The Pressurized Bouncing Ball

A simple model

Gaëtan Landry
Alex Georgallas

Dalhousie University
Faculty of Agriculture
Truro, NS

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Context – Physics

- Introductory physics – Motion
  - Parabolic motion
  - Energy transformations, losses
  - Air resistance, other sources of errors

- Intermediate/advanced physics – Impact
  - Mechanics of impact
  - Impulse forces, deformation, etc.
  - **No simple model available!**
Context – Sports rules

- NBA – “The ball shall be an officially approved NBA ball between $7\frac{1}{2}$ and $8\frac{1}{2}$ pounds pressure [51.7 to 58.6 kPa].”

- FIBA – “[The ball shall] be inflated to an air pressure such that, when it is dropped onto the playing floor from a height of approximately 1,800 mm measured from the bottom of the ball, it will rebound to a height of between 1,200 mm and 1,400 mm, measured to the top of the ball.”
The coefficient of restitution $e$ of a ball impacting against an immovable body is

$$e = \left| \frac{v_f}{v_i} \right| \quad (1)$$

For balls, $e$ ranges between 0 [no bounce] and 1 [perfectly bouncy].
Q: How does internal pressure affect the bouncing of a ball?
Q: What is $e(P)$?
A: No model exists!

- Polynomial?
- Exponential?
- Something else?
Our Answer

- Pressure forces
- Wall forces
- Dissipative forces
- Final model
Pressure Forces 1

Geometry

\[ A = \pi \left[ R^2 - (R - x)^2 \right] \]  \hspace{1cm} (2)

\[ V = \frac{4}{3} \pi R^3 - \frac{1}{3} \pi x^2 (3R - x) \]  \hspace{1cm} (3)

Pressure force

\[ F_P = (P - P_0) A \]  \hspace{1cm} (4)

Isothermal compression

\[ PV = P_i V_i \]  \hspace{1cm} (5)
Pressure Forces 2

Combining (2)–(5) together yields

\[ F_P = \left[ \frac{4R^3}{4R^3 - x^2 (3R - x)} P_i - P_0 \right] \pi x (2R - x) \]  

(6)

Taylor expansion in terms of the gauge pressure \( P_G = P_i - P_0 \)

\[ F_P = 2\pi R P_G x \left[ 1 - \frac{1}{2} \left( \frac{x}{R} \right) + \frac{3}{4} \left( 1 + \frac{P_0}{P_G} \right) \left( \frac{x}{R} \right)^2 + \ldots \right] \]  

(7)

If \( x \ll R \) and \( P_G \gg 0 \)

\[ F_P \approx 2\pi R P_G x \]  

(8)

\( F_P \) is linear in \( x \), with a force constant of

\[ k_P = 2\pi R P_G \]  

(9)
Wall Forces 1

Wall forces (shear forces)

\[ F_W = A_p G \theta \] (10)

Cross-sectional area of perimeter

\[ A_p = 2\pi D_W \sqrt{R^2 - (R - x)^2} \] (11)

Angle of contact

\[ \theta = \arccos \left( \frac{R - x}{R} \right) \] (12)
Combining (10)–(12) yield

\[ F_W = 2\pi GD_W \sqrt{R^2 - (R - x)^2} \arccos \left( \frac{R - x}{R} \right) \]  

(13)

Taylor expansion

\[ F_W = 2\pi GD_W x \left[ 2 - \frac{1}{3} \left( \frac{x}{R} \right) - \frac{1}{15} \left( \frac{x}{R} \right)^2 + \ldots \right] \]  

(14)

If \( x \ll R \)

\[ F_W \approx 4\pi GD_W x \]  

(15)

\( F_W \) is linear in \( x \), with a force constant of

\[ k_W = 4\pi GD_W \]  

(16)
Total Restoring Force

According to our model, the combined restoring effect of wall strength and pressure is

\[ F_R = F_P + F_W \]  \hspace{1cm} (17)

\[ \approx (2\pi R P_G + 4\pi G D_W) x \]

and the ball will effectively have a spring constant of

\[ k = 2\pi R P_G + 4\pi G D_W \]  \hspace{1cm} (18)
Let us consider a ball of spring-like restoring force $F_R$, with spring constant $k$, subject to a uniform dissipative force $F_D$.

In compression $(A \rightarrow B)$

$$K_i = \frac{1}{2} k x_0^2 - m g x_0 + F_D x_0$$  \hspace{1cm} (19)$$

In decompression $(B \rightarrow C)$

$$K_f = e^2 K_i = \frac{1}{2} k x_0^2 - m g x_0 - F_D x_0$$  \hspace{1cm} (20)$$
Combining (20) and (21), we obtain

\[
\frac{(1 + e^2)}{(1 - e^2)^2} = \frac{kK_i}{4F_D^2} - \frac{mgK_i}{2F_D^2 x_0^2}
\]  \hspace{1cm} (21)

If \( \frac{1}{2}kx_0^2 \gg mgx_0 \), we can ignore the last term, and

\[
\frac{(1 + e^2)}{(1 - e^2)^2} \approx \frac{kK_i}{4F_D^2}
\]  \hspace{1cm} (22)
Incorporating (19) into (23), we obtain

\[
\frac{(1 + e^2)}{(1 - e^2)^2} = \frac{(2\pi RP_G + 4\pi GD_W) K_i}{4F_D^2}
\]  

(23)

Or

\[
\frac{(1 + e^2)}{(1 - e^2)^2} = A P_G + B
\]  

