FROM DREG TO NLO COMPUTATIONS IN 4D

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Introduction to Loop-tree duality (LTD)

IR regularization
- Threshold [Buchta’s talk] and IR singularities
- Finite real+virtual integration

UV renormalization

Conclusions

Based on:
Catani et al., JHEP 09 (2008) 065
Bierenbaum et al., JHEP 1010 (2010) 073; JHEP 03 (2013) 025
Buchta et al., JHEP 11 (2014) 014
Rodrigo et al., in preparation (to be published soon…)
Loop-tree duality (LTD)

- KLN theorem suggests that *virtual* and *real* contributions have the same *IR divergent structure* (because they cancel in IR-safe observables)
- *Cut contributions* are similar to *tree-level* scattering amplitudes, if all the loops are cut. At one-loop, *1-cuts are tree-level objects* (higher-cuts are products of unconnected graphs)
- **Objective:** Combine real and virtual contributions at *integrand level* and perform the *computation in four-dimensions* (take $\varepsilon$ to 0 with DREG)
  - Write virtual contributions as real radiation phase-space integrals of «tree-level» objects \(1\)-cut = sum over «tree level» contributions
  - *Loop measure is related with extra-radiation phase-space*

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*Loop-Tree Duality*  
Catani et al, JHEP 09 (2008) 065
Loop-tree duality (LTD)

Dual representation of one-loop integrals

\[ L^{(1)}(p_1, \ldots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i) = \int_\ell \prod_{i=1}^{N} \frac{1}{q_i^2 - m_i^2 + i0} \]

\[ L^{(1)}(p_1, \ldots, p_N) = -\sum_{i=1}^{N} \int_\ell \tilde{\delta}(q_i) \prod_{j=1, j \neq i}^{N} G_D(q_i; q_j) \]

\[ G_D(q_i, q_j) = \frac{1}{q_j^2 - m_j^2 - i0\eta(q_j - q_i)} \]

\[ \tilde{\delta}(q_i) = i2\pi \theta(q_i,0) \delta(q_i^2 - m_i^2) \]

Catani et al, JHEP 09 (2008) 065
LTD at one-loop

- Feynman integrands develop singularities when propagators go on-shell. LTD allows to understand it as soft/collinear divergences of real radiation.

- Forward (backward) on-shell hyperboloids associated with positive (negative) energy solutions.

  \[ G_F^{-1}(q_i) = q_i^2 - m_i^2 + i0 = 0 \]

  \[ q_i(\pm) = \pm \sqrt{q_i^2 + m_i^2} - i0 \]

- LTD equivalent to integrate along the forward on-shell hyperboloids.
- Degenerate to light-cones for massless propagators.
- Dual integrands become singular at intersections (two or more on-shell propagators)

For massive case, see Buchta´s talk!
IR regularization in LTD

Reference example: Massless scalar three-point function in the time-like region

\[ L^{(1)}(p_1, p_2, -p_3) = \int \prod_{i=1}^{3} G_F(q_i) = -\frac{c_F}{\epsilon^2} \left( -\frac{s_{12}}{\mu^2} - i0 \right)^{-1 - \epsilon} = \sum_{i=1}^{3} I_i \]

\[ I_1 = \frac{1}{s_{12}} \int d[\xi_{1,0}] d[v_1] \xi_{1,0}^{-1} \left( v_1 (1 - v_1) \right)^{-1} \]

\[ I_2 = \frac{1}{s_{12}} \int d[\xi_{2,0}] d[v_2] \frac{(1 - v_2)^{-1}}{1 - \xi_{2,0} + i0} \]

\[ I_3 = \frac{1}{s_{12}} \int d[\xi_{3,0}] d[v_3] \frac{v_3^{-1}}{1 + \xi_{3,0} - i0} \]

This integral is UV-finite; there are only IR-singularities, associated to soft and collinear regions

OBJECTIVE: Define a IR-regularized loop integral by adding real corrections at integrand level (i.e. no epsilon should appear, 4D representation)

IR regularization in LTD

- Analize the integration region. Application of LTD converts loop-integrals into PS: integrate in forward light-cones.

