Universal aspects in the equation of state for Yang-Mills theories [1,2]

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[1] M. Caselle, A. Nada, M. Panero, arXiv:1505.01106, accepted for publication in JHEP

[2] M. Bruno, M. Caselle, M. Panero and R. Pellegrini, JHEP **1503** (2015) 057

Introduction and motivations - SU(N)

One of the main features of pure SU(N) non-abelian gauge theories is the existence of a **deconfinement phase transition**, i.e. a temperature above which gluons are "deconfined". Our goal is to study the thermodynamics of pure gauge theories in the confining phase where the only degrees of freedom are the **glueballs**.

Looking at the thermodynamics of the theory in the confining phase we have a tool to explore the glueball spectrum of the theory. Our main result is that the thermodynamics of the model can only be described assuming a **string-like** description of glueballs (and thus a Hagedorn spectrum).

This analysis was performed in the 3+1 dimensional SU(3) model in 2009 in the pioneering work of Meyer[1]. Now, using high precision lattice data for SU(3)[2] and a new set of SU(2) data on (3+1) dimensions[3], we are in the position to refine the effective string analysis and test its predictive power.

[1] H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

[2] Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

[3] M. Caselle, A. Nada, M. Panero, *Hagedorn spectrum and thermodynamics of SU(N) Yang-Mills theories*, arXiv:1505.01106, accepted for publication in JHEP

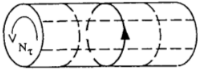
Why SU(2)?

The SU(2) model is a perfect laboratory to test these results.

- It is easy to simulate: very precise results may be obtained with a reasonable amount of computing power.
- The deconfinement transition is of second order and thus it is expected to coincide with the Hagedorn temperature, i.e. $T_c \equiv T_H$.
- The infrared physics of the model is very similar to that of the SU(3) theory, with one important difference: the gauge group is real and thus only $C = 1$ glueballs exist. Thus the glueball exponential spectrum contains only **half** of the states with respect to SU(3).
- The behaviour of the system is supposed to be dominated by a gas of non-interacting glueballs and the prediction of an ideal relativistic Bose gas can be used to describe it.

Thermodynamic quantities

On a $N_t \times N_s^3$ lattice the volume is $V = (aN_s)^3$ (where a is the lattice spacing), while the temperature is determined by the inverse of the temporal extent (with periodic boundary conditions): $T = (aN_t)^{-1}$.



The thermodynamic quantities taken into account will be:

- the pressure p, that in the thermodynamic limit (i.e. for large and homogenous systems) can be written as
$$p \simeq \frac{T}{V} \log Z(T, V)$$
- the trace of the energy-momentum tensor Δ, that in units of T^4 is
$$\frac{\Delta}{T^4} = \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right)$$

Energy density $\epsilon = \Delta + 3p$ and entropy density $s = \frac{\epsilon + p}{T} = \frac{\Delta + 4p}{T}$ can be easily calculated.

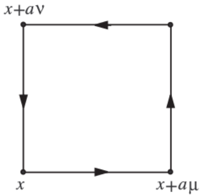
The **pressure** can be estimated by the means of the so-called "integral method"[1]

$$\frac{p(T)}{T^4} \simeq \frac{T^3}{V} \log Z(T,V) = -{N_t}^4 \int_0^\beta d\beta' \left[3(P_\sigma + P_\tau) - 6P_0\right]$$

where P_σ and P_τ are the expectation values of spacelike and timelike **plaquettes** respectively and P_0 is the expectation value at zero T.

