Holomorphic blocks and q-deformed correlators

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In recent years many exact results for gauge theories on compact manifolds have been obtained by the method of SUSY localisation.

The idea is that by adding a Q-exact term to the action it is possible to reduce the path integral to a finite dimensional integral:

Localisation: $Z_{\mathcal{M}} = \int D\psi e^{-S[\Psi]} = \int D\Psi_0 e^{-S[\Psi_0]} Z_{\text{1-loop}}[\Psi_0]$

- Ψ_0: field configurations satisfying localising (saddle point) equations
- with a clever localisation scheme, Ψ_0 is a finite dimensional set
- $Z_{\text{1-loop}}[\Psi_0]$ is due to the quadratic fluctuation around Ψ_0

$\Rightarrow$ useful to study holography

$\Rightarrow$ connect to exactly solvable models such as 2d CFTs and TQFTs

So far exact results have been obtained for
$S^2, S^2 \times S^1, S^3/\mathbb{Z}_r, S^3/\mathbb{Z}_r \times S^1, S^4, S^4 \times S^1, S^5, Y_{p,q} \cdots$
[Benini-Cremonesi],[Droud-Gomis-LeFloch-Lee], [Kapustin-Willett-Yaakov], [Imamura-Yokoyama],[Kapustin-Willet],[Gadde-Pomoni-Rastelli-Razamat],[Kim-Kim-Lee],[Terashima],[Iqbal-Vafa],[Kallen-Zabzine],[Kallen-Qiu-Zabzine],[Hosomichi-Seong-Terashima],[Imamura],[Lockhart-Vafa],[Kim-Kim-Kim]...

Comprehensive formalism for SUSY theories formulated on curved manifolds initiated in [Festuccia-Seiberg]. Recent developments:

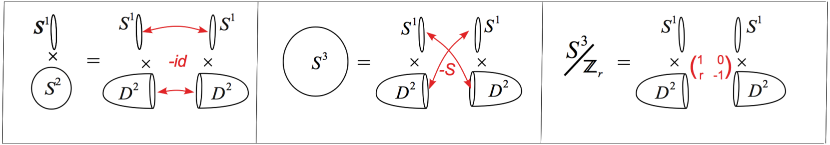
$4d$, $\mathcal{N} = 1$ theories with $U(1)_R$ can be defined on complex manifolds with an Hermitian metric. Partition functions are "topological quantities": they are metric independent and compute complex structures and holomorphic vector bundles (defined by background gauge fields) invariants. Similar results for $3d$, $\mathcal{N} = 2$ theories [Closset-Dumitrescu-Festuccia-Komargodski].

... localisation computations are long, geometric data hidden.

Is there a set of building blocks to construct partition functions?

Are there new integrable structures associated to these blocks?

3-manifolds from solid tori $D^2 \times S^1$ gluing



Is there a QFT analogue of this decomposition?

Yes, $\mathcal{N} = 2$ partition functions can be factorised into holomorphic-blocks [SP],[Beem-Dimofte-SP],[Nieri-SP, to appear]

$$Z_{\mathcal{M}_g} = \mathcal{P} \sum_{\alpha} \mathcal{B}_{\alpha}^{D^2 \times S^1}(\vec{x}, q)\, \mathcal{B}_{\alpha}^{D^2 \times S^1}(\tilde{\vec{x}}, \tilde{q}) = \mathcal{P} \sum_{\alpha} \left|\left| \mathcal{B}_{\alpha}^{D^2 \times S^1}(\vec{x}, q) \right|\right|_g^2$$

α labels SUSY vacua, $\vec{x}$, flavor parameters, $q = e^{2\pi i \tau}$ is the boundary torus complex structure and

$$\tau \to \tilde{\tau} = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot SL(2, \mathbb{Z}).$$

Factorisation$\leftrightarrow$dynamical parity anomaly cancellation. $\mathcal{P}$ is the contribution of background mixed Chern-Simons terms.

Example: the lens space

The Coulomb branch localisation of $\mathcal{N}=2$ theories on the lens space $S^3/\mathbb{Z}_r$ yields [Benini-Nishioka-Yamazaki],[Imamura-Matsuno-Yokoyama]

$$Z_{S^3/\mathbb{Z}_r} = \sum_{\vec{l}=0}^{r-1} \int d\vec{z}\; Z_{cl}(\vec{z},\vec{l};\vec{m},\vec{H},\omega_1,\omega_2,r)\cdot Z_{1\text{loop}}(\vec{z},\vec{l};\vec{m},\vec{H},\omega_1,\omega_2,r)$$

$\vec{l},\vec{H}$ are dynamical and flavor holonomies, $\vec{m}$ are mass parameters, $\omega_{1,2}$ are complex structure parameters (squashing).