- Only forward-backward interference originate threshold or IR poles.
- Forward-forward cancel among dual contributions
- Threshold and IR singularities associated with finite regions (i.e. contained in a compact region)
- No threshold or IR singularity at large loop momentum

- This structure suggests how to perform real-virtual combination! Also, how to overcome threshold sing. (integrable but numerically inestable Buchta´s talk)

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From the previous plots, we define three contributions:

**IR-divergent contributions ($\xi_0 < 1+w$)**
- Originated in a **finite region** of the loop three-momentum
- All the IR singularities of the original loop integral

IR integral ($I^{IR}$):

$$I^{IR} = I_1^{(s)} + I_1^{(c)} + I_2^{(c)} = \frac{c_\Gamma}{s_{12}} \left( \frac{-s_{12} - i0}{\mu^2} \right)^{-\epsilon} \times \left[ \frac{1}{\epsilon^2} + \left( \ln(2) \ln(w) - \frac{\pi^2}{3} - 2Li_2 \left( -\frac{1}{w} \right) + \nu \ln(2) \right) \right] + O(\epsilon)$$

**Forward integrals ($v < 1/2, \xi_0 > 1$)**
- Free of IR/UV poles
- Integrable in 4-dimensions!

Forward integral ($I^{(f)}$):

$$I^{(f)} = \sum_{i=1}^{3} I_i^{(f)} = \frac{1}{s_{12}} \left[ \frac{\pi^2}{3} - \nu \pi \log(2) \right] + O(\epsilon)$$

**Backward integrals ($v > 1/2, \xi_0 > 1+w$)**
- Free of IR/UV poles
- Integrable in 4-dimensions!

Backward integral ($I^{(b)}$):

$$I^{(b)} = c_\Gamma \frac{1}{s_{12}} \left[ 2Li_2 \left( -\frac{1}{w} \right) - \ln(2) \ln(w) \right] + O(\epsilon)$$

IR regularization in LTD

Let’s stop and make some remarks about the structure of these expressions:

- Introduction of an **arbitrary cut** \( w \) to **include threshold regions**.
- Forward and backward integrals can be performed in 4D because the sum does not contain poles.
- Presence of extra Log’s in (F) and (B) integrals. They are originated from the expansion of the measure in DREG, i.e.

\[
\xi_r^{1-2\epsilon} = -\frac{Q_s^{-2\epsilon}}{2\epsilon} \delta(\xi_r) + \left( \frac{1}{\xi_r} \right)_C - 2\epsilon \left( \ln \left( \frac{\xi_r}{\xi_f} \right) \right)_C + \mathcal{O}(\epsilon^2)
\]

for both \( v \) and \( \xi \) (keep finite terms only). **It is possible to avoid them!**

- IR-poles isolated in \( I^{IR} \) **IR divergences originated in compact region of the three-loop momentum!!!**

\[
L^{(1)}(p_1, p_2, -p_3) = I^{IR} + I^{(b)} + I^{(f)}
\]

Explicit poles still present… Can be done in 4D!

IR regularization in LTD

- Now, we must add real contributions. Suppose one-loop scalar scattering amplitude given by the triangle

\[ |\mathcal{M}(0)(p_1, p_2; p_3)\rangle = ig \]
\[ |\mathcal{M}(1)(p_1, p_2; p_3)\rangle = -ig^3 L^{(1)}(p_1, p_2, -p_3) \]
\[ \Rightarrow \text{Re} \langle \mathcal{M}(0) | \mathcal{M}(1) \rangle \]

- 1->2 one-loop process \( \rightarrow \) 1->3 with unresolved extra-parton

- Add scalar tree-level contributions with one extra-particle; consider interference terms:

\[ |\mathcal{M}_{ir}^{(0)}(p'_1, p'_2, p'_r; p_3)\rangle = -ig^2 / s'_{ir} \Rightarrow \text{Re} \langle \mathcal{M}_{ir}^{(0)} | \mathcal{M}_{jr}^{(0)} \rangle = \frac{g^4}{s'_{ir} s'_{jr}} \]

- Generate 1->3 kinematics starting from 1->2 configuration plus the loop three-momentum \( \vec{l} \) !!!

IR regularization in LTD

**Mapping of momenta:** generate 1->3 real emission kinematics (3 external on-shell momenta) starting from the variables available in the dual description of 1->2 virtual contributions (2 external on-shell momenta and 1 free three-momentum)

\[
p_{1}^{\mu} = q_{1}^{\mu} \quad p_{2}^{\mu} = (1 - \alpha_{1}) p_{2}^{\mu} \quad \alpha_{1} = \frac{q_{3}^{2}}{2q_{3} \cdot p_{2}} \quad q_{1} = \ell + p_{1}
\]

\[
p_{3} \rightarrow p_{1} + p_{2} \Rightarrow p_{3} \rightarrow p_{1}' + p_{2}' + p_{r}' + \ell
\]

- Mapping optimized for \(y_{1r}' < y_{2r}'\); analogous expression in the complement
- Express interference terms using this map

