Scale Invariance and Conformal Invariance in Quantum Field Theory

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with Benjamín Grinstein and Andreas Stergiou
The big picture

- **Theory space**
  - Quantum field theory (QFT)
  - Scale field theory (SFT)
  - Conformal field theory (CFT)

- **Renormalization group (RG) flow**
  - Limit cycle
  - Fixed point

- **Motivations**
  - Scale and conformal invariance
  - Weyl consistency conditions
  - Discussion and conclusion
Why is it interesting?

QFT phases

- Infrared (IR) free
  - With mass gap ⇒ Exponentially-decaying correlation functions (e.g. Higgs phase)
  - Without mass gap ⇒ Trivial power-law correlation functions (e.g. Abelian Coulomb phase)

- IR interacting
  - CFTs ⇒ Power-law correlation functions (e.g. non-Abelian Coulomb phase)
  - SFTs ⇒ ?

Possible types of RG flows

- Strong coupling
- Weak coupling
  - Fixed points (e.g. Banks-Zaks fixed point \textbf{Banks, Zaks (1982)})
  - Recurrent behaviors (e.g. limit cycles or ergodic behaviors)
1 Motivations

2 Scale and conformal invariance
   - Preliminaries
   - Scale invariance and recurrent behaviors

3 Weyl consistency conditions
   - $c$-theorem
   - Scale invariance implies conformal invariance

4 Discussion and conclusion
   - Features and future work
Preliminaries ($d > 2$)

- **Dilatation current** [Wess (1960)]
  - $\mathcal{D}_\mu(x) = x^\nu T^\mu_{\nu}(x) - V^\mu(x)$
  - $T^\mu_{\nu}(x)$ any symmetric EM tensor following from spacetime nature of scale transformations
  - $V^\mu(x)$ local operator (virial current) contributing to scale dimensions of fields
  - Freedom in choice of $T^\mu_{\nu}(x)$ compensated by freedom in choice of $V^\mu(x)$

- **Scale invariance** $\Rightarrow T^\mu_{\mu}(x) = \partial_\mu V^\mu(x)$
- Conformal current \textit{Wess (1960)}

\[ C_\nu^\mu (x) = \nu_\nu^\lambda (x) T_\lambda^\mu (x) - (\partial_\lambda \nu_\nu^\lambda (x)) V''^\mu (x) + (\partial_\rho \partial_\lambda \nu_\nu^\lambda (x)) L^\rho^\mu (x) \]

- \( T_\lambda^\mu (x) \) any symmetric EM tensor following from spacetime nature of conformal transformations
- \( V''^\mu (x) \) local operator corresponding to ambiguity in choice of dilatation current
- \( L^\rho^\mu (x) \) local symmetric operator correcting position dependence of scale factor
- \( (\partial_\lambda \nu_\nu^\lambda (x)) \) scale factor (general linear function of \( x_\nu \))
- Freedom in choice of \( T_\lambda^\mu (x) \) compensated by freedom in choice of \( V''^\mu (x) \) and \( L^\rho^\mu (x) \)

- Conformal invariance \( \Rightarrow \) \( T_\mu^\mu (x) = \partial_\mu V''^\mu (x) = \partial_\mu \partial_\nu L^\nu^\mu (x) \)

- Conformal invariance \( \Rightarrow \) Existence of symmetric traceless energy-momentum tensor \textit{Polchinski (1988)}
Non-conformal scale-invariant QFTs  Polchinski (1988)

- Scale invariance $\Rightarrow T_\mu^\mu(x) = \partial_\mu V_\mu(x)$
- Conformal invariance $\Rightarrow T_\mu^\mu(x) = \partial_\mu \partial_\nu L_\nu^\mu(x)$
- Scale without conformal invariance
  $\Rightarrow T_\mu^\mu(x) = \partial_\mu V_\mu(x)$ where $V_\mu(x) \neq J_\mu(x) + \partial_\nu L_\nu^\mu(x)$ with $\partial_\mu J_\mu(x) = 0$

