\kappa\sigma^2 = \kappa_4/\kappa_2$	1	$(1+6\mu_t/k+6\mu_t^2/k^2)$
$\mu/\sigma^2 = \kappa_1/\kappa_2$	1	$1/(1+\mu_t/k)$
$S\sigma^3/\mu = \kappa_3/\kappa_1$	1	$(1+3\mu_t/k+2\mu_t^2/k^2)$

Use the NBD Cumulant Theorem allowing $\varepsilon=p$ to be different for N^+ and N^-

$$\boxed{\kappa_j = \kappa_j^+ + (-1)^j \kappa_j^-}$$

Efficiency-Corrected NBD Cumulant Ratios

$$\frac{\mu}{\sigma^2}=\frac{\kappa_1^+-\kappa_1^-}{\kappa_2^++\kappa_2^-}=\frac{\mu_t^+-\mu_t^-}{\mu_t^+[1+(\frac{\mu_t^+}{k^+})]+\mu_t^-[1+(\frac{\mu_t^-}{k^-})]}$$

$$\mu_t=\frac{\mu}{\varepsilon}$$

$$\frac{S\sigma^3}{\mu}=\frac{\kappa_3^+-\kappa_3^-}{\kappa_1^+-\kappa_1^-}=\frac{\mu_t^+[1+3(\frac{\mu_t^+}{k^+})+2(\frac{\mu_t^+}{k^+})^2]-\mu_t^-[1+3(\frac{\mu_t^-}{k^-})+2(\frac{\mu_t^-}{k^-})^2]}{\mu_t^+-\mu_t^-}$$

$$S\sigma=\frac{\kappa_3^+-\kappa_3^-}{\kappa_2^++\kappa_2^-}=\frac{\mu_t^+[1+3(\frac{\mu_t^+}{k^+})+2(\frac{\mu_t^+}{k^+})^2]-\mu_t^-[1+3(\frac{\mu_t^-}{k^-})+2(\frac{\mu_t^-}{k^-})^2]}{\mu_t^+[1+\frac{\mu_t^+}{k^+}]+\mu_t^-[1+\frac{\mu_t^-}{k^-}]}$$

$$\kappa\sigma^2=\frac{\kappa_4^++\kappa_4^-}{\kappa_2^++\kappa_2^-}=\frac{\mu_t^+[1+7(\frac{\mu_t^+}{k^+})+12(\frac{\mu_t^+}{k^+})^2+6(\frac{\mu_t^+}{k^+})^3]+\mu_t^-[1+7(\frac{\mu_t^-}{k^-})+12(\frac{\mu_t^-}{k^-})^2+6(\frac{\mu_t^-}{k^-})^3]}{\mu_t^+[1+\frac{\mu_t^+}{k^+}]+\mu_t^-[1+\frac{\mu_t^-}{k^-}]}$$

The NBD only uses 4 quantities for this calculation: μ_t^+ and μ_t^- $(\mu_t/k)^+$ and $(\mu_t/k)^-$
The error on μ_t <<than the error on μ_t/k so is neglected. The errors are highly correlated for the sums of powers of μ_t/k in both the numerator and denominator. These correlations are handled by varying the $(\mu_t/k)^+$ and $(\mu_t/k)^-$ by $\pm1\sigma$ independently and adding the variations in quadrature

Bzdak-Koch standard Binomial efficiency correction PRC 86 (2012) 044904

Efficiency corrected cumulants in terms of corrected double Factorial moments

$$N = \langle N_+ \rangle + \langle N_- \rangle$$

$$\kappa_1 = \langle N_+ \rangle - \langle N_- \rangle = \frac{\langle n_+ \rangle}{\epsilon_+} - \frac{\langle n_- \rangle}{\epsilon_-},$$

$$F_{ik} = \sum_{N_1=i}^{\infty} \sum_{N_2=k}^{\infty} P(N_1, N_2) \frac{N_1!}{(N_1-i)!} \frac{N_2!}{(N_2-k)!}$$