A chiral multiplet in the fundamental representation contributes as

$$Z^{chiral}_{1\text{loop}} = \prod_{a=1}^{N_f}\prod_{n=1}^{N} \hat{s}_{b,-\ell_n-H_a}\left(i\frac{Q}{2}-z_n-\mu_a\right)\ ,$$

The function $\hat{s}_{b,h}$ is the projection of the double Sine function:

$$\hat{s}_{b,-h}(z) = e^{\frac{i\pi}{2r}([h](r-[h])-(r-1)h^2)} \prod_{\substack{n_1,n_2\geq 0\\ n_2-n_1=h \mod r}} \frac{n_1\omega_1+n_2\omega_2+Q/2-iz}{n_2\omega_1+n_1\omega_2+Q/2+iz}\ .$$

Performing the integration (residues computation) and summing over the holonomies we find [Nieri-SP, to appear],[Imamaura-Yokoyama]

$$Z_{S^3/\mathbb{Z}_r} = \mathcal{P} \sum_{\alpha} \left|\left| \mathcal{B}_{\alpha}^{D^2 \times S^1}(\vec{x}; q) \right|\right|_r^2 ,$$

with

$$q = e^{2\pi i \tau} = e^{2\pi i \frac{\omega_1+\omega_2}{r\omega_1}} , \qquad \vec{x} = e^{\vec{X}} e^{-\frac{2\pi i \vec{H}}{r}} = e^{2\pi i \frac{\vec{m}}{r\omega_1}} e^{-\frac{2\pi i \vec{H}}{r}} ,$$

and

$$\tilde{\tau} = \frac{\tau}{r\tau - 1} , \quad \tilde{X} = \frac{X}{r\tau - 1} , \quad \tilde{H} = r - H .$$

This gluing rule is consistent with the realisation of $L(r,1)$ from a pair of solid tori $[0,1] \times T^2$.

3d holomorphic blocks

- can be defined as basis of solutions to difference equations.

  Example: the SQED, $U(1)$ theory with N_f flavours and FI parameter u, are solutions ($\alpha = 1, \cdots N_f$) of the q-hypergeometric difference equation

$$\mathcal{B}_\alpha^{D^2\times S^1}(\vec{x};q) = \frac{\theta(x_\alpha u;q)}{\theta(u;q)\theta(x_\alpha;q)} \prod_{j,k}^{N_f} \frac{(qx_j x_\alpha^{-1};q)_\infty}{(y_k x_\alpha^{-1};q)_\infty} \mathcal{Z}_V^{(\alpha)} ,$$

$$Z_V^{(\alpha)} = \sum_{p=1}^{\infty} \prod_{j,k=1}^{N_f} \frac{(x_\alpha y_k^{-1};q)_p}{(qx_\alpha x_j^{-1};q)_p} u^p = {}_{N_f}\Phi_{N_f-1}(x_\alpha y_j{-}1, qx_\alpha x_j^{-1};u) .$$

- difference equations are solved by block integrals: [Beem-Dimofte-SP]

$$\mathcal{B}_\alpha^{D^2\times S^1}(\vec{x};q) = \int_{\mathcal{C}_\alpha} \frac{ds}{2\pi i s} \Upsilon(s,\vec{x};q)$$

  - $\mathcal{C}_\alpha$ are all convergent (downward-flow) contours.
  - at special values of $(\vec{x},q)$, Stokes walls contours can jump.
  - the block integrand $\Upsilon(s,\vec{x};q)$ has been recently rederived via localisation on $D^2 \times S^1$ [Yoshida-Sugiyama]

Analytic properties

- Holomorphic blocks are defined by q-series (defined for $|q| \neq 1$).
- When $|q| < 1$ we have $|\tilde{q}| > 1$, the q-series and the $\tilde{q}$-series converge to different functions.

  Example: the free chiral

$$\mathcal{B}_{chiral}^{D^2\times S^1}(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} x^{-n}}{(q)_n} = \begin{cases} \prod_{n=0}^{\infty}\left(1 - q^{n+1}x^{-1}\right) & |q| < 1\,, \\ \prod_{n=0}^{\infty}\left(1 - q^{-n}x^{-1}\right)^{-1} & |q| > 1\,. \end{cases}$$

  $\mathcal{B}_{\alpha}^{D^2\times S^1}(x;q)$ and $\mathcal{B}_{\alpha}^{D^2\times S^1}(\tilde{x};\tilde{q})$ transform as independent functions!

  At Stokes walls we have:

$$\mathcal{B}_{\alpha}^{D^2\times S^1}(\vec{x},q) \to M_{\alpha}^{\beta}\mathcal{B}_{\beta}^{D^2\times S^1}(\vec{x},q)\,,\ \mathcal{B}_{\alpha}^{D^2\times S^1}(\vec{\tilde{x}},\tilde{q}) \to (M^{-1\,T})_{\alpha}^{\beta}\mathcal{B}_{\beta}^{D^2\times S^1}(\vec{\tilde{x}},\tilde{q})\,,$$

  while partition functions stay invariant.

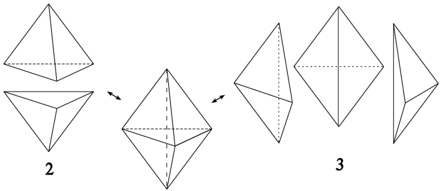
3d-3d correspondence

Arises by wrapping $M5$ branes on $M \times N$, where M=hyperbolic 3-manifold and $N = S_b^3, S^2 \times S^1,\ D^2 \times S^1$ and states that:

[Dimofte-Gukov],[Dimofte-Gukov-GaiottoI,II]

$\mathcal{T}(M)$, 3d $\mathcal{N} = 2$ theory on $N \leftrightarrow$ complex Chern-Simons on M

- ideal tetrahedron $\leftrightarrow$ free chiral + half Chern-Simons unit
- gluing tetrahedra $\leftrightarrow$ gauging flavour symmetries
- internal edges $\leftrightarrow$ superpotential couplings
- change of triangulation $\leftrightarrow$ 3d mirror symmetry
- fundamental move



gauged $U(1)$ with 2 chirals $\leftrightarrow$ 3 chirals with superpotential (XYZ)

$\rightarrow$ geometric classification of a large class of abelian mirror symmetries

3d blocks and analytically continued Chern-Simons

Take the 3d $\mathcal{N} = 2$ theory on the solid torus $N = D^2 \times S^1$

- SUSY vacua α in $\mathcal{T}(M) \leftrightarrow$ flat $SL(2, \mathbb{C})$ connections A^α on M
- holomorphic blocks $\mathcal{B}_\alpha^{D^2 \times S^1}(x, q)$ $\leftrightarrow$ analytically continued Chern-Simons partition functions $Z_\alpha^{CS}(x, q)$ introduced by Witten.