The **trace of energy-momentum tensor** is simply

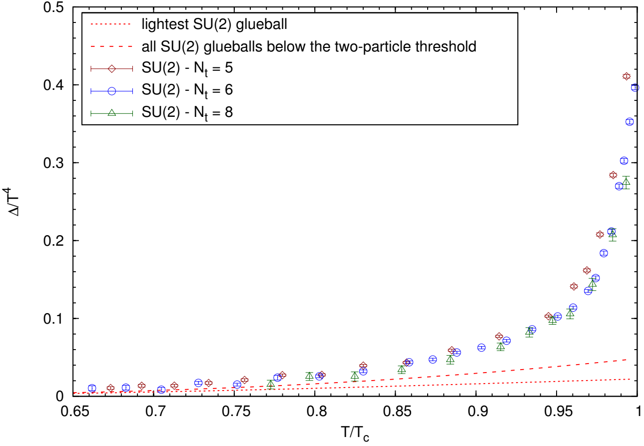
$$\frac{\Delta(T)}{T^4} = T\frac{\partial}{\partial T}\left(\frac{p}{T^4}\right) = -{N_t}^4 T \frac{\partial \beta}{\partial T}\left[3(P_\sigma + P_\tau) - 6P_0\right].$$



Plaquette in the directions μ and ν.

[1] J. Engels *et al.*, *Nonperturbative thermodynamics of SU(N) gauge theories*, 1990

SU(2): trace of energy-momentum tensor



The contribution of all glueball states with mass $m < 2m_{0^{++}}$.

A few important observations

- The large gap between the $m_{0^{++}}$ and the $m < 2m_{0^{++}}$ curves and those between them and the data show that the spectrum must be of the **Hagedorn** type, i.e. that the number of states increases exponentially with the mass. This is typically the signature of a **string-like** origin of the spectrum.
- The thermal behaviour of the model in the confining phase is thus a perfect laboratory to study the nature of this spectrum and of the underlying string model.
- Effective string theory suggests that, with a very good approximation, this model should be a **Nambu-Goto** string.

Let us see the consequences of this assumption.

Glueballs as rings of glue

A **closed string model**[1,2] for the full glueball spectrum can be introduced to account for the values of thermodynamic quantities near the transition. In the closed-string approach glueballs are described in the limit of large masses as **"rings of glue"**, that is closed tubes of flux modelled by closed bosonic string states.

[1] N. Isgur and J. Paton, *A Flux Tube Model for Hadrons in QCD*, 1985

[2] R. Johnson and M. Teper, *String models of glueballs and the spectrum of SU(N) gauge theories in (2+1)-dimensions*, 2002

Spectral density

The **spectral density** as a function of the mass is

$$\hat{\rho}(m) = \frac{(D-2)^{D-1}}{m} \left(\frac{\pi T_H}{3m} \right)^{D-1} e^{m/T_H}$$

where the **Hagedorn temperature**[1] is defined as

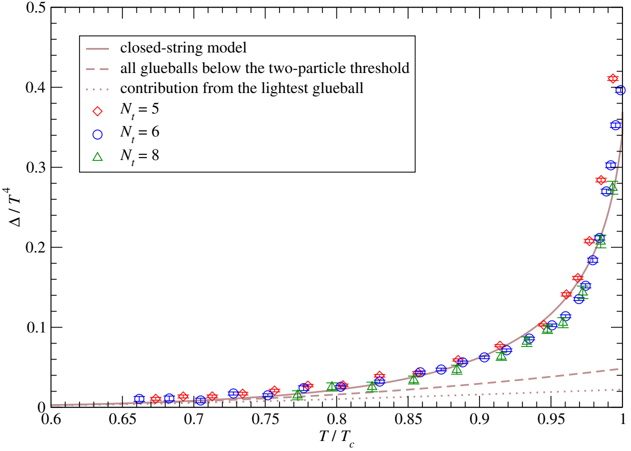
$$T_H = \sqrt{\frac{3\sigma}{\pi(D-2)}}.$$

Finally, the spectral density is used to account for all the states above the mass threshold $2m_{0^{++}}$:

$$\Delta = \sum_{m<2m_{0^{++}}} (2J+1)\Delta(m,T) + \int_{2m_{0^{++}}}^{\infty} \mathrm{d}m\, \hat{\rho}(m)\, \Delta(m,T)$$