(24)

where

\[
A = \frac{\pi RK_i}{2F_D^2}
\]  

(25) \quad B = \frac{\pi GD_W K_i}{F_D^2}

(26) \quad \frac{B}{A} = \frac{2GD_W}{R}

(27)
### Reality check – Basketball

<table>
<thead>
<tr>
<th>Height (m)</th>
<th>$A$ (10$^{-4}$ Pa$^{-1}$)</th>
<th>$B$</th>
<th>$G$ (10$^5$ Pa)</th>
<th>$F_D$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>● 0.75</td>
<td>2.576 ± 0.052</td>
<td>1.35 ± 0.28</td>
<td>1.01 ± 0.27</td>
<td>55.92 ± 0.92</td>
</tr>
<tr>
<td>○ 1.50</td>
<td>1.916 ± 0.036</td>
<td>1.29 ± 0.20</td>
<td>1.29 ± 0.27</td>
<td>91.7 ± 1.5</td>
</tr>
</tbody>
</table>

![Graph showing the relationship between height and gauge pressure](image)
Reality check – Soccerball

<table>
<thead>
<tr>
<th>Height (m)</th>
<th>A ($10^{-4}$ Pa$^{-1}$)</th>
<th>B (10$^5$ Pa)</th>
<th>G (10$^5$ Pa)</th>
<th>$F_D$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>● 0.75</td>
<td>2.76 ± 0.11</td>
<td>6.60 ± 0.65</td>
<td>2.90 ± 0.49</td>
<td>43.8 ± 1.2</td>
</tr>
<tr>
<td>○ 1.50</td>
<td>2.277 ± 0.088</td>
<td>3.24 ± 0.46</td>
<td>2.52 ± 0.44</td>
<td>55.7 ± 1.4</td>
</tr>
</tbody>
</table>
## Reality check – Volleyball

![Graph showing the relationship between gauge pressure and height](image)

<table>
<thead>
<tr>
<th>Height (m)</th>
<th>$A$ ($10^{-4}$ Pa$^{-1}$)</th>
<th>$B$ ($10^5$ Pa)</th>
<th>$G$ ($10^5$ Pa)</th>
<th>$F_D$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>3.14 ± 0.14</td>
<td>3.50 ± 0.80</td>
<td>1.18 ± 0.39</td>
<td>32.16 ± 0.93</td>
</tr>
<tr>
<td>1.50</td>
<td>2.091 ± 0.074</td>
<td>4.74 ± 0.50</td>
<td>1.62 ± 0.39</td>
<td>68.1 ± 1.8</td>
</tr>
</tbody>
</table>
If $P_G > 25$ kPa, spread <2.5%, individual points <1.5%!
Conclusions 1

- Model is very accurate at $P_G > 25$ kPa.
- $F_D$ increases by a factor of 1.3 to 2.1 when height is doubled.
- $G$ is constant within error when height is doubled.
- $G$ has correct order of magnitude.
  - $G_{\text{exp}} \approx 10^5$ Pa vs $G_{\text{rubber}} = 3 \times 10^5$ Pa.
Conclusions 2

- Could include higher-order correction terms in the analysis if greater accuracy is desired at $P_G < 25$ kPa.
- Ultimately, could go back to the specific forms of $A(x)$, $V(x)$, $A_p(x)$ and $\theta(x)$ for more accurate $F_P(x)$ and $F_W(x)$.
- Non-uniform dissipative forces?
- Non-isothermal compressions?
Acknowledgments

- Dr Alex Georgallas, Dalhousie University (Truro) co-author
- Dr Simon de Vet, Dalhousie University (Halifax) discussions related to time-of-flight methods

Experimental Method 1

Several methods exist to probe $e$

$$e = \left| \frac{v_f}{v_i} \right|$$  \hspace{1cm} (28)

$v_f$ and $v_i$ are related to several other quantities, like the height of bounces, times of flight, etc. In terms of typical accuracy

Time methods $>$ Height methods $>$ Velocity methods
Experimental Method 2

Since $e$ depend on $K_i$, we need to control for $K_i$. Easiest way is to control for $H_i$, and study first impact.

Assuming no air resistance

$$v_i = \sqrt{2gH} \quad (29)$$

$$v_f = \frac{1}{2}gt_f \quad (30)$$

Therefore

$$e = \sqrt{\frac{gt_f}{8H}} \quad (31)$$
Experimental Method 3

- Manual release ($\pm < 1 \text{ cm}$) at 0.75 m and 1.50 m
- Sound-based time-of-flight measurement ($\pm < 1 \text{ ms}$)
- Inflated with bike pump, but accurate sensor ($\pm < 0.5 \text{ kPa}$)
- At least 5 trials per pressure, per ball, per height

<table>
<thead>
<tr>
<th>Ball &amp; Model</th>
<th>$R$ (cm)</th>
<th>$m$ (g)</th>
<th>$D_W$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Bask.) Wilson WTB0935</td>
<td>11.75 ± 0.15</td>
<td>592.9 ± 0.1</td>
<td>3.10 ± 0.09</td>
</tr>
<tr>
<td>(Socc.) Nike SC2400-471</td>
<td>10.80 ± 0.15</td>
<td>422.2 ± 0.1</td>
<td>4.51 ± 0.08</td>
</tr>
<tr>
<td>(Voll.) Wilson WTH3501</td>
<td>10.35 ± 0.15</td>
<td>271.2 ± 0.1</td>
<td>5.02 ± 0.28</td>
</tr>
</tbody>
</table>