- Constraints on possible virial current candidates
  - Gauge invariant (spatial integral)
  - Fixed $d - 1$ scale dimension in $d$ spacetime dimensions

- No suitable virial current $\Rightarrow$ Scale invariance implies conformal invariance (examples: $\phi^p$ in $d = n - \epsilon$ for $(p, n) = (6, 3), (4, 4)$ and $(3, 6)$)
Virial current candidates \((d = 4)\)

Most general classically scale-invariant renormalizable theory in \(d = 4 - \epsilon\) spacetime dimensions \(\text{Jack, Osborn (1985)}\)

\[
\mathcal{L} = - \mu^{-\epsilon} Z_A \frac{1}{4g_A^2} F^A_{\mu\nu} F^{A\mu\nu} + \frac{1}{2} Z^{1/2}_{ab} Z^{1/2}_{ac} D_\mu \phi_b D^\mu \phi_c \\
+ \frac{1}{2} Z^{1/2*}_{ij} Z^{1/2*}_{ik} \bar{\psi}_j i\bar{\sigma}^\mu D_\mu \psi_k - \frac{1}{2} Z^{1/2*}_{ij} Z^{1/2}_{ik} D_\mu \bar{\psi}_j i\bar{\sigma}^\mu \psi_k \\
- \frac{1}{4!} \mu^\epsilon (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d \\
- \frac{1}{2} \mu^2 (yZ^y)_{a|ij} \phi_a \psi_i \psi_j - \frac{1}{2} \mu^2 (yZ^y)^*_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j
\]

- \(\phi_a(x)\) real scalar fields
- \(\psi_i^\alpha(x)\) Weyl fermions
- \(A^A_\mu(x)\) gauge fields

Dimensional regularization with minimal subtraction
Virial current candidates and new improved EM tensor

- **Virial current** \( V^\mu(x) = Q_{ab} \phi_a D^\mu \phi_b - P_{ij} \bar{\psi}_i \sigma^\mu \psi_j \)
  - \( Q_{ba} = -Q_{ab} \)
  - \( P_{ji}^* = -P_{ij} \)

- **New improved energy-momentum tensor** \( [\Theta_{\nu}^\mu(x)] \)
  - Callan, Coleman, Jackiw (1970)
  - Finite and not renormalized (vanishing anomalous dimension)

\[
[\Theta_{\nu}^\mu(x)] = \frac{B_A}{2g_A^3} [F^A_{\mu\nu} F^{A\mu\nu}] - \frac{1}{4!} B_{abcd} [\phi_a \phi_b \phi_c \phi_d] \\
- \frac{1}{2} (B_{a|ij} [\phi_a \psi_i \psi_j] + \text{h.c.}) - ((\delta + \Gamma)f) \cdot \frac{\delta}{\delta f} A
\]
Anomalous trace

\[
[\Theta_{\mu}(x)] = B^I [\mathcal{O}_I(x)] - ((\delta + \Gamma)f) \cdot \frac{\delta}{\delta f} A
\]

Conserved dilatation current

\[
\partial_{\mu} D^\mu(x) = 0 \quad \text{(up to EOMs)}
\]

\[
B^I = Q^I \equiv -(gQ)^I
\]

Conserved conformal current

\[
\partial_{\mu} C^\mu_\nu(x) = 0 \quad \text{(up to EOMs)}
\]

\[
B^I = 0
\]

⇒ Both SFT \((Q \neq 0)\) and CFT \((Q = 0)\) can be treated simultaneously
Virial current and unitarity bounds

- New improved energy-momentum tensor ⇒ Finite and not renormalized  
  Callan, Coleman, Jackiw (1970)

- Operators related to EOMs ⇒ Finite and not renormalized  

- Virial current ⇒ Finite and not renormalized  
  - Unconserved current with scale dimension exactly 3

- Unitarity bounds for conformal versus scale-invariant QFTs  
  Grinstein, Intriligator, Rothstein (2008)

- Non-trivial virial current ⇒ Non-conformal scale-invariant QFTs
**RG flows along scale-invariant trajectories**