$$\kappa_2 = N - \kappa^2{}_1 + F_{02} - 2F_{11} + F_{20},$$

$$\kappa_3 = \kappa_1 + 2\,\kappa^3{}_1 - F_{03} - 3F_{02} + 3\,F_{12} + 3\,F_{20} - 3F_{21} + F_{30}$$
$$- 3\kappa_1(N + F_{02} - 2F_{11} + F_{20}),$$

$$\kappa_4 = N - 6\kappa^4{}_1 + F_{04} + 6\,F_{03} + 7\,F_{02} - 2F_{11} - 6F_{12} - 4F_{13}$$
$$+ 7F_{20} - 6F_{21} + 6F_{22} + 6F_{30} - 4F_{31} + F_{40}$$
$$+ 12\kappa^2{}_1(N + F_{02} - 2F_{11} + F_{20}) - 3(N + F_{02} - 2F_{11} + F_{20})^2$$
$$- 4\kappa_1(\kappa_1 - F_{03} - 3F_{02} + 3\,F_{12} + 3\,F_{20} - 3F_{21} + F_{30})$$

Here you can see the nice subtraction of the lower order moments; but new quantities, double Factorial Moments are introduced and very difficult to compute $P(13^+, 11^-)$=? so you need to know both N_+ and N_- distributions and their correlations. The F_{ik} can be calculated from the data by making a 3d Lego plot with base axes N_+ and N_- and height $P(N_+, N_-)$ which costs statistical error but other methods, e.g. Monte Carlo, are used.

Are acceptance corrections possible?

$$\frac{S\sigma^3}{\mu} = \frac{\kappa_3^+ - \kappa_3^-}{\kappa_1^+ - \kappa_1^-} = \frac{\mu_t^+[1+3(\frac{\mu_t^+}{k^+})+2(\frac{\mu_t^+}{k^+})^2] - \mu_t^-[1+3(\frac{\mu_t^-}{k^-})+2(\frac{\mu_t^-}{k^-})^2]}{\mu_t^+ - \mu_t^-}$$

$$R_{32} - R_{12} = S\sigma - \frac{\mu}{\sigma^2} = \frac{\mu_t^+[3(\frac{\mu_t^+}{k^+})+2(\frac{\mu_t^+}{k^+})^2] - \mu_t^-[3(\frac{\mu_t^-}{k^-})+2(\frac{\mu_t^-}{k^-})^2]}{\mu_t^+[1+\frac{\mu_t^+}{k^+}] + \mu_t^-[1+\frac{\mu_t^-}{k^-}]}$$

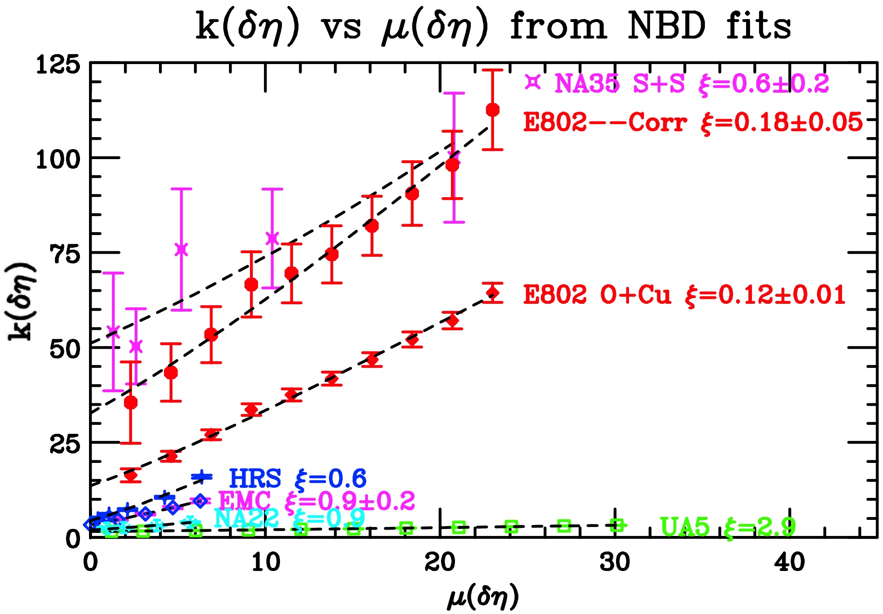
Bzdak and Koch (and likely many others) have expressed concern about what is the "required acceptance" for an experimental result e.g. on the above quantities to compare with Lattice QCD calculations