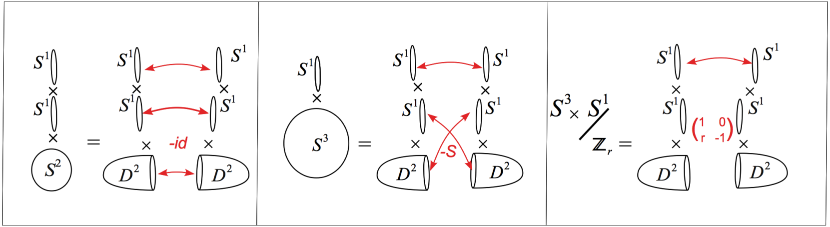
Example: If M is the 4_1 knot complement $\mathcal{T}(4_1)$ is the $U(1)$ theory with 2 chirals (for particular value of the masses).

- two vacua, two blocks $\leftrightarrow$ two CS irreducible flat connections.
- asymptotics for $q = e^\hbar$, $\hbar \to 0$:

$$Z^{CS}_{\alpha=1,2}(x, q) = \mathcal{B}_{\alpha=1,2}^{D^2 \times S^1}(x, q) \sim \exp\left(\frac{i}{\hbar}[\pm 2.0298]\right)$$

$\rightarrow$ our block integrals are the first concrete examples of non-perturbative path integrals in analytically continued CS along "exotic" integration cycles (labelled by irreducible flat $SL(2, \mathbb{C})$ connections).

4-manifolds from solid tori $D^2 \times T^2$ gluing



There is a corresponding factorisation of $\mathcal{N} = 1$ partition functions [Peelaers],[Yoshida],[Nieri-SP,to appear]. Example:

$$Z_{S^3 \times S^1/\mathbb{Z}_r} = \mathcal{A} \sum_{\alpha} \left|\left| \mathcal{B}_{\alpha}^{D^2 \times T^2}(\vec{x}; q_\tau, q_\sigma) \right|\right|_r^2$$

$$q_\tau = e^{2\pi i \frac{\omega_1+\omega_2}{r\omega_1}} = e^{2\pi i \tau}, \quad q_\sigma = e^{-2\pi i \frac{\omega_3}{r\omega_1}} = e^{-2\pi i \sigma}, \quad \vec{x} = e^{2\pi i \frac{\vec{m}}{r\omega_1}} e^{\frac{2\pi i \vec{H}}{r}} = e^{\vec{X}} e^{\frac{2\pi i \vec{H}}{r}}$$

$$\tilde{\tau} = \frac{\tau}{r\tau - 1}, \qquad \tilde{\sigma} = \frac{\sigma}{r\tau - 1}, \quad \tilde{X} = \frac{X}{r\tau - 1}, \quad \tilde{H} = r - H$$

Factorisation↔dynamical anomaly cancellation. $\mathcal{A}$, extracted by an $SL(3, Z)$ transformation, is the contribution of background anomalies.

In 3d and 4d the factorisation can be understood in various ways:

- as the result of an alternative localisation scheme, Higgs branch localisation, where the localising loci are vortices [Benini-Peelaers], [Fujitsuka-Honda-Yoshida].
- as a consequence of the quasi-topological nature of partition functions left invariant by deformation to cigars connected by infinitively long tubes: effective projection [Alday-Martelli-Richmond-Sparks].
- in the more general tt^* setup, developed for 3d and 4d theories [Cecotti-Gaiotto-Vafa].

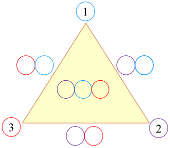
In 5d the factorisation is already present on the Coulomb branch.

Example: $\mathcal{N} = 1$ theories on S^5

Localisation on $\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 + \omega_3^2|z_3|^2 = 1$ yields:

[Kallen-Zabzine],[Hosomichi-Seong-Terashima],[Kim-Kim-Kim],[Lockart-Vafa]

S^5:



$$\Rightarrow Z_{S^5} = \int d\sigma \; Z_{cl} Z_{1loop} \left|\left| \mathcal{Z}^{5d}_{inst} \right|\right|^3_S$$

- $\mathbb{R}^4 \times S^1$ instantons $\mathcal{Z}^{5d}_{inst}(e^{2\pi\sigma/e_3}, e^{2\pi\vec{m}/e_3}; q, t)$ are localized at fixed points of the Hopf fibration and are glued as:

$$\left|\left| f(e^{2\pi z/e_3}; q, t) \right|\right|^3_S := \prod_{k=1}^{3} f(e^{2\pi z/e_3}; q, t)_k \,, \quad q = e^{2\pi i e_1/e_3}, t = e^{2\pi i e_2/e_3}$$

$$(e_1, e_2, e_3) = (\omega_3, \omega_2, \omega_1)\,, (\omega_1, \omega_3, \omega_2)\,, (\omega_1, \omega_2, \omega_3) \quad \text{for} \quad k = 1, 2, 3\,.$$