Scale-invariant solution \((\lambda_{abcd}, y_{a|ij}, g_A)\) \(\Rightarrow\) RG trajectory

\[
\begin{align*}
\bar{\lambda}_{abcd}(t) &= \hat{Z}_{a'a}(t)\hat{Z}_{b'b}(t)\hat{Z}_{c'c}(t)\hat{Z}_{d'd}(t)\lambda_{a'b'c'd'} \\
\bar{y}_{a|ij}(t) &= \hat{Z}_{a'a}(t)\hat{Z}_{i'i}(t)\hat{Z}_{j'j}(t)y_{a'|i'j'} \\
\bar{g}_A(t) &= g_A
\end{align*}
\]

\[
\begin{align*}
\hat{Z}_{a'a}(t) &= (e^{Qt})_{a'a} \\
\hat{Z}_{i'i}(t) &= (e^{Pt})_{i'i}
\end{align*}
\]

\[t = \ln(\mu_0/\mu) \quad (\text{RG time})\]

- \((\bar{\lambda}_{abcd}(t, g, \lambda, y), \bar{y}_{a|ij}(t, g, \lambda, y), \bar{g}_A(t, g, \lambda, y))\) also scale-invariant solution

- \(Q_{ab}\) and \(P_{ij}\) constant along RG trajectory

- \(\hat{Z}_{ab}(t)\) orthogonal and \(\hat{Z}_{ij}(t)\) unitary \(\Rightarrow\) Always non-vanishing beta-functions along scale-invariant trajectory
Scale invariance and recurrent behaviors

RG flows along scale-invariant trajectories ⇒ Recurrent behaviors!


- Virial current ⇒ Transformation in symmetry group of kinetic terms \((SO(N_S) \times U(N_F))\)
  - \(\hat{Z}_{ab}(t)\) and \(\hat{Z}_{ij}(t)\) in \(SO(N_S) \times U(N_F)\)
  - \(Q_{ab}\) antisymmetric and \(P_{ij}\) antihermitian ⇒ Purely imaginary eigenvalues

⇒ Periodic (limit cycle) or quasi-periodic (ergodicity) scale-invariant trajectories
Motivations

Scale and conformal invariance

Weyl consistency conditions

Discussion and conclusion

Recurrent behaviors

Intuition from $\mathcal{D}^\mu(x) = x^\nu T_\nu{}^\mu(x) - V^\mu(x)$

- RG flow $\Rightarrow$ Generated by scale transformation ($x^\nu T_\nu{}^\mu(x)$)
- RG flow $\Rightarrow$ Related to virial current through conservation of dilatation current
- Virial current $\Rightarrow$ Generates internal transformation of the fields
  - Internal transformation in compact group $SO(N_S) \times U(N_F)$
  $\Rightarrow$ Rotate back to or close to identity
- RG flow return back to or close to identity $\Rightarrow$ Recurrent behavior
Dilatation generators do not generate dilatations in non-scale-invariant QFTs  Coleman, Jackiw (1971)

- Quantum anomalies at low orders
  - Anomalous dimensions
  ⇒ Possible to absorb into redefinition of scale dimensions of fields
  ✓ Preserve scale invariance

- Quantum anomalies at high orders
  - Beta-functions
  ⇒ Not possible to absorb
  × Break scale invariance
Why dilatation generators generate dilatations in scale-invariant QFTs?

- Beta-functions on scale-invariant trajectories
  - Both vertex correction and wavefunction renormalization contributions
  - Very specific form for vertex correction contribution
  - Equivalent in form to wavefunction renormalization contribution (redundant operators)
  ⇒ Also possible to absorb into redefinition of scale dimensions of fields
  ✔ Preserve scale invariance!
- Beta-functions from vertex corrections and wavefunction renormalizations \((d = 4\) spacetime dimensions)\

\[
B_{abcd} = - \frac{d\lambda_{abcd}}{dt} = -(\lambda\gamma^\lambda)_{abcd} + \lambda_{a'b'cd}\Gamma_{a'a} + \lambda_{ab'cd}\Gamma_{b'b} + \lambda_{abc'd}\Gamma_{c'c} + \lambda_{abcd'}\Gamma_{d'd}
\]

\[
B_{a|ij} = - \frac{dy_{a|ij}}{dt} = -(y\gamma^y)_{a|ij} + y_{a'|ij}\Gamma_{a'a} + y_{a|i'j}\Gamma_{i'i} + y_{a|ij'}\Gamma_{j'j}
\]

\[
B_A = - \frac{dg_A}{dt} = \gamma Ag_A \quad \text{(no sum)}
\]

