Restricted Weyl Invariance in Four Dimensional Curved Spacetime

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Consider a dimensionless action with a scalar field nonminimally coupled to gravity and with the usual kinetic term containing two derivatives.

A generic action of this type is

\[
S = \int d^4x \sqrt{|g|} \left( -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \eta R \phi^2 + a G - c \text{Weyl}^2 + \frac{b}{9} R^2 - \frac{\lambda \phi^4}{4!} \right)
\]

where

\[
G \equiv R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - 4R_{\mu\nu}R^{\mu\nu} + R^2
\]

is the Gauss-Bonnet topological invariant and

\[
\text{Weyl}^2 \equiv C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3} R^2
\]

is the Weyl tensor squared.
The above action is not invariant under $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$ and $\phi \rightarrow \Omega^{-1}\phi$ i.e. it is not Weyl invariant.

It is clearly scale invariant (when $\Omega=$constant).

However, as we will see, the dimensionless action in 4D has a symmetry that goes beyond scale invariance.
Transformation rules

Under a Weyl transformation we have the following transformation rules (in 4D)

\[
\sqrt{|g|} \rightarrow \Omega^4 \sqrt{|g|} \\
R \rightarrow \Omega^{-2} R - 6 \Omega^{-3} \Box \Omega.
\]

It follows then that under a Weyl transformation we have

\[
\sqrt{|g|} R \phi^2 \rightarrow \sqrt{|g|} R \phi^2 - \sqrt{|g|} 6 \phi^2 \Omega^{-1} \Box \Omega,
\]
\[
\sqrt{|g|} R^2 \rightarrow \sqrt{|g|} R^2 - \sqrt{|g|} 12 R \Omega^{-1} \Box \Omega + \sqrt{|g|} 36 \Omega^{-2} (\Box \Omega)^2,
\]
\[
\sqrt{|g|} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \rightarrow \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \sqrt{|g|} \phi^2 \Omega^{-1} \Box \Omega
\]
\[- \sqrt{|g|} \nabla_\mu (\phi^2 \nabla^\mu (\ln \Omega)).\]
The dimensionless action is Weyl invariant if $\Omega(x)$ obeys the constraint

$$\Box \Omega \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \Omega = 0.$$ 

We say it is restricted Weyl invariant.

$\Omega(x)$ is not restricted to being a constant: symmetry is larger than scale invariance.
Consider the two consecutive restricted Weyl transformations in general $d$ space-time dimensions

$$
\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \\
\tilde{\tilde{g}}_{\mu\nu} = \tilde{\Omega}^2 \tilde{g}_{\mu\nu} = \tilde{\Omega}^2 \Omega^2 g_{\mu\nu}
$$

(1)

where $\Box \Omega = 0$ and $\Box \tilde{\Omega} = 0$.

We now show that $\Box (\tilde{\Omega} \Omega) = 0$ (obeys composition law).

First, $\Box \tilde{\Omega} = 0$ implies

$$
\Omega^{-2} \Box \tilde{\Omega} + (d - 2) \Omega^{-3} g^{\mu\nu} \partial_\mu \Omega \partial_\nu \tilde{\Omega} = 0 .
$$

(2)
Thus,

\[
\Box(\tilde{\Omega} \Omega) = \Omega \Box \tilde{\Omega} + 2g^{\mu \nu} \partial_\mu \tilde{\Omega} \partial_\nu \Omega + \tilde{\Omega} \Box \Omega = -(d - 2)g^{\mu \nu} \partial_\mu \Omega \partial_\nu \tilde{\Omega} + 2g^{\mu \nu} \partial_\mu \tilde{\Omega} \partial_\nu \Omega
\]

(3)

so that in (and only in) \( d = 4 \) dimension, the consecutive restricted Weyl transformation generates the restricted Weyl transformation by a composition law.

The “inverse” of the restricted Weyl transformation exists in \( d = 4 \) dimensions only. For

\[
\tilde{g}_{\mu \nu} = \Omega^2 g_{\mu \nu}
\]

(4)

with \( \Box \Omega = 0 \), we may define

\[
g_{\mu \nu} = \Omega^{-2} \tilde{g}_{\mu \nu}
\]

(5)

as an inverse of the restricted Weyl transformation. It can be readily checked that \( \Box \Omega^{-1} = 0 \) only in four dimensions.
If a metric $g_{\mu\nu}$ is Weyl flat (conformal to flat) so that $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, then the Weyl tensor is zero.

If $\Omega$ obeys $\Box \Omega = 0$, where $\Box$ here is the flat space d’Alembertian, then the Ricci scalar is also zero.

We will call a Weyl flat metric which obeys $\Box \Omega = \eta^{\mu\nu} \partial_\mu \partial_\nu \Omega = 0$, a restricted Weyl flat metric.

Both the Weyl tensor and the Ricci scalar are zero in a restricted Weyl flat metric.
Moreover, the Ricci tensor cannot be zero in a restricted Weyl flat spacetime (except for the trivial case of flat spacetime i.e. if $\Omega$ is a constant).

In the context of General Relativity,

restricted Weyl flat spacetimes are never vacuum spacetimes but are spacetimes with traceless matter since $R = 0$. 
Examples of restricted Weyl flat metrics

Examples of well-known spacetimes that are restricted Weyl flat include

- $\text{AdS}_2 \times S^2$ (near-horizon limit of an extremal black hole)
- Radiation-dominated era of FLRW cosmology

The metric of $\text{AdS}_2 \times S^2$ is given by

$$\left( \begin{array}{c} ds^2 = -\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + (d\theta^2 + \sin^2 \theta d\phi^2) = \frac{1}{r^2} \left( -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right). \right.$$ 

It is Weyl flat with Weyl (conformal) factor $\Omega = 1/r$. Since $\Box \Omega = 0$, this is a restricted Weyl flat metric.
The flat space FLRW metric can be expressed in the following equivalent forms

\[ ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) = \Omega^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2) \]

where \( a(t) \) is the scale factor and \( \Omega(\tau) \) is the conformal factor. In the radiation dominated era, \( a(t) \propto t^{1/2} \) and \( \Omega(\tau) \propto \tau \). Since \( \Box \Omega = 0 \), this is a restricted Weyl flat metric.

One can readily check that the above two cases have a Weyl tensor and Ricci scalar of zero.
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