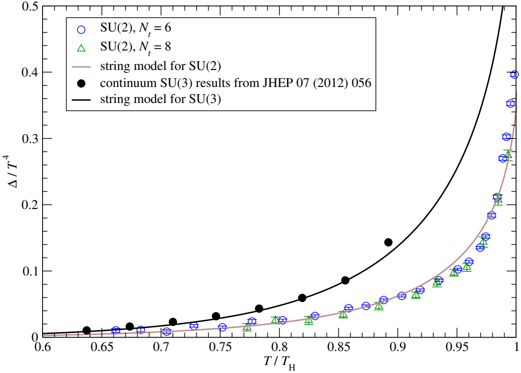
[1] R. Hagedorn, Nuovo Cim. Suppl. **3**, 147 (1965)

SU(2): trace of energy-momentum tensor



SU(2) *vs.* SU(3)

For $N = 3$ the closed flux tube has two possible orientations that account for the $\mathcal{C} = +1/-1$ sectors and so a further twofold degeneracy must be introduced in the string spectrum. Plus, the transition is first order, so $T_c < T_H = 0.691\sqrt{\sigma}$.



The doubling of the Hagedorn spectrum is clearly visible in the data.

Results for G_2 YM theory in (3+1) dimensions

Why G_2?

Why study G_2?

- Many similarities with QCD: **no colored states** in the physical spectrum; **linear potential** between color charges up to intermediate distances; **dynamical string-breaking** at large distances (due to gluons).

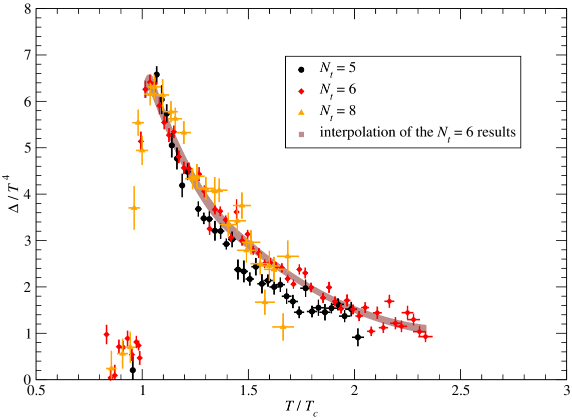
- In SU(N) theories the deconfinement at finite T is associated with the breaking of the global $\mathbb{Z}_N$ center symmetry; the center of G_2 is **trivial** but a first-order deconfinement phase transition is present nonetheless.

- **No sign problem**: simulations at finite densities are possible as the determinant of the Dirac operator is real and the resulting hadrons have a fermionic nature (as opposed to SU(2), where they are bosons).

With a new set of data[1] from simulations in the $T \lesssim 3T_c$ region it is possible to point out qualitative and quantitative analogies with SU(N) theories. The results push for the idea of a **universal thermal behaviour** between confining gauge theories, irrespective of the gauge group that is considered.

[1] M. Bruno, M. Caselle, M. Panero and R. Pellegrini, JHEP **1503** (2015) 057

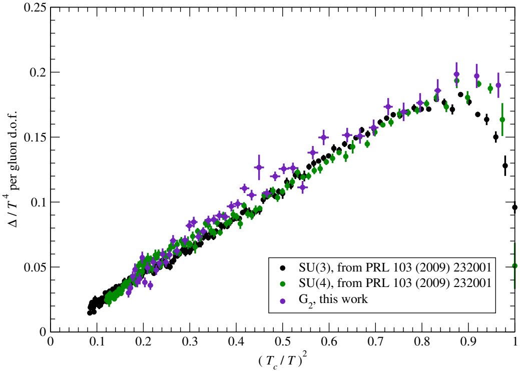
G_2: trace of the energy-momentum tensor



Results for the trace anomaly for the G_2 YM theory.

G_2: Δ/T^4 per gluon degrees of freedom

Here the results are divided by $2 \times d_a$, with $d_a = 14$ for G_2 and $d_a = N^2 - 1$ for SU(N).



G_2 and SU(N) data collapse on each other. Moreover a linear dependence on $1/T^2$ clearly appears.