- Beta-functions on scale-invariant trajectories\

\[
B_{abcd} = - \lambda_{a'b'cd}Q_{a'a} - \lambda_{ab'cd}Q_{b'b} - \lambda_{abc'd}Q_{c'c} - \lambda_{abcd'}Q_{d'd}
\]

\[
B_{a|ij} = - y_{a'|ij}Q_{a'a} - y_{a|i'j}P_{i'i} - y_{a|ij'}P_{j'j}
\]

\[
B_A = 0
\]
Ward identity for scale invariance

Callan-Symanzik equation for effective action

\[
\left[ M \frac{\partial}{\partial M} + B^I \frac{\partial}{\partial g^I} + \Gamma^I \int d^4 x f_I(x) \frac{\delta}{\delta f_J(x)} \right] \Gamma[f(x), g, M] = 0
\]

- In non-scale-invariant QFTs
  - ✓ Anomalous dimensions
  - ✗ Beta-functions

\[
\left[ M \frac{\partial}{\partial M} + (\Gamma + Q)^I \int d^4 x f_I(x) \frac{\delta}{\delta f_J(x)} \right] \Gamma[f(x), g, M] = 0
\]

- In scale-invariant QFTs
  - ✓ Anomalous dimensions
  - ✓ Beta-functions (redundant operators)

Beta-functions on scale-invariant trajectories
- Quantum-mechanical generation of scale dimensions
- Appropriate scale dimensions required by virial current
  ⇒ Conserved dilatation current $D^\mu(x)$

Poincaré algebra with dilatation charge $D = \int d^3x \, D^0(x)$

\[
\begin{align*}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}) \\
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[D, P_\mu] &= -iP_\mu
\end{align*}
\]

Algebra action on fields $O_I(x)$

\[
\begin{align*}
[M_{\mu\nu}, O_I(x)] &= -i(x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{\mu\nu})O_I(x) \\
[P_\mu, O_I(x)] &= -i\partial_\mu O_I(x) \\
[D, O_I(x)] &= -i(x \cdot \partial + \Delta)O_I(x)
\end{align*}
\]
New classical scale dimensions of fields due to virial current

\[ [D, \phi_a(x)] = -i(x \cdot \partial + 1)\phi_a(x) - iQ_{ab}\phi_b(x) \]
\[ [D, \psi_i(x)] = -i(x \cdot \partial + \frac{3}{2})\psi_i(x) - iP_{ij}\psi_j(x) \]

- How do non-conformal scale-invariant QFTs know about new scale dimensions?
  \(\Rightarrow\) Generated by beta-functions!

Quantum-mechanical scale dimensions of fields

\[ \Delta_{ab} = \delta_{ab} + \Gamma_{ab} + Q_{ab} \]
\[ \Delta_{ij} = \frac{3}{2}\delta_{ij} + \Gamma_{ij} + P_{ij} \]
c-theorem

Barnes, Intriligator, Wecht, Wright (2004)

- RG flow $\Rightarrow$ Irreversible process (integrating out DOFs)
- $c(g) \sim$ measure of number of massless DOFs
- c-theorem and implications for SFT
  - weak ($c_{IR} < c_{UV}$) Komargodski, Schwimmer (2011) & Luty, Polchinski, Rattazzi (2012)
  - stronger ($\frac{dc}{dt} \leq 0$) Osborn (1989,1991) & Jack, Osborn (1990)
  - strongest (RG flows as gradient flows)
Gradient Flow