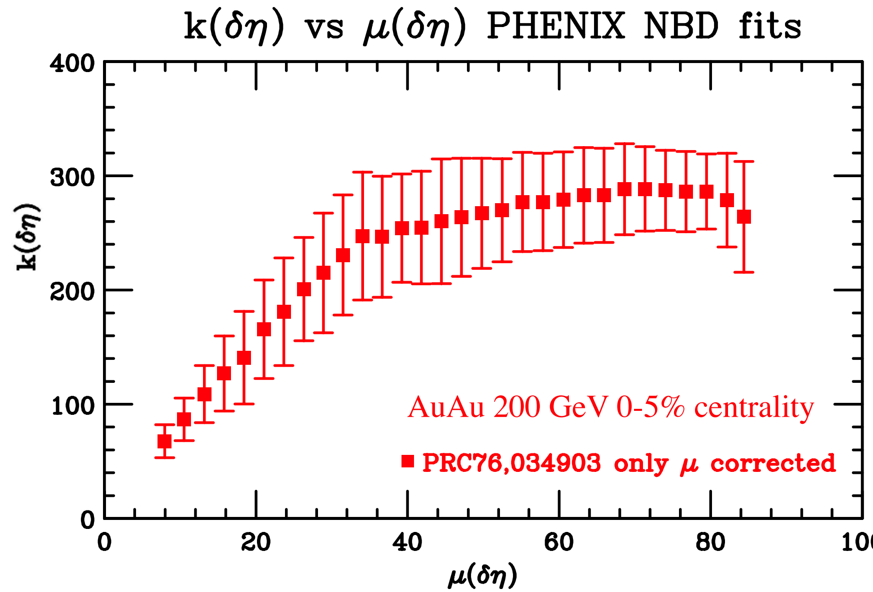
The good news from the above equations and those on the previous pages is that if the ratios $(\mu_t/k)^+$ and $(\mu_t/k)^-$ don't change with the acceptance and if μ_t^+ and μ_t^- scale by the same amount with the acceptance (e.g. dn/dη constant in rapidity and azimuth) then the above formulas remain unchanged. What does nature say?

Recall the NBD slide from E802

E802 PRC**52**,2663(1995)

PHENIX PRC**76**,0349033(2007)





The nice examples of short range correlation with ξ, indicated in the E802 plot, change dramatically in the newer PHENIX Au+Au (200 GeV) measurement with the abrubt flattening of k(δη) for μ(δη)>30, |η|>0.15. This as far as I know is the only such measurement at RHIC or LHC. The E802 data has perfect centrality, all nucleons interact as measured in a ZDC, so the suggestion is that the flattening could be a long range correlation due to fluctuations in the number of participants in a centrality bin.

Conclusions

- The NBD cumulant theorem brings a huge simplification to calculating the efficiency correction and statistical errors on net-charge cumulants.

- Acceptance corrections are much more difficult because of short range correlations in $\delta\eta$ and $\delta\phi$, but in certain cases discussed above the cumulant ratios will remain constant independent of acceptance, so would be one possible resolution to the question of the "required acceptance" to compare experiments with Lattice QCD calculations

- Fortunately, the two above issues can be further investigated by both experiment and theory. For instance if the STAR NBD data for net charge were available, I could calculate the corrected values and the errors for $\kappa\sigma^2$, etc. Similarly STAR could make cuts in acceptance in their measurements to determine the variation in the results and whether or where the "required acceptance" is satisfied.

Extras

- NBD fit plots
- 4 generating functions
- k(δη) PRC76,0349033(2007)

4 Generating functions

Moment generating fn

$$M'_x(t) = \left\langle e^{tx} \right\rangle$$

Cumulant generating function

$$g_x(t) = \ln M'_x(t) = \ln \left\langle e^{tx} \right\rangle$$

Factorial moment gen fn.