- 1-loop contributions are: expressed in terms of triple-sine functions:

$$S_3(x) = \prod_{i,j,k} (i\omega_1 + j\omega_2 + k\omega_3 + x)(i\omega_1 + j\omega_2 + k\omega_3 + E - x)\,, \qquad E = \omega_1 + \omega_2 + \omega_3\,.$$

It is possible to factorise the classical (Yang-Mills and Chern-Simons terms) and 1loop parts

$$Z_{cl} Z_{1loop} = \left|\left|\mathcal{Z}_{cl} \mathcal{Z}_{1loop}\right|\right|_S^3$$

using that [Felder-Varchenko]:

$$e^{-\frac{2\pi i}{3!} B_{33}(x,\vec{\omega})} = \left|\left|\Gamma_{q,t}(x/e_3)\right|\right|_S^3, \qquad \Gamma_{q,t}(z) = \frac{(e^{-2\pi i z}\, q\, t\,; q, t)}{(e^{2\pi i z}; q, t)}$$

and

$$S_3(iz) = e^{-\frac{\pi i}{3!} B_{33}(iz)} \left|\left|(e^{-\frac{2\pi}{e_3} z}; q, t)\right|\right|_S^3$$

and obtain the block factorized form which respects periodicity (invariance under shift $z \to z + ik\omega_i$) in each sector:

$$\boxed{Z_{S^5} = \int \ d\sigma \ \left|\left|\mathcal{B}^{5d}\right|\right|_S^3, \qquad \mathcal{B}^{5d} := \mathcal{Z}_{\text{cl}} \ \mathcal{Z}_{\text{1-loop}} \ \mathcal{Z}^{5d}_{\text{inst}}}$$

$\rightarrow$ these blocks are universal!

5-manifolds from solid tori $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1$ gluing

It is possible to introduce a set of 5d holomorphic blocks, such that:
[Nieri-SP-Passerini-Torrielli]

$$Z_{S^4 \times S^1} = \int d\sigma \, \left|\left|\mathcal{B}^{5d}\right|\right|^2_{id}, \qquad Z_{S^5} = \int d\sigma \, \left|\left|\mathcal{B}^{5d}\right|\right|^3_{S}, \qquad \cdots$$

Now id, S are elements in $SL(3, Z)$ to glue the three boundary circles.

Generalisation to $\mathcal{N} = 1$ theories on toric Sasaki-Einstein manifolds, T^3 fibrations over an n-gon, [Qiu-Tizzano-Winding-Zabzine]

$$Z_n = \int d\sigma \, \prod_{k=1}^{n} \, (\mathcal{B}^{5d})_k \, .$$

So far: all the exact results for SUSY partition functions on compact manifolds in various dimensions, derived via localisation, can be re-obtained by gluing a small set of building blocks.
Next: construct new results, add defect operators, explore more general backgrounds ...

There are two more very important examples:

- 4d $\mathcal{N}$=2 theories on S^4: [Pestun]

$$Z_{S^4} = \int d\sigma \; Z_{cl} Z_{1loop} \left| \mathcal{Z}^{4d}_{inst} \right|^2 = \int d\sigma \; \left| \mathcal{B}^{4d} \right|^2$$

- $\mathcal{N} = (2,2)$ theories on S^2: [Droud-Gomis-Le Floch-Lee],[Benini-Cremonesi]

$$Z_{S^2} = \sum_{\alpha} \left| \mathcal{B}^{2d}_{\alpha} \right|^2$$

Remarkably $\mathcal{B}^{4d}$, $\mathcal{B}^{2d}$ are the building blocks of another theory: they are (normalised) Toda CFT conformal blocks. This is the main statement of the AGT correspondence.

AGT correspondence

The Alday-Gaiotto-Tachikawa correspondence relates:

- 4d “class S” $\mathcal{N}=2$ gauge theories $\mathcal{T}_{g,n}$, obtained wrapping M5 on $C_{g,n}$ [Gaiotto]. These theories enjoy S-duality corresponding to different pant-decompositions of $C_{g,n}$.
- Liouville theory on $C_{g,n}$. It is a non-rational 2d CFT, characterised by 3-point functions and spectrum. Consistency requires modular invariance of correlators.

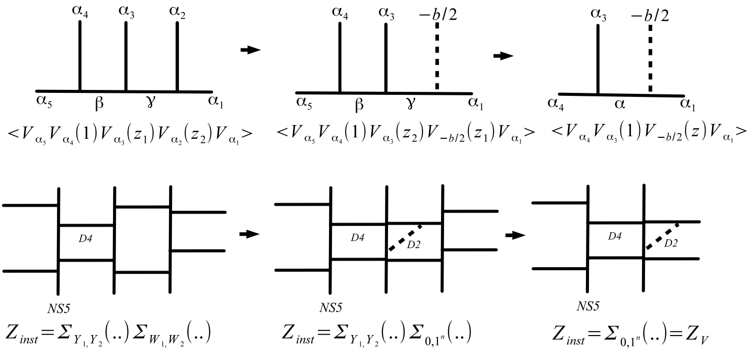
$$\langle \prod_i^n V_{\alpha_i} \rangle_{C_{g,n}} = \int D\alpha \; C \cdots C \; |\mathcal{F}_{\alpha}^{\alpha_i}|^2 = \int [Da] \; Z_{1loop} \left| \mathcal{Z}_{cl} \mathcal{Z}_{inst}^{4d} \right|^2 = Z_{S^4}[\mathcal{T}_{g,n}]$$