- **Gradient flow**
  \[ B^I(g) = -\frac{dg^I}{dt} = G^{IJ}(g) \frac{\partial c(g)}{\partial g^J} \]
  
  - \( G^{IJ} \) positive-definite metric
  - Potential \( c(g) \) function of couplings

- Potential \( c(g) \) monotonically decreasing along RG trajectory
  \[ \frac{dc(g(t))}{dt} = -G_{IJ}(g) B^I B^J \leq 0 \]
  
  - Recurrent behaviors (scale-invariant trajectories) ↯ Gradient flows (scale implies conformal invariance) Wallach, Zia (1975)

\[ \Rightarrow \] Another way to prove scale implies conformal invariance

- Different than proof for \( d = 2 \) unitarity interacting QFTs with well-defined correlation functions Zamolodchikov (1986) & Polchinski (1988)
c-theorem and gradient flow at weak coupling


\[ \frac{\partial c(g)}{\partial g^I} = (G_{IJ} + A_{IJ})\beta^J \Rightarrow \frac{dc(g(t))}{dt} = -\beta^I G_{IJ}(g)\beta^J \]

- Curved spacetime  Background metric with spacetime-dependent couplings

  \Rightarrow (Weak-coupling) RG flow recurrent behaviors forbidden at all loops
Local and global renormalized operators

*Global* renormalized operator \( \mathcal{O}_I(x) = \partial \mathcal{L}(x) / \partial g^I \)

- Finite global insertion in Green functions \( \Rightarrow -i \partial \langle \ldots \rangle / \partial g^I = \langle \int d^d x \mathcal{O}_I(x) \ldots \rangle \)
- Infinite local insertion in Green functions \( \Rightarrow \langle \mathcal{O}_I(x) \ldots \rangle \)

*Local* renormalized operator \([\mathcal{O}_I(x)] = \delta A / \delta g^I(x)\)

- Finite local insertion in Green functions \( \Rightarrow \langle [\mathcal{O}_I(x)] \ldots \rangle = \langle (\mathcal{O}_I(x) - \partial \mu J^\mu_I(x)) \ldots \rangle \)
- Infinite current \( J^\mu_I(x) = -(N_I)_{ab} \phi_a D^\mu \phi_b + (M_I)_{ij} \bar{\psi}_i \sigma^\mu \psi_j \)
  - \((N_I)_{ba} = -(N_I)_{ab}\) and \((M_I)^*_{ji} = -(M_I)_{ij}\)
  - \(N_I = \sum_{i \geq 1} \frac{N^{(i)}_I}{\epsilon^i}\) and \(M_I = \sum_{i \geq 1} \frac{M^{(i)}_I}{\epsilon^i}\)
Computations of new divergences

\( (N_{c|ij})_{ab} = - \frac{1}{16\pi^2\epsilon} \frac{1}{2} (y^*_a|ij\delta_{bc} - y^*_b|ij\delta_{ac}) + \text{h.c.} + \text{finite} \)
Finite contributions to EM tensor


\[ [\Theta_{\mu}(x)] = \beta^I[O_I] - D_\mu[S_{ab}\phi_aD^\mu\phi_b - R_{ij}\bar{\psi}_i\imath\sigma^\mu\psi_j] - ((\delta + \gamma)f) \cdot \frac{\delta}{\delta f} A \]

\[
  \begin{align*}
    f_0 &= \mu^{\frac{1}{2} - \delta} \epsilon Z^{\frac{1}{2}}(g)f \\
    \hat{\gamma} &= (\frac{1}{2} - \delta)\epsilon - k_Ig^I\partial_IZ^{\frac{1}{2}(1)} \\
    S &= -k_Ig^IN^{(1)}_I \\
    g_0^I &= \mu^{k_I\epsilon}(g^I + L^I(g)) \\
    \hat{\beta}^I &= -k_Ig^I\epsilon - k_IL^{(1)} + k_Jg^J\partial_JL^{(1)} \\
    R &= -k_Ig^IM^{(1)}_I
  \end{align*}
\]
Ambiguities in RG functions