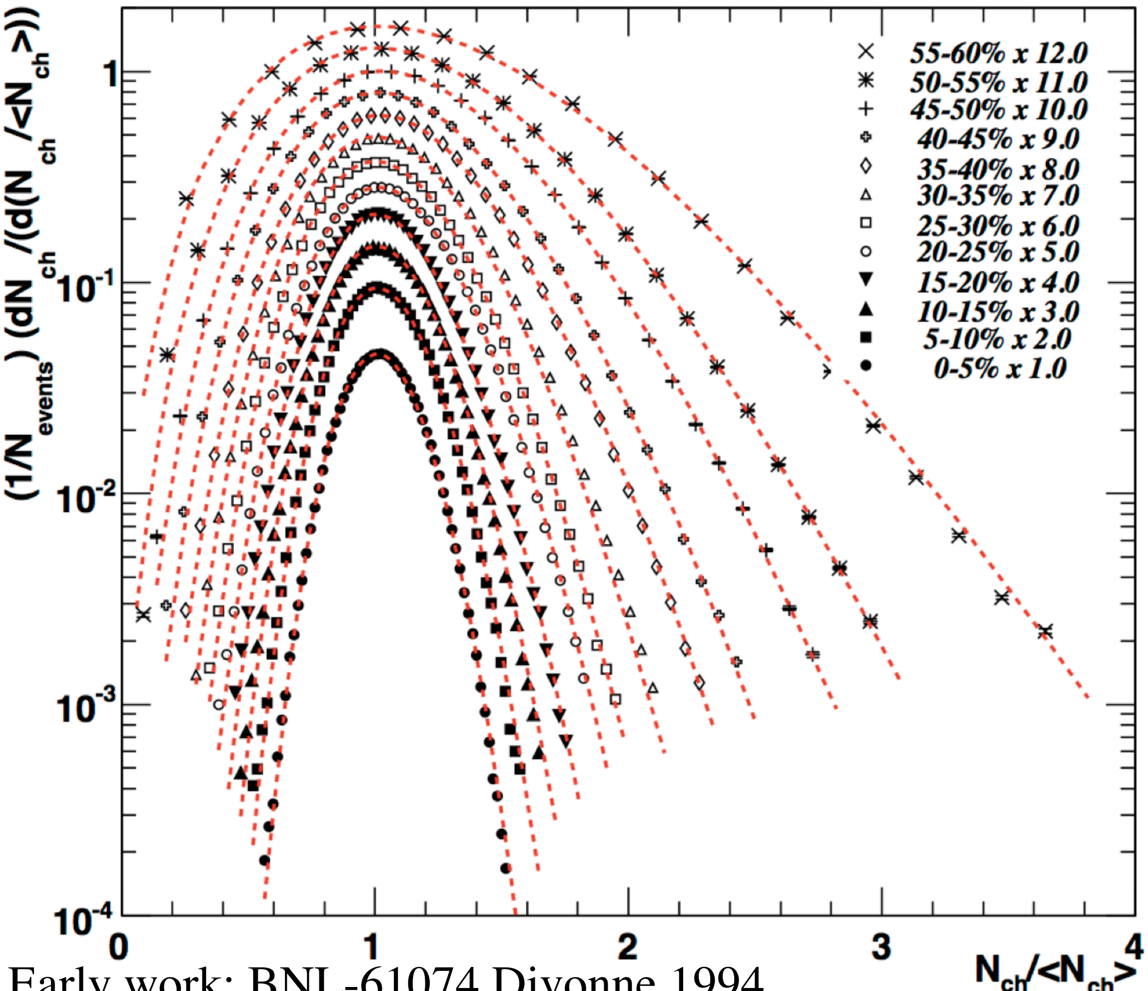
$$M_x(t) = \left\langle (1+t)^x \right\rangle$$

Factorial cumulant gen fn.