$2d$CFT	$4d$ gauge theory
Virasoro conf block : $\mathcal{F}_{\alpha}^{\alpha_i}$	$\mathcal{Z}_{\text{inst}}^{4d}$
3point functions : $C(\alpha_1, \alpha_2, \alpha_3)$	Z_{1loop}
cross ratio z	$e^{2\pi i \tau}$
external momenta α_i	masses m_i
internal momentum α	coulomb branch a

CFT modular invariance $\Leftrightarrow$ generalised $\mathcal{N}=2$ S-duality

Simple surface operators⇔degenerate primaries $\left(L_{-2}+\frac{1}{b^2}L_{-1}^2\right)V_{-b/2}=0$

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde]



- degenerate conformal blocks ↔ vortex counting
  [Dimofte-Gukov-Hollands],[Kozcaz-Pasquetti-Wyllard].
- degenerate correlators ↔ S^2 partition functions
  [Droud-Gomis-LeFloch-Lee],[Gomis-LeFloch]

$$\langle V_{\alpha_4}V_{\alpha_3}(1)V_{-b/2}(z_1)\cdots V_{-b/2}(z_k)V_{\alpha_1}\rangle = Z_{S_2}$$

flop symmetry ⇔ crossing symmetry

Is there a $5d - 3d$ analogue?

Hint 1: $\mathcal{Z}^{5d}_{inst} \leftrightarrow q$-deformed Virasoro chiral blocks. [Awata-Yamada],[many others]

Hint 2: $5d \to 3d$ degeneration of $\mathcal{N} = 1$ partition functions.

Conjecture: S^5 and $S^4 \times S^1$ partition functions are captured by q-deformed Liouville correlators. [Nieri-SP-Passerini]

But what is q-deformed Liouville?

q-deformed Virasoro algebra $\mathcal{V}ir_{q,t}$

$\mathcal{V}ir_{q,t}$ has two complex parameters q, t and generators T_n with $n \in \mathbb{Z}$.
[Shiraishi-Kubo-Awata-Odake],[Lukyanov-Pugai],[Frenkel-Reshetikhin],[Jimbo-Miwa]

$$
\begin{aligned}
[T_n\,,\,T_m] \quad = \quad & -\sum_{l=1}^{+\infty} f_l\,(T_{n-l}\,T_{m+l} - T_{m-l}\,T_{n+l}) \\
& -\frac{(1-q)(1-t^{-1})}{1-p}((q/t)^n - (q/t)^{-n})\delta_{m+n,0}
\end{aligned}
$$

where $f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp\left[\sum_{l=1}^{+\infty} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} z^n\right]$

- For $t = q^{-b_0^2}$ and $q \to 1$, $\mathcal{V}ir_{q,t}$ reduces to Virasoro.
- Emerge as symmetries of solvable 2d lattice models.
- Verma module construction (singular states), Dotsenko-Fateev like integral representation are known [Mironov-Morozov-Shakirov], [Aganagic-Haouzi-Kozcaz-Shakirov].

$\to$ "5d AGT": $\mathcal{Z}_{inst}^{5d} \leftrightarrow \mathcal{V}ir_{q,t}$ chiral blocks [Awata-Yamada,$\cdots$]

Degeneration of 5d partition function

For special values of mass parameters integrals defining partition functions *localize* to discrete sums and satisfy difference equations.

Poles in $Z^{S^5}_{\text{1-loop}}$ and $Z^{S^4\times S^1}_{\text{1-loop}}$ move and pinch the integration contour; the (meromorphic) continuation of partition functions requires taking residues of poles crossing the integration path.

$\rightarrow$ A similar mechanisms reduces non-degenerate Liouville correlators to degenerate ones, which satisfy differential equations. [Ponsot-Teschner]

Example: consider the $SU(2)$, $N_f = 4$ theory on S^5. The poles structure of $Z^{S^5}_{\text{1-loop}}$ is such that:

for $m_1 + m_2 = -i\omega_3$ the integral localizes $\int d\sigma \Rightarrow \sum_{\{\sigma_1,\sigma_2\}}$

When evaluated on $\sigma = \{\sigma_1, \sigma_2\}$, instantons degenerate to vortices:

$$\mathcal{Z}^{5d}_{inst,1} = \sum_{Y_1,Y_2} (\cdots) \to \sum_{0,1^n} (\cdots) = \mathcal{Z}^{(i)}_V \,, \qquad \mathcal{Z}^{5d}_{inst,2} = \sum_{W_1,W_2} (\cdots) \to \sum_{0,n} (\cdots) = \tilde{\mathcal{Z}}^{(i)}_V \,,$$

$$\mathcal{Z}^{5d,III}_{inst,3} = \sum_{X_1,X_2} (\cdots) \to \sum_{0,0} (\cdots) = 1$$

and:

$$Z^{SCQCD}_{S^5} = \int d\sigma \; Z_{cl} Z^{S^5}_{\text{1-loop}} \left|\left| \mathcal{Z}^{5d}_{inst} \right|\right|^3_S \Rightarrow \sum_i^2 \left|\left| \mathcal{B}_i^{D^2 \times S^1} \right|\right|^2_S = Z^{SQED}_{S^3}$$

An identical degeneration works for permutations of $\omega_1, \omega_2, \omega_3$, corresponding to the three big S^3 inside S^5.

A similar mechanisms for $m_1 + m_2 = -ib_0$ leads to

$$Z^{SCQCD}_{S^4 \times S^1} \Rightarrow Z^{SQED}_{S^2 \times S^1}$$

$\rightarrow$ to be reinterpreted as degenerations of q-correlators.

