Aspects of Quantum Gravity Phenomenology

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Overview

- Motivation for Quantum Gravity
- Quantum Gravity Phenomenology
- Generalized Uncertainty Principle
- Discreteness of Space in Flat Spacetime
- Discreteness in Curved Spacetime: Non-relativistic Case
- Discreteness in Curved Spacetime: Relativistic Case
- Future Work
- Bibliography

Why A Quantum Theory of Gravity

- **Motivation**
  - Standard Model as a successful quantum theory of the fundamental interactions except for gravity
  - Classical nature of General Relativity
- **Problems with gravity as a quantum field theory**
  Infinities, Singularities of the Feynman diagrams, Renormalization failure.
- **Candidate Theories**
  - String Theory
  - Loop Quantum Gravity
  - Causal Set Theory

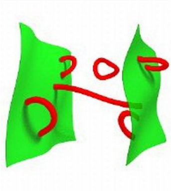

Figure: D-branes and strings[4]

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Domain of Quantum Gravity

- Planck Length:
  $\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35} m$
- Planck Mass:
  $m_{Pl} = \sqrt{\frac{\hbar c}{G}} \sim 10^{-8} kg$
- Planck Time:
  $t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-44} s$

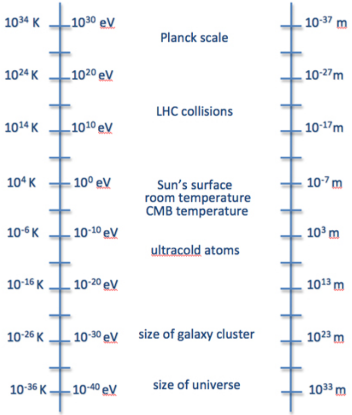

Figure: Planck length vs size of the Universe
$[www.learner.org/courses/physics/visual/img_lrg/planck_scale2.jpg]$

Quantum Gravity Phenomenology (QGP)

- Conceptual consistency guides the theory because of lack of experimental evidence.
- Quantum Gravity effects are significant at Planck scale: $E_p \sim 10^{28} eV$, $\ell_{Pl} \sim 10^{-35} m$.
- Working energy scale at LHC is the order of $10^{12} eV$.
- We hope to predict quantum gravity signature at low energy or macroscopic length scale.
- Breakdown of classical notion of spacetime continuum

- The simplest generalized uncertainty relation: $\Delta x \Delta p \geqslant \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + \gamma)$
- RHS $(\beta \Delta p)^2$ grows faster than the LHS (Δp) for arbitrarily small Δx
- Minimum length uncertainty

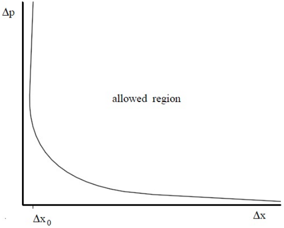

Figure: Modified uncertainty relation, implying a minimal length $\Delta x_0 > 0$[6]

Generalized Uncertainty Principle (GUP) and Schrödinger Equation

- Modification in $[x, p]$ leads to the generalized uncertainty principle [31, 32, 33],

$$\begin{aligned} \Delta x \Delta p &\geqslant \frac{\hbar}{2}\left[1 - 2\alpha < p > + 4\alpha^2 < p^2 >\right] \\ &\geqslant \frac{\hbar}{2}\left[1 + \left(\frac{\alpha}{\sqrt{< p^2 >}} + 4\alpha^2\right)\Delta p^2 + 4\alpha^2 < p >^2 - 2\alpha\sqrt{< p >^2}\right], \alpha_0 \sim 1. \end{aligned}$$

- Modified position and momenta: $x_i = x_{0i},\ p_i = p_{0i}(1 - \alpha p_0 + 2\alpha^2 p_0^2),\ i = 1, .., 3,$
- GUP-corrected Schrödinger equation for a non-relativistic particle in a one dimensional box of length L with boundaries at $x = 0$ and at $x = L$

$$\frac{d^2\psi}{dx^2} + 2i\alpha\hbar\frac{d^3\psi}{d^3x} + \sqrt{\frac{2mE}{\hbar^2}}\psi = 0. \tag{1}$$

- Solution: $\psi = Ae^{ik'x} + Be^{-ik''x} + Ce^{ix/2\alpha\hbar}$, where $k' = k(1 + k\alpha\hbar)$ and $k'' = k(1 - k\alpha\hbar)$ and $k_0 = \sqrt{2mE/\hbar^2}$.
- Boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$ give[34],

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0 l_{Pl}} = p\pi,\ p \in \mathbb{N} \tag{2}$$

- All measurable lengths are quantized in units of $\alpha_0 l_{Pl}$.
- GUP-corrected Klein-Gordon and Dirac equations lead to quantizations of higher dimensions.

