Restricted Weyl Invariance in Four Dimensional Curved Spacetime

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Dimensionless action in four spacetime dimension

Consider a dimensionless action with a scalar field nonminimally coupled to gravity and with the usual kinetic term containing two derivatives.

A generic action of this type is

$$S = \int d^4x\sqrt{|g|}\Big(- g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \eta R\phi^2 + a\,\mathrm{G} - c\,\mathrm{Weyl}^2 + \tfrac{b}{9}\,R^2 - \frac{\lambda\phi^4}{4!}\Big)$$

where

$$\mathrm{G} \equiv R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

is the Gauss-Bonnet topological invariant and

$$\mathrm{Weyl}^2 \equiv C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}\,R^2$$

is the Weyl tensor squared.

- The above action is not invariant under $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$ and $\phi \to \Omega^{-1}\phi$ i.e. it is not Weyl invariant.
- It is clearly scale invariant (when Ω=constant).
- However, as we will see, the dimensionless action in 4D has a symmetry that goes beyond scale invariance.

Transformation rules

Under a Weyl transformation we have the following transformation rules (in 4D)

$$\sqrt{|g|} \to \Omega^4 \sqrt{|g|}$$
$$R \to \Omega^{-2} R - 6\,\Omega^{-3}\,\Box\,\Omega\,.$$

It follows then that under a Weyl transformation we have

$$\sqrt{|g|}\,R\,\phi^2 \to \sqrt{|g|}\,R\,\phi^2 - \sqrt{|g|}\,6\,\phi^2\,\Omega^{-1}\,\Box\Omega\;,$$
$$\sqrt{|g|}\,R^2 \to \sqrt{|g|}\,R^2 - \sqrt{|g|}\,12\,R\,\Omega^{-1}\,\Box\Omega + \sqrt{|g|}\,36\,\Omega^{-2}\,(\Box\,\Omega)^2\,,$$
$$\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \to \sqrt{|g|}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \sqrt{|g|}\,\phi^2\,\Omega^{-1}\,\Box\,\Omega$$
$$- \sqrt{|g|}\,\nabla_\mu\big(\phi^2\,\nabla^\mu(\ln\Omega)\big)\,.$$

Restricted Weyl Invariance

The dimensionless action is Weyl invariant if $\Omega(x)$ obeys the constraint

$$\Box\Omega \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega = 0\,.$$

We say it is restricted Weyl invariant.

$\Omega(x)$ is not restricted to being a constant: symmetry is larger than scale invariance.

Properties of Restricted Weyl Invariance: composition law

Consider the two consecutive restricted Weyl transformations in general d space-time dimensions

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu} \\ \tilde{\tilde{g}}_{\mu\nu} &= \tilde{\Omega}^2 \tilde{g}_{\mu\nu} = \tilde{\Omega}^2 \Omega^2 g_{\mu\nu} \end{aligned} \tag{1}$$

where $\Box\Omega = 0$ and $\tilde{\Box}\tilde{\Omega} = 0$.

We now show that $\Box(\tilde{\Omega}\Omega) = 0$ (obeys composition law).

First, $\tilde{\Box}\tilde{\Omega} = 0$ implies

$$\Omega^{-2}\Box\tilde{\Omega} + (d-2)\Omega^{-3} g^{\mu\nu}\partial_\mu\Omega\partial_\nu\tilde{\Omega} = 0 \ . \tag{2}$$

Thus,

$$\Box(\tilde{\Omega}\Omega) = \Omega\Box\tilde{\Omega} + 2g^{\mu\nu}\partial_\mu\tilde{\Omega}\partial_\nu\Omega + \tilde{\Omega}\Box\Omega$$
$$= -(d-2)g^{\mu\nu}\partial_\mu\Omega\partial_\nu\tilde{\Omega} + 2g^{\mu\nu}\partial_\mu\tilde{\Omega}\partial_\nu\Omega \quad (3)$$

so that in (and only in) $d = 4$ dimension, the consecutive restricted Weyl transformation generates the restricted Weyl transformation by a composition law.

The "inverse" of the restricted Weyl transformation exists in $d = 4$ dimensions only. For

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad (4)$$

with $\Box\Omega = 0$, we may define

$$g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu} \quad (5)$$

as an inverse of the restricted Weyl transformation. It can be readily checked that $\tilde{\Box}\Omega^{-1} = 0$ only in four dimensions.

Restricted Weyl flat metrics

- If a metric $g_{\mu\nu}$ is Weyl flat (conformal to flat) so that $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, then the Weyl tensor is zero.
- If Ω obeys $\Box\Omega = 0$, where $\Box$ here is the flat space d'Alembertian, then the Ricci scalar is also zero.
- We will call a Weyl flat metric which obeys $\Box\Omega = \eta^{\mu\nu}\partial_\mu\partial_\nu\Omega = 0$, a restricted Weyl flat metric.

Both the Weyl tensor and the Ricci scalar are zero in a restricted Weyl flat metric.

Moreover, the Ricci tensor cannot be zero in a restricted Weyl flat spacetime (except for the trivial case of flat spacetime i.e. if Ω is a constant).

In the context of General Relativity,

restricted Weyl flat spacetimes are never vacuum spacetimes but are spacetimes with traceless matter since $R = 0$.

Examples of restricted Weyl flat metrics

Examples of well-known spacetimes that are restricted Weyl flat include

- $AdS_2 \times S^2$ (near-horizon limit of an extremal black hole)
- Radiation-dominated era of FLRW cosmology

The metric of $AdS_2 \times S^2$ is given by

$$ds^2 = -\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + (d\theta^2 + \sin^2\theta d\phi^2) = \frac{1}{r^2}\left(-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) .$$

It is Weyl flat with Weyl (conformal) factor $\Omega = 1/r$. Since $\Box\Omega = 0$, this is a restricted Weyl flat metric.

The flat space FLRW metric can be expressed in the following equivalent forms

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) = \Omega^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2)$$

where $a(t)$ is the scale factor and $\Omega(\tau)$ is the conformal factor. In the radiation dominated era, $a(t) \propto t^{1/2}$ and $\Omega(\tau) \propto \tau$. Since $\Box\Omega = 0$, this is a restricted Weyl flat metric.

One can readily check that the above two cases have a Weyl tensor and Ricci scalar of zero.

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