Liouville CFT correlators can be defined and computed in a purely axiomatic fashion, without using the Lagrangian.

- conformal blocks are determined by the Virasoro algebra
- 3-point functions can be obtained using degenerate reps of the Virasoro algebra + crossing symmetry=bootstrap approach
  [Belavin-Polyakov-Zamolodchikov],[Teschner]

We define and compute q-deformed Liouville correlators in a purely axiomatic fashion, without knowing the Lagrangian:

- chiral blocks are determined by the $\mathcal{V}ir_{qt}$ algebra
- we determine 3-point function using degenerate reps of the $\mathcal{V}ir_{qt}$ algebra + crossing symmetry+gluing prescription (inspired by gauge theory)= q-deformed bootstrap approach

q-deformed Bootstrap Approach

Consider a 4-point correlator with a degenerate insertion

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z, \tilde{z}) V_{\alpha_1}(0) \rangle \sim G(z, \tilde{z})$$

Correlators with degenerate primaries (singular states in the Verma module) satisfy difference equations. For the lowest degenerate we find:
[Awata-Kubo-Morita-Odake-Shiraishi],[Awata-Yamada], [Schiappa-Wyllard]

$$D(A, B; C; q; z) G(z, z) = 0 \,, \qquad D(\tilde{A}, \tilde{B}; \tilde{C}; \tilde{q}; \tilde{z}) G(z, \tilde{z}) = 0 \,,$$

where $D(A, B; C; q; z)$ is the q-hypergeometric operator.

$\rightarrow G(z, \tilde{z})$ is a bilinear combination of solutions

Around $z = 0$

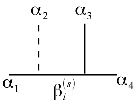
$$I_1^{(s)} = {}_2\Phi_1(A, B; C; z)\,, \qquad I_2^{(s)} = \frac{\theta(q^2C^{-1}z^{-1}; q)}{\theta(qC^{-1}; q)\theta(qz^{-1}; q)}\, {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2C^{-1}; z)$$

For $q \to 1$ becomes the undeformed s-channel basis.

s-channel correlator:

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(r)V_{\alpha_2}(z)V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(s)} K_{ij}^{(s)} I_j^{(s)}$$

$$= \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_*^2 = \sum_i$$



$K_{ij}^{(s)}$ is diagonal with elements related to 3-point functions:

$$K_{ii}^{(s)} = C(\alpha_4, \alpha_3, \beta_i^{(s)})\, C(Q_0 - \beta_i^{(s)}, -b_0/2, \alpha_1)\,, \quad \beta_i^{(s)} = \alpha_1 \pm \frac{b_0}{2}\,, \;\; i = 1, 2$$

$\to$ we will need to prescribe the gluing $\left|\left|(\cdots)\right|\right|_*^2$

Around $z = \infty$

$$I_1^{(u)} = \frac{\theta(qA^{-1}z^{-1};q)}{\theta(A^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(A, qAC^{-1}; qAB^{-1}; q^2z^{-1}),$$

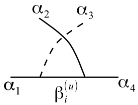
$$I_2^{(u)} = \frac{\theta(qB^{-1}z^{-1};q)}{\theta(B^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(B, qBC^{-1}; qBA^{-1}; q^2z^{-1})$$

For $q \to 1$ limit becomes the undeformed u-channel basis.

u-channel correlator:

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(r)V_{\alpha_2}(z)V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(u)} K_{ij}^{(u)} I_j^{(s)}$$

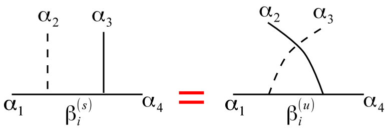
$$= \sum_{i=1}^{2} K_{ii}^{(u)} \left|\left| I_i^{(u)} \right|\right|_*^2 = \sum_i$$



$K_{ij}^{(u)}$ is diagonal with elements related to 3-point functions

$$K_{ii}^{(u)} = C(\alpha_1, \alpha_3, \beta_i^{(u)})\, C(Q_0 - \beta_i^{(u)}, -b_0/2, \alpha_4), \quad \beta_i^{(u)} = \alpha_4 \pm \frac{b_0}{2}, \quad i = 1, 2$$

impose crossing symmetry



$$K_{11}^{(s)}\left|\left|I_1^{(s)}\right|\right|_*^2 + K_{22}^{(s)}\left|\left|I_2^{(s)}\right|\right|_*^2 = K_{11}^{(u)}\left|\left|I_1^{(u)}\right|\right|_*^2 + K_{22}^{(u)}\left|\left|I_2^{(u)}\right|\right|_*^2$$

analytic continuation $I_i^{(s)} = \sum_{j=1}^2 M_{ij} I_j^{(u)}$, $\tilde{I}_i^{(s)} = \sum_{j=1}^2 \tilde{M}_{ij} \tilde{I}_j^{(u)}$ yields:

$$\sum_{k,l=1}^2 K_{kl}^{(s)} \tilde{M}_{ki} M_{lj} = K_{ij}^{(u)}$$

Now we need an ansatz for the gluing rule:

- S^5 gluing rule $\rightarrow$ 3-point function $C_S(\alpha_1, \alpha_2, \alpha_3)$
- $S^4 \times S^1$ gluing rule $\rightarrow$ 3-point function $C_{id}(\alpha_1, \alpha_2, \alpha_3)$

3-point functions

- $S^4 \times S^1$ gluing rule $\rightarrow$ *id*-correlators:

$$C_{id}(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{\Upsilon^\beta(2\alpha_T - Q_0)} \prod_{i=1}^{3} \frac{\Upsilon^\beta(2\alpha_i)}{\Upsilon^\beta(2\alpha_T - 2\alpha_i)}$$

where $2\alpha_T = \alpha_1 + \alpha_2 + \alpha_3$, $Q_0 = b_0 + 1/b_0$, b_0 is the S^4 squashing parameter and β is the S^1 radius.