Discreteness of Space from GUP in Curved Spacetime: Non-relativistic Case

- Weak non-fluctuating background gravitational field

$$V(x) = \begin{cases} kx & \text{if } 0 \leqslant x \leqslant L \\ \infty & \text{otherwise.} \end{cases} \quad (3)$$

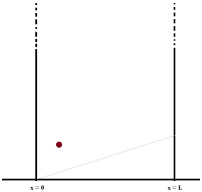

Figure: *particle in a one-dimensional box of length L*

- GUP-corrected Schrödinger equation

$$2i\alpha\hbar \frac{d^3}{dx^3}\psi + \frac{d^2}{dx^2}\psi + \frac{2m}{\hbar^2}(E - kx)\psi = 0 \quad (4)$$

- Trial solution: $\psi_1 = \psi_0(E + c\alpha, k, x) = \psi_0(E, k, x) + c\alpha \frac{d}{dE}\psi_0(E, k, x)$,

$$\psi_0(x) = C_1 Ai\left[\frac{\frac{2m}{\hbar^2}(kx - E)}{(\frac{2m}{\hbar^2}k)^{\frac{2}{3}}}\right] + C_2 Bi\left[\frac{\frac{2m}{\hbar^2}(kx - E)}{(\frac{2m}{\hbar^2}k)^{\frac{2}{3}}}\right]$$

Perturbative Solution of GUP-corrected Schrödinger equation

- c is found to be

$$c = \Bigg[(2i\hbar)\frac{3}{4}\left(\frac{2m}{\hbar^2}\right)^{11/12} k^{7/6}E^{-1/4}\left(C_1\sin(\xi_0+\frac{\pi}{4}) - C_2\cos(\xi_0+\frac{\pi}{4})\right) + \alpha(2i\hbar)\left(\frac{2m}{\hbar^2}\right)^{17/12} k^{1/6}E^{5/4}\left(C_2\sin(\xi_0+\frac{\pi}{4}) - C_1\cos(\xi_0+\frac{\pi}{4})\right)\Bigg] \div \Bigg[\left(\frac{2m}{\hbar^2}\right)^{11/12} k^{1/6}E^{-1/4} \times \left(C_1\sin(\xi_0+\frac{\pi}{4}) - C_2\cos(\xi_0+\frac{\pi}{4})\right)\Bigg]. \quad (5)$$

- Pertubative solution

$$\psi_1 = \psi_0(E+c\alpha,k,x) = \psi_0(E,k,x) + c\alpha\frac{d}{d\xi}\psi_0(E,k,x)\frac{d\xi}{dE},$$

where

- $\psi_0(E,k.x) = \frac{C_1}{\sqrt{\pi}}\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right) + \frac{C_2}{\sqrt{\pi}}\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)$
- $\frac{d\psi_0}{d\xi} = \frac{C_1}{\sqrt{\pi}}\left[-\frac{1}{4}\xi^{-5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right) + \xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right] + \frac{C_2}{\sqrt{\pi}}\left[-\xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right) - \frac{1}{4}\xi^{-5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right]$
- $\frac{d\xi}{dE} = \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{3}}k^{-\frac{2}{3}}.$

General Solution and Length Quantization

- Non-pertubative solution: $\psi_0^{III} = e^{ix/2\alpha_0\ell_{Pl}} = e^{ix/2\hbar\alpha}$
- General solution of GUP-corrected Schrödinger equation

$$\begin{aligned}\psi(x) \;=\; & \frac{A}{\sqrt{\pi}}\left[\xi^{-1/4}\sin\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)+\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\frac{1}{4}\xi^{-5/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)+\right.\right.\\ & \left.\left.\xi^{1/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right]+\frac{B}{\sqrt{\pi}}\left[\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{\frac{3}{2}}+\frac{\pi}{4}\right)+\right.\\ & \left.\left(\frac{2m}{\hbar^2}\right)^{1/3}k^{-2/3}c\alpha\left(-\xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)-\frac{1}{4}\xi^{-5/4}\cos\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}\right)\right)\right]+Ce^{ix/2\hbar\alpha}\end{aligned} \tag{6}$$