Conclusions

The thermodynamics of SU(2) and SU(3) Yang-Mills theories in $D = (3 + 1)$ is well described by a gas of **non-interacting** glueballs.

- The agreement is obtained only assuming a **Hagedorn spectrum** for the glueballs.
- The fine details of the spectrum, in particular the Hagedorn temperature, agree well with the predictions of the **Nambu-Goto effective string**.
- The results agree with previous findings in $D = (2 + 1)$ SU(N) Yang Mills theories with $N = 2, 3, 4, 5, 6$.

The results for the equation of state for the exceptional Lie group G_2 present two features

- The results for Δ/T^4 *per gluon d.o.f.* are compatible with SU(N) ($N = 3, 4$), pointing to a **universal behaviour** between (very) different confining gauge theories.
- A peculiar behaviour already observed for SU(N) theories appears, as Δ goes with $1/T^2$, suggesting non-perturbative effects.

Acknowledgements

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[3]NIC, DESY

Scale setting

The SU(2) scale setting is fixed by calculating the string tension via the computation of **Polyakov loop correlators** with the multilevel algorithm.

The range of the parameter β which has been considered ($\beta \in [2.27, 2.6]$) covers approximately the temperature region analyzed in the finite temperature simulations.

The string tension is obtained with a two-parameter fit of

$$V = -\frac{1}{N_t} \log\langle PP \rangle$$

with the first order effective string prediction for the potential

$$V = \sigma r + V_0 - \frac{\pi}{12r}.$$

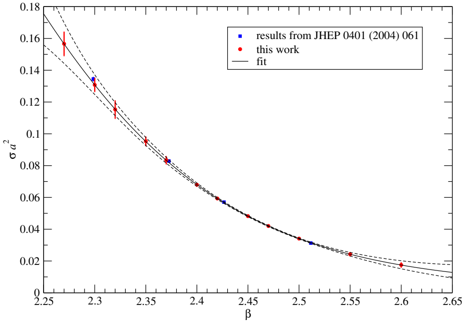
Higher order effective string corrections turned out to be negligible within the precision of our data.

Scale setting

The values of the string tension are interpolated by a fit to

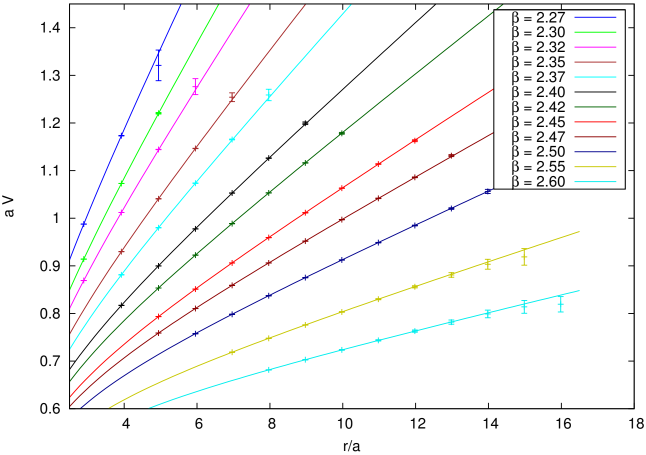
$$\log(\sigma a^2) = \sum_{j=0}^{n_{\mathrm{par}}-1} a_j(\beta - \beta_0)^j \qquad \text{with } \beta_0 = 2.35$$

which yields a χ^2_{red} of 0.01. It is presented below along with older data[1].