- S^5 gluing rule $\rightarrow$ S-correlators:

$$C_S(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{S_3(2\alpha_T - E)} \prod_{i=1}^{3} \frac{S_3(2\alpha_i)}{S_3(2\alpha_T - 2\alpha_i)}$$

where $E = \omega_1 + \omega_2 + \omega_3$ and $\omega_1, \omega_2, \omega_3$ are the S^5 squashing parameters.

$$\Upsilon^\beta(X) \propto \prod_{n_1, n_2 = 0}^{\infty} \sinh\left[\frac{\beta}{2}\left(X + n_1 b_0 + \frac{n_2}{b_0}\right)\right] \sinh\left[\frac{\beta}{2}\left(-X + (n_1+1) b_0 + \frac{(n_2+1)}{b_0}\right)\right]$$

$$S_3(X) \propto \prod_{n_1, n_2, n_3 = 0} (\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + X)(\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + E - X)$$

With a suitable dictionary (akin to the AGT dictionary) we can map q-correlators to 5d partition functions.

Examples:

- 5d SQCD, $SU(2)$, $N_f = 4$ theory $\Leftrightarrow$ 4-point correlator

$$Z^{SQCD}_{S^4 \times S^1} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{id}$$

$$Z^{SQCD}_{S^5} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{S}$$

- 5d $\mathcal{N} = 1^*$ $SU(2)$ theory $\Leftrightarrow$ 1-point torus correlator

$$Z^{\mathcal{N}=1^*}_{S^4 \times S^1} = \langle V_{\alpha_1} \rangle_{id}$$

$$Z^{\mathcal{N}=1^*}_{S^5} = \langle V_{\alpha_1} \rangle_{S}$$

Brief summary and open questions

The factorisation of 5d partition functions in terms of 5d holomorphic blocks $\mathcal{B}^{5d}$ and their identification with chiral $\mathcal{V}ir_{qt}$ blocks, suggest to map 5d partition functions to q-deformed Liouville correlators.

We defined q-deformed Liouville correlators in terms of $\mathcal{V}ir_{qt}$ blocks and 3-point functions and showed that indeed they can be mapped to 5d partition functions.

–We need to investigate the full duality group of q-deformed correlators (what is the q-deformation of the Moore-Seiberg groupoid?)

So far we know that degenerate q-correlators are crossing symmetry invariant (we imposed this in the bootstrap).

–What is the 5d gauge theory interpretation of q-correlators dualities?

–Can we define Verlinde-loop operators in the q-deformed case? What is their gauge theory dual?

Our approach so far has been purely axiomatic/algebraic. To construct q-correlators we only used representations of $\mathcal{V}ir_{qt}$ and imposed associativity of the operator algebra (crossing symmetry).

It'd be interesting to have a more geometric/semiclassical description of the these theories.

In Liouville theory an interesting object which bridges between the axiomatic and the semiclassical approach is the reflection coefficient.

Reflection coefficient

- exact reflection coefficient from DOZZ 3-point function:

$$R^L = \frac{C(Q_0 - \alpha_1, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} \sim \frac{\Gamma(2iPb_0)\Gamma(2iP/b_0)}{\Gamma(-2iPb_0)\Gamma(-2iP/b_0)}\,, \quad P = i(\alpha - Q_0/2)$$

- semiclassical ($b_0 \to 0$) reflection coefficient from mini-superspace

$$\left(-\partial^2_{\phi_0} + e^{2b_0\phi_0}\right)\Psi = E\Psi$$

with solution

$$\psi \sim e^{2iP\phi_0} + R(P)e^{-2iP\phi_0}\,,$$

yielding

$$R(P) \sim \frac{\Gamma(2iP/b_0)}{\Gamma(-2iP/b_0)}$$

$\rightarrow$ captures only *half* of the exact result.

1d Schrödinger problems can be mapped to free motion in curved spaces.

The radial part of Laplace-Beltrami operator on the Lobachevsky space $\simeq SL(2,\mathbb{C})/SU(2)$ reduces to the Liouville wall problem with asymptotics

$$\psi_\lambda(x) \sim c(\lambda)e^{i\lambda x} + c(-\lambda)e^{-i\lambda x} \quad \text{as} \quad x \to -\infty$$

The coefficients $c(\pm\lambda)$ are the Harish-Chandra c-functions:

$$c(\lambda) = \frac{1}{\Gamma(1+2i\lambda)}\,, \qquad R = \frac{c(-\lambda)}{c(\lambda)} = -\frac{\Gamma(2i\lambda)}{\Gamma(-2i\lambda)}$$

with the identification of the spectral parameter $\lambda = P/b_0$ reproduces the semiclassical Liouville result.

c-functions of any classical symmetric space, can be expressed as

$$c(\lambda) = \prod_{\alpha\in\Delta^+} \frac{1}{\Gamma(l+\lambda\cdot\alpha)}\,,$$

where $l = 1, 1/2$ for finite dimensional or affine Lie algebras.