- Imposing boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$, the following relation is obtained

$$\frac{L}{2\hbar\alpha} = f(k)p_1\pi + p\pi, \tag{7}$$

$f(k)$ being a polynomial in k.

- Fine structure of length quantization similar to energy quantization is obtained.

Figure: Comparison between quantized lengths without gravity(solid lines) and with gravity(dotted lines)

Length quantization in curved spacetime: Relativistic Case

- High-energy particles are required to probe the nature of space near Planck-length.
- Use of linear potential is still justified for small distances.
- Relativistic particle in a one-dimensional box follows GUP-corrected **Klein-Gordon equation** $-2i\alpha\hbar\frac{d^3\psi}{dx^3} - \frac{d^2\psi}{dx^2} + m^2c^4\psi = (E - V(x))^2\,\psi$ or,

$$2i\alpha\hbar\frac{d^3\psi}{dx^3} + \frac{d^2\psi}{dx^2} + \frac{1}{\hbar^2c^2}\left(E^2 - m^2c^4 - 2Ekx\right)\psi = 0 \tag{8}$$

- Variable transformation according to

$$\frac{2m}{\hbar^2}E \rightarrow \frac{1}{\hbar^2c^2}(E^2 - m^2c^4),$$
$$\frac{2Ek}{\hbar^2c^2} \rightarrow \frac{2mk}{\hbar^2}$$

converts Klein-Gordon equation to GUP-corrected Schrödinger equation.

- Quantization of length similar to the case of Schrödinger equation follows.

Dirac Equation in one dimension

- Three dimensional generalization of Klein-Gordon equation suffers from non-locality of differential operators in the form $2i\alpha\hbar^3\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^{3/2}$.
- Most fundamental particles are fermions - **Dirac equation** is more appropriate.
- Dirac equation for a relativistic particle confined in a box $i\frac{\partial\Psi}{\partial t}=\left(\beta mc^2+c\tilde{\alpha}.\vec{P}+V(\vec{r})\mathbf{I_4}\right)\Psi$ where $\alpha^i=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}$, $\beta\equiv\gamma^0=\begin{pmatrix}\mathbf{I}_2 & 0\\ 0 & -\mathbf{I}_2\end{pmatrix}$.
- GUP-corrected Dirac equation for one spatial dimension z and choosing $V(\vec{r})=kz$

$$\left(-ic\hbar\alpha_z\frac{d}{dz}+c\alpha\hbar^2\frac{d^2}{dz^2}+\beta mc^2+kz\mathbf{I}_4\right)\psi(Z)=E\psi(Z). \tag{9}$$

- Four linearly independent solutions are

$$\psi_1=N_1\left(1-\frac{4ik\alpha\kappa z}{c/z+2i\alpha\kappa\left(c(1-2\alpha\kappa\hbar^2)-2E\right)}\right)e^{i\kappa z}\begin{pmatrix}\chi\\ r\sigma_z\chi\end{pmatrix}$$
$$\psi_2=N_2e^{iz/\alpha\hbar}\begin{pmatrix}\chi\\ \sigma_z\chi\end{pmatrix} \tag{10}$$

where χ is a normalized spinor.

Boundary Conditions and Length Quantization

- Imposition of boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$ directly leads to Klein paradox.
- MIT bag model is used: mass of the relativistic particle is considered as a function of z,

$$m(z) = \begin{cases} M & \text{if } z \leqslant 0 \ (I) \\ m & 0 \leqslant z \leqslant L \ (II) \\ M & z \geqslant L \ (III), \end{cases}$$