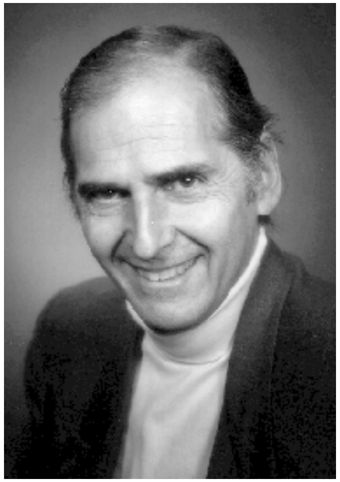
$$g_x(t) = \ln \left\langle (1+t)^x \right\rangle$$

From one of Jeff Mitchell's talks 2001: "Multiplicity Fluctuations"

PHENIX AuAu Multiplicity N_{ch} PRC **78**, (2008) 044902



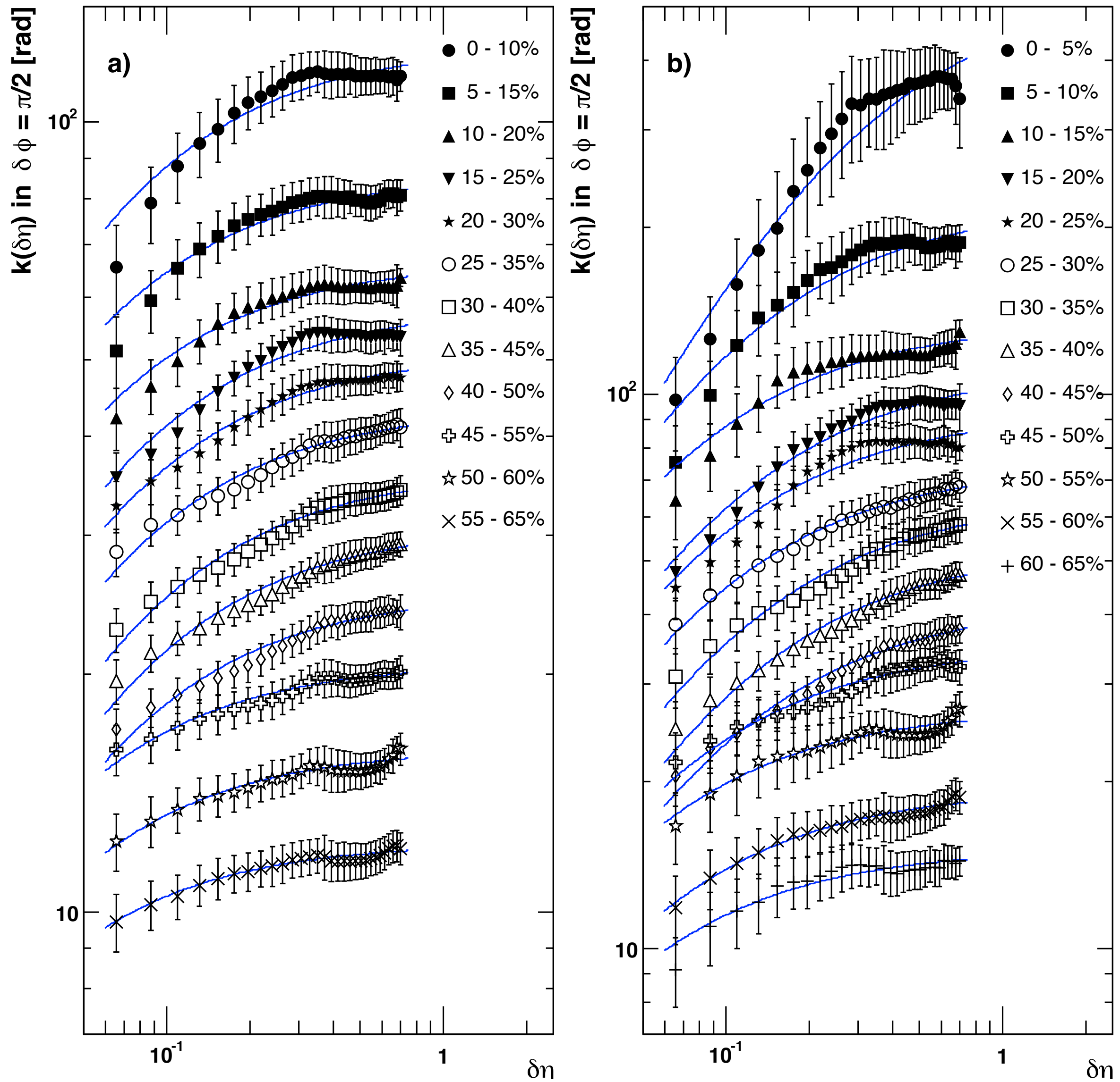
It's not Poisson
It's a Negative Binomial distribution!