[1] B. Lucini, M. Teper, U. Wenger, *The high temperature phase transition in SU(N) gauge theories*, 2003

Scale setting



Ideal glueball gas

The behaviour of the system is supposed to be dominated by a gas of non-interacting glueballs.
The prediction of an ideal relativistic Bose gas can be used to describe the thermodynamics of such gas. Its partition function for 3 spatial dimensions is

$$\log Z = (2J+1)\frac{2V}{T}\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

where m is the mass of the glueball, J is its spin and K_2 is the modified Bessel function of the second kind of index 2.
Observables such as Δ and p thus can be easily derived:

$$p = \frac{T}{V}\log Z = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

$$\Delta = \epsilon - 3p = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right) K_1\left(k\frac{m}{T}\right)$$

Lattice setup for finite temperature

N_s^4 at $T = 0$	$N_s^3 \times N_t$ at $T \neq 0$	n_β	β-range	n_{conf}
32^4	$60^3 \times 5$	17	$[2.25, 2.3725]$	1.5×10^5
40^4	$72^3 \times 6$	25	$[2.3059, 2.431]$	1.5×10^5
40^4	$72^3 \times 8$	12	$[2.439, 2.5124]$	10^5

Setup of our simulations. The first two columns show the lattice sizes (in units of the lattice spacing a) for the $T = 0$ and finite-temperature simulations, respectively. In the third column, n_β denotes the number of β-values simulated within the β-range indicated in the fourth column. Finally, in the fifth column we report the cardinality n_{conf} of the configuration set for the $T = 0$ and finite-T simulations.

The **mass spectrum** of a closed strings gas in D spacetime dimensions is given by

$$m^2 = 4\pi\sigma\left(n_L + n_R - \frac{D-2}{12}\right)$$

where $n_L = n_R = n$ are the total contribution of left- and right-moving phonons on the string. Every glueball state corresponds to a given phonon configuration, but associated to each fixed n there are multiple different states whose number is given by $\pi(n)$, i.e. the **partitions** of n.

The **density of states** $\rho(n)$ is expressed through the square of $\pi(n)$:

$$\rho(n) = \pi(n_L)\pi(n_R) = \pi(n)^2 \simeq 12\,(D-2)^{\frac{D-1}{2}}\left(\frac{1}{24n}\right)^{\frac{D+1}{2}}\exp\left(2\pi\sqrt{\frac{2(D-2)n}{3}}\right)$$

SU(2) *vs.* SU(3)

The SU(3) case was studied for the first time in 2009 in the pioneering work of Meyer[1]. Now, using the high precision lattice data for SU(3) of [2] we are in the position to test the Hagedorn behaviour in a very stringent way.

The main differences between SU(2) and SU(3) are

- SU(3) has a first order deconfining transition, so $T_c < T_H$.
- SU(3) has complex representations, thus glueballs have an additional quantum number C and the glueball spectrum contains twice the number of glueballs than in the SU(2) case

In principle we could consider in this case T_H as a free parameter, but in the effective string framework we may safely fix it at the expected Nambu-Goto value $T_H = \sqrt{3\sigma/2\pi} \simeq 0.691\sqrt{\sigma}$. Lorentz invariance of the effective string tells us that this should be a very good approximation of the exact result and that we should expect only small deviations from this value.

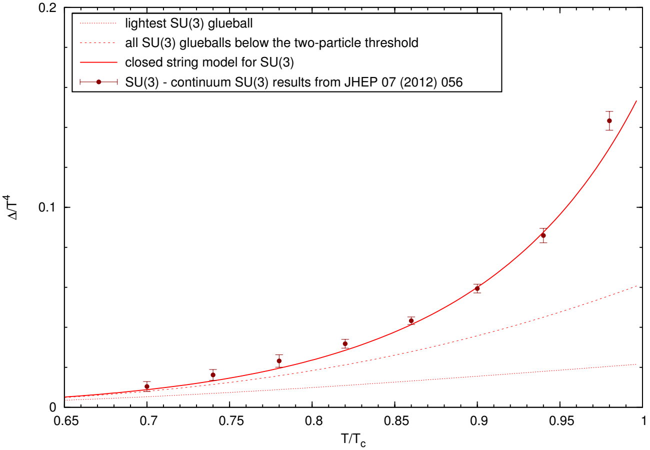
The relation between T_H and T_c is:

$$\frac{T_H}{T_c} = 1.098$$

[1] H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

[2] Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

SU(3): trace of energy-momentum tensor



Also in this case the $m < 2m_{0^{++}}$ sector of the glueball spectrum is not enough to fit the behaviour of Δ/T^4, while including the whole Hagedorn spectrum we find again a remarkable agreement with no free parameter!