- Square root of wavefunction renormalization $Z^{\frac{1}{2}}$
  - Freedom $Z^{\frac{1}{2}} \rightarrow \tilde{Z}^{\frac{1}{2}} = OZ^{\frac{1}{2}}$ with $Z = Z^{\frac{1}{2}} T Z^{\frac{1}{2}} \rightarrow Z^{\frac{1}{2}} T O^T OZ^{\frac{1}{2}}$
  - $O^T O = 1$ and $O = 1 + \sum_{i \geq 1} \frac{O^{(i)}}{\epsilon^i}$

- Extra freedom with $\omega = k_I g^I \partial_I O^{(1)}$
  - $Z^{\frac{1}{2}}(1) \rightarrow Z^{\frac{1}{2}}(1) + O^{(1)}$
  - $L^{(1)} \rightarrow L^{(1)} - (gO^{(1)})^I$
  - $N^{(1)}_I \rightarrow N^{(1)}_I - \partial_I O^{(1)}$
  - $\hat{\gamma} \rightarrow \hat{\gamma} - \omega$
  - $\hat{\beta}^I \rightarrow \hat{\beta}^I - (g \omega)^I$
  - $S \rightarrow S + \omega$

- Invariant anomalous trace
  $$[\Theta_{\mu}^\mu(x)] = (\beta^I + (gS)^I)[O_I] - ((\delta + \gamma + S)f) \cdot \frac{\delta}{\delta f} A$$
  $$= B^I [O_I] - ((\delta + \Gamma)f) \cdot \frac{\delta}{\delta f} A$$
“Correct” RG flow ⇒ $B^I = \beta^I + (gS)^I = -(gQ)^I$
  - SFTs ($Q \neq 0$) ⇒ limit cycles ($B^I = -(gQ)^I \neq 0$)
  - CFTs ($Q = 0$) ⇒ fixed points ($B^I = 0$)

“Old” RG flow ⇒ $\beta^I = -(g(S + Q))^I$
  - SFTs ($Q \neq 0$) ⇒ fixed points ($S = -Q$) and limit cycles ($S \neq -Q$)
  - CFTs ($Q = 0$) ⇒ fixed points ($S = 0$) and limit cycles ($S \neq 0$)

⇒ Systematic understanding of SFTs and CFTs through “correct” RG flow (unless $S$ vanishes identically)
Computation of $S$

$S^{\text{(one-loop)}} = S^{\text{(two-loop)}} = 0$ due to symmetry of contributions to $N_i$

\[ (16\pi^2)^3 S_{ab} = \frac{5}{8} \text{tr}(y_a y_c^* y_d y_e^*) \lambda_{bcde} + \frac{3}{8} \text{tr}(y_a y_c^* y_d y_b^* y_e^*) - \{a \leftrightarrow b\} + \text{h.c.} \]

$S \neq 0 \Rightarrow$ Examples of CFTs with $S \neq 0$ exist \text{JFF, Grinstein, Stergiou (2012)}
Generalized \( c \)-theorem

- **Weyl consistency conditions and local current conservation**  

  \[
  \frac{\partial c(g)}{\partial g^I} = (G_{IJ} + A_{IJ})B^J \Rightarrow \frac{dc(g(t))}{dt} = -B^I G_{IJ} B^J
  \]

  - Curved spacetime \( \Rightarrow \) Background metric with spacetime-dependent couplings
  - Spin-one operator of dimension 3 \( \Rightarrow \) Background gauge fields with gauge-dependent couplings
  \( \Rightarrow \) (Weak-coupling) RG flow recurrent behaviors allowed at all loops

- **Scale invariance implies conformal invariance**  
Features and future work

Features of SFTs and CFTs

- Correct RG flow
  - SFTs ⇒ Recurrent behaviors
  - CFTs ⇒ Fixed points

- Generalized c-theorem ⇒ Only CFTs allowed
  ⇒ Scale invariance implies conformal invariance
  - Unexpected CFTs with expected behaviors

Future work

- Proof at strong coupling  
  Farnsworth, Luty, Prelipina (2013)

- 6d analysis  

Thank you!