Early work: BNL-61074 Divonne 1994

http://www.osti.gov/scitech/servlets/purl/10108142

PHENIX k(δη) PRC76,0349033(2007)

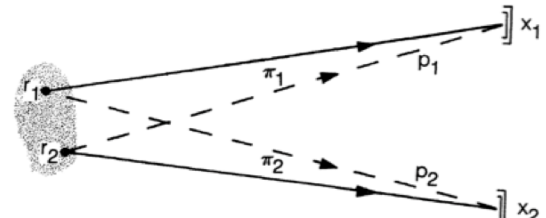


Short range multiplicity correlations do not vanish in A+A collisions!

- Short range multiplicity correlations in p-p collisons come largely from hadron decays such as $\rho\rightarrow\pi\,\pi$, $\Lambda\rightarrow\pi^-\,p$, etc., with correlation length ξ~1 unit of rapidity

- In A+A collisions the chance of getting two particles from the same ρ meson is reduced by~$1/N_{part}$ so that **the only remaining correlations are Bose-Einstein Correlations---**when two identical Bosons, e.g. $\pi^+\,\pi^+$, occupy nearly the same coordinates in phase space so that constructive interference occurs due to the symmetry of the wave function from Bose statistics---a quantum mechanical effect, which remains at the same strength in A+A collisions:the amplitudes from the two different points add giving a large effect also called Hanbury-Brown Twiss (HBT).

See W.A.Zajc, et al, PRC 29 (1984) 2173



HBT effects in 2-particle Correlations

- The normalized two-particle short range rapidity correlation $R_2(y_1,y_2)$ is defined as

$$R_2(y_1,y_2) \equiv \frac{C_2(y_1,y_2)}{\rho_1(y_1)\rho_1(y_2)} \equiv \frac{\rho_2(y_1,y_2)}{\rho_1(y_1)\rho_1(y_2)} - 1 = R(0,0)\, e^{-|y_1-y_2|/\xi} \quad , \qquad (8)$$

where $\rho_1(y)$ and $\rho_2(y_1,y_2)$ are the inclusive densities for a single particle (at rapidity y) or 2 particles (at rapidities y_1 and y_2), $C_2(y_1,y_2) = \rho_2(y_1,y_2) - \rho_1(y_1)\rho_1(y_2)$ is the Mueller correlation function for 2 particles (which is zero for the case of no correlation), and ξ is the two-particle short-range rapidity correlation length[3] for an exponential parameterization.

$$\boxed{K_2(\delta\eta) = 2R(0,0)\frac{(\delta\eta/\xi - 1 + e^{-\delta\eta/\xi})}{(\delta\eta/\xi)^2}}$$

for NBD: $k(\delta\eta) = 1/K_2(\delta\eta)$

The rapidity correlation length $\xi = 0.2$ for Si+Au E802, PRC56(1977) 1544 is from HBT.

if $\delta\eta \ll \xi$, $k \rightarrow 1/R(0,0)$=constant if $\delta\eta \gg \xi$, $k/\delta\eta \approx k/\mu \rightarrow$ constant

- For HBT analyses of two particles with $\mathbf{p}_1$ and $\mathbf{p}_2$, $C^{HBT}{}_2(\mathbf{q}) = R_2(\mathbf{p}_1 - \mathbf{p}_2)+1$ and the random (un-correlated) distribution is taken from particles with $\mathbf{p}_1$ and $\mathbf{p}_2$ on different events. The HBT correlation function is taken as a Gaussian not an exponential as in (8) and is written:

$$\boxed{C_2^{HBT} = 1 + \lambda \exp -(R_{side}^2 q_{side}^2 + R_{out}^2 q_{out}^2 + R_{long}^2 q_{long}^2)}$$