- MIT bag model boundary conditions [43]: $i\gamma^3\psi = \psi, \ at \ z = 0$ and $i\gamma^3\psi = -\psi, \ at \ z = L.$
- Length quantization follows

$$\frac{L}{\alpha\hbar} = -\frac{\pi}{4} + arg\left[\frac{\rho_1(ir-1)\left(e^{i(\delta-\kappa L)-e^{i\left(\kappa L - tan^{-1}\left(\frac{2r}{r^2-1}\right)\right)}}\right)}{F'}\right] + 2n\pi, \ n \in \mathbb{N} \tag{11}$$

where $\kappa = \kappa_0 + \alpha\hbar\kappa_0^2$, $\kappa_0 = \frac{1}{\hbar}\sqrt{E^2 - (mc^2)^2}, r = \frac{\hbar\kappa_0 c}{E+mc^2}$ and $\rho_1 = \left(1 - \frac{4ik\alpha\kappa z}{c/z + 2i\alpha\kappa(c(1-2\alpha\kappa\hbar^2)-2E)}\right)$.

Dirac Equation in Three Dimensions

- Particle in a box defined by $0 \leq x_i \leq L_i,\ i = 1, .., d,\ d = 1, 2$ or 3.
- Dirac Hamiltonian for a relativistic particle in a three-dimensional box with a linearized potential inside the box

$$H = c\vec{\alpha}.\vec{p_0} - c\alpha\left(\vec{\alpha}.\vec{p_0}\right)\left(\vec{\alpha}.\vec{p_0}\right) + \beta mc^2 + kxI$$

- Wavefunction inside the box

$$\psi = \begin{pmatrix} \left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}} e^{i\kappa_i x_i} + \rho_2^{\delta_{i1}} e^{-i(\kappa_i x_i - \delta_i)}\right) + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\right]\chi \\ \sum_{j=1}^{d}\left[\prod_{i=1}^{d}\left(\rho_1^{\delta_{i1}} e^{i\kappa_i x_i} + (-1)^{\delta_{ij}}\rho_2^{\delta_{i1}} e^{-i(\kappa_i x_i - \delta_i)}\right) r\hat{\kappa_j} + Fe^{i\frac{\hat{q}.\vec{r}}{\alpha\hbar}}\hat{q_j}\right]\sigma_j\chi \end{pmatrix}, \tag{12}$$

Discreteness of space from Dirac Equation

- MIT bag model boundary conditions $i\gamma^l\psi = \psi,\ at\ x_l = 0$ and $i\gamma^l\psi = -\psi,\ at\ x_l = L$ imply
  - Length quantization along x-axis

$$\frac{\hat{q}_1 L_1}{\alpha\hbar} = \frac{\hat{q}_1 L_1}{\alpha_0 \ell_{Pl}} = -\theta_1 + arg\left(\frac{\rho_1(ir\hat{\kappa_1} - 1) - \rho_2(ir\hat{k_1} + 1)e^{i\delta_1}}{F'} f_{\bar{l}}\right) + 2n_1\pi,\ n_1 \in \mathbb{N} \tag{13}$$

  - Length quantization along y and z axes

$$\frac{\hat{q}_l L_l}{\alpha\hbar} = \frac{\hat{q}_l L_l}{\alpha_0 \ell_{Pl}} = -2\theta_l + 2n_l\pi,\ n_l \in \mathbb{N}, \tag{14}$$

$l = 2, 3.$

- Area and Volume quantization

$$A_N = \prod_{l=1}^{N} \frac{\hat{q}_l L_l}{\alpha_0 \ell_{Pl}} = \prod_{l=2}^{N} (2n_l\pi - 2\theta_l)\left(2n_1\pi - \theta_1 + arg\left(\frac{\rho_1(ir\hat{\kappa_1} - 1) - \rho_2(ir\hat{\kappa_1} + 1)e^{i\delta_1}}{F'} f_{\bar{l}}\right)\right), \quad n_l \in \mathbb{N} \tag{15}$$

- In the limit $k \to 0$, these reduce to quantization results without the presence of gravity [35].

Conclusions

- Discreteness of space continues to hold for slightly curved spacetime.
- Presence of a background field leads to fine structure of quantization of length.
- Results support the claim of the existence of a minimum measurable length.
- Numerical analysis is required for solving the quantization equations for explicit length.
- Application of results towards the search for experimental signature of quantum gravity at microscopic length scale.
- Extension of the methods can be used for arbitrarily curved spacetime.

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