Affinisation

[Gerasimov et al.] observed that the exact Liouville reflection coefficient can be obtained considering the affine version of the group

$$\mathfrak{sl}(2) \to \hat{\mathfrak{sl}}(2)\,.$$

Adding the affine root α_0 to the positive root α_1 and choosing

$$\lambda \cdot (\alpha_0 + \alpha_1) = \tau, \quad \lambda \cdot \alpha_1 = 2iP/b_0 - 1/2, \quad \lambda \cdot \alpha_0 = \tau - 2iP/b_0 + 1/2\,,$$

the c-function becomes

$$c(P)^{-1} \equiv \Gamma(2iP/b_0) \prod_{n \geq 1} \Gamma(2iP/b_0 + n\tau)\Gamma(1 - 2iP/b_0 + n\tau)\Gamma(1/2 + n\tau)\,,$$

yielding

$$\frac{c(-P)}{c(P)} \sim -\frac{\Gamma(2iP/b_0)}{\Gamma(-2iP/b_0)} \frac{\Gamma(2iP/b_0\tau)}{\Gamma(-2iP/b_0\tau)}\,,$$

with $\tau = 1/b_0^2$ matches the exact Liouville result R_L.

Message: affinisation$\leftrightarrow$ effective 2nd quantisation.

id-reflection coefficient

From 3-point functions we find the exact reflection coefficient:

$$R_{id} = \frac{C_{id}(Q_0 - \alpha, \alpha_2, \alpha_1)}{C_{id}(\alpha, \alpha_2, \alpha_1)} \sim \frac{\Gamma_q(2iPb_0)\,\Gamma_t(2iP/b_0)}{\Gamma_q(-2iPb_0)\,\Gamma_t(-2iP/b_0)}\,,$$

with $q = e^{\beta/b_0}$, $t = e^{\beta b_0}$ and $\Gamma_q(x) \equiv \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$.

Analogy with semiclassical Liouville suggests to take

$$c(\lambda) = \Gamma_t(I + \lambda \cdot \alpha)^{-1}\,,$$

this is the c-function of the quantum Lobachevsky space where $\mathfrak{sl}(2) \to U_q(\mathfrak{sl}(2))$ [Olshanetsky-Rogov].

Taking the affine version and defining $q = t^\tau$, $\tau = 1/b_0^2$, we recover the exact reflection coefficient:

$$c(P) \sim \frac{(t^{2iP/b_0}; q, t)(t^{1-2iP/b_0}; q, t)}{(t^{1-2iP/b_0}; t)}\,, \quad \frac{c(-P)}{c(P)} \sim \frac{\Gamma_q(2iPb_0)\,\Gamma_t(2iP/b_0)}{\Gamma_q(-2iPb_0)\,\Gamma_t(-2iP/b_0)}.$$

S-reflection coefficient

From 3-point functions we find the exact reflection coefficient:

$$R_S = \frac{C_S(E-\alpha, \alpha_2, \alpha_1)}{C_S(\alpha, \alpha_2, \alpha_1)} = \frac{S_3(-2iP|\vec{\omega})}{S_3(2iP|\vec{\omega})}\,, \quad P = i\alpha - iE/2\,.$$

We take a c-function given in terms of the double-Gamma function:

$$c(P) = \Gamma_2(2iP/\kappa|e_1, e_2)^{-1}\,, \quad e_1 + e_2 = 1\,.$$

This c-function have been argued to arise in generalised symmetric spaces and to be part of a hierarchy of integrable systems, whose S-matrix building blocks are Γ_n functions [Freund-Zabrodin].

Finally the affinisation prescription yields the exact reflection coefficient:

$$\begin{aligned} c(P)^{-1} &= \prod_{n\geq 0} \Gamma_2(2iP/\kappa + n\tau|e_1, e_2)\Gamma_2(1 - 2iP/\kappa + (n+1)\tau|e_1, e_2) \\ &\quad \prod_{n\geq 1} \Gamma_2(1/2 + n\tau|e_1, e_2) = S_3(2iP|\vec{\omega})^{-1}\,, \end{aligned}$$

with $\omega = \kappa(e_1, e_2, \tau)$.

Relation to S-matrices

XYZ

$$J(u)=\Pi_{k=0}\frac{\Gamma_q(iu+rk)\Gamma_q(iu+rk+r+1)}{\Gamma_q(iu+rk+1/2)\Gamma_q(iu+rk+r+1/2)}$$

$$q=e^{-4\gamma},\ r=\frac{-i\pi\tau}{2\gamma}$$

$q\rightarrow 1,\ r=const$ $\qquad$ $q=const,\ r\rightarrow\infty$

XXZ *disordered*

$$J(u)\sim\Gamma_2$$

affinisation

$$R^S(\alpha_1)=\frac{C_S(E-\alpha_1,\alpha_2,\alpha_3)}{C_S(\alpha_1,\alpha_2,\alpha_3)}$$

XXZ *anti-ferro*

$$J(u)\sim\Gamma_q$$

affinisation

$$R^{id}(\alpha_1)=\frac{C_{id}(Q_0-\alpha_1,\alpha_2,\alpha_3)}{C_{id}(\alpha_1,\alpha_2,\alpha_3)}$$

$\rightarrow$ The appearance of the XXZ spin-chain is not surprising: SUSY vacua of 3d $\mathcal{N}=2$ theories can be mapped to eigenstates of spin-chain Hamiltonians [Nekrasov-Shatashivili], 3d blocks satisfy the Baxter equation for the XXZ spin chain [Gadde-Gukov-Putrov].

THANK YOU!