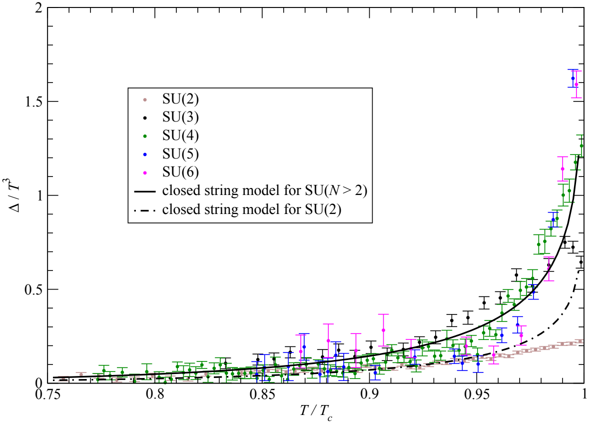
The same picture is confirmed by a study we performed a few years ago[1] in (2+1) dimensional SU(N) Yang-Mills theories for $N = 2, 3, 4, 6$. Also in (2+1) dimensions we found that:

- a Hagedorn spectrum was mandatory to fit the thermodynamic data
- there was a jump between the SU(2) and the SU($N > 2$) case due to the doubling of the spectrum
- we had to fix the Hagedorn temperature to the Nambu-Goto value which, due to the different number of trensverse degrees of freedom is different from the (3+1) dimensional one: $T_H = \sqrt{3\sigma/\pi} = 0.977..\sqrt{\sigma}$

Moreover we found that in the vicinity of the critical point there was an excess of Δ/T^4 with respect to our predictions for $N = 4, 5$ and 6 and that this excess increases with N. This could be understood as due to the k-string glueballs

[1] M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

SU(N) Yang-Mills theories in $(2+1)$ dimensions



For SU(N) pure gauge theories on the lattice the dynamics is described by the standard Wilson action

$$S_W = \beta \sum_{p=sp,tp} (1 - \frac{1}{N} \mathrm{ReTr} U_p)$$

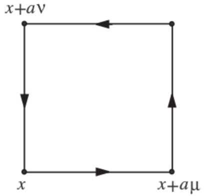
where U_P is the product of four U_μ SU(N) variables on the space-like or time-like plaquette P and $\beta = \frac{2N}{g^2}$.

The partition function is

$$Z = \int \prod_{x,\mu} \mathrm{d}U_\mu(x) e^{-S_W}$$

the expectation value of an observable A

$$\langle A \rangle = \frac{1}{Z} \int \prod_{n,\mu} \mathrm{d}U_\mu(n)\, A(U_\mu(n))\, e^{-S_W}$$



Decomposition of a G_2 tensor product

Following the rules for the G_2 algebra, the product of 3 adjoint representations can be decomposed

$$\begin{aligned}\mathbf{14} \otimes \mathbf{14} \otimes \mathbf{14} = & \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{14} \oplus \mathbf{14} \oplus \mathbf{14} \oplus \mathbf{14} \oplus \mathbf{27} \oplus \mathbf{27} \oplus \mathbf{27} \oplus \mathbf{64} \oplus \mathbf{64} \\ & \oplus \mathbf{77} \oplus \mathbf{77} \oplus \mathbf{77} \oplus \mathbf{77} \oplus \mathbf{77}' \oplus \mathbf{77}' \oplus \mathbf{77}' \oplus \mathbf{182} \oplus \mathbf{189} \oplus \mathbf{189} \oplus \mathbf{189} \\ & \oplus \mathbf{273} \oplus \mathbf{448} \oplus \mathbf{448}\end{aligned}$$

and the representation 7 appears on the right hand side. This implies that a fundamental colour source can be screened by three gluons.