**Real and virtual contributions are described using the same integration variables!**

Only required for I₁ and I₂ (I₃ singularities cancel among dual terms)

\[
\bar{\sigma}_{i,R} = \sigma_{0}^{-1} 2\text{Re} \int d\Phi_{1 \rightarrow 3} \langle \mathcal{M}_{2r}^{(0)} | \mathcal{M}_{1r}^{(0)} \rangle \theta(y_{jr}' - y_{ir}')
\]

\[
\bar{\sigma}_{i,V} = \sigma_{0}^{-1} 2\text{Re} \int d\Phi_{1 \rightarrow 2} \langle \mathcal{M}^{(0)} | \mathcal{M}_{i}^{(1)} \rangle \theta(y_{jr}' - y_{ir}')
\]

\[
\bar{\sigma}_{1} = \bar{\sigma}_{1,V} + \bar{\sigma}_{1,R} = \mathcal{O}(\epsilon)
\]

\[
\bar{\sigma}_{2} = \bar{\sigma}_{2,V} + \bar{\sigma}_{2,R} = -c_{\Gamma} \frac{g^2}{s_{12}} \frac{\pi^2}{6} + \mathcal{O}(\epsilon)
\]

UV renormalization in LTD

Reference example: two-point function with massless propagators

\[ L^{(1)}(p, -p) = \int \prod_{i=1}^{2} G_F(q_i) = \frac{c_T}{\epsilon(1 - 2\epsilon)} \left( -\frac{p^2}{\mu^2} - i0 \right)^{-\epsilon} = \sum_{i=1}^{2} I_i \]

In this case, the integration regions of dual integrals are two energy-displaced forward light-cones. This integral contains UV poles only.

**OBJECTIVE:** Define a UV-regularized loop integral by adding unintegrated UV counter-terms, and find a purely 4-dimensional representation of the loop integral.

Divergences arise from the high-energy region (UV poles) and can be cancelled with a suitable renormalization counter-term. For the scalar case, we use

\[ I_{\text{UV}}^{\text{cnt}} = \int_{\ell} \frac{1}{(q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0)^2} \]

Dual representation (new: double poles in the loop energy) Bierenbaum et al. JHEP 03 (2013) 025

\[ I_{\text{UV}}^{\text{cnt}} = \int_{\ell} \frac{\tilde{\delta}(q_{\text{UV}})}{2 \left(q_{\text{UV},0}^{(+)}}\right)^2} \]

\[ q_{\text{UV},0}^{(+)} = \sqrt{q_{\text{UV}}^2 + \mu_{\text{UV}}^2 - i0} \]

Loop integration for loop energies larger than \( \mu_{\text{UV}} \)

Cancellation of UV singularities

- Using the standard parametrization we define

\[
L^{(1)}(p, -p) - I_{\text{UV}}^{\text{cnt}} = c_T \left[ -\log \left( -\frac{p^2}{\mu_{\text{UV}}^2} - i0 \right) + 2 \right] + O(\varepsilon)
\]

- Since it is finite, we can express the regularized two-point function in terms of 4-dimensional quantities (i.e. no epsilon required!!)

- **Physical interpretation of renormalization scale**: Separation between on-shell hyperboloids in UV-counterterm is \(2/\mu_{\text{UV}}\). To avoid intersections with forward light-cones associated with \(l_1\) and \(l_2\), the renormalization scale has to be larger or of the order of the hard scale. So, the minimal choice that fulfills this agrees with the standard choice (i.e. \(1/2\) of the hard scale).

Conclusions

- Introduced new method based on the Loop-Tree Duality (LTD) that allows to treat virtual and real contributions in the same way: simultaneous implementation and no need of IR subtraction
- Physical interpretation of IR/UV singularities in loop integrals
- Presented proof of concept of LTD with reference examples

- Perspectives:
  - Apply the technique to compute full NLO physical observables
  - Extend the procedure to higher orders: NNLO and beyond
Thanks!!!
**Loop-tree duality (LTD)**

Feynman integrals and propagators

\[ L_R^{(N)}(p_1, p_2, \ldots, p_N) = -i \int \frac{d^d \ell}{(2\pi)^d} \prod_{i=1}^{N} G_R(q_i). \]

**Generic one-loop Feynman integral**

**Momenta definition**

\[ q_i = \ell + \sum_{k=1}^{i} p_k \]

**Feynman propagator**

\[ G(q) \equiv \frac{1}{q^2 + i0} \]

\[ G(q) \]

\[ q_0(q_\pm) \text{ plane} \]

\[ L^{(N)} \]

\[ C_L \]

\[ q_0 \]

+i0 prescription guarantees that positive (negative) frequencies are propagated forward (backward) in time

Location of Feynman’s integrand poles
Loop-tree duality (LTD)

**Idea:** «Sum over all possible 1-cuts» (but with a **modified i0 prescription...»

- Apply Cauchy residue theorem to the Feynman integral:

\[
L^{(N)}(p_1, p_2, \ldots, p_N) = \int_q \int dq_0 \prod_{i=1}^{N} G(q_i) = \int_q \int_{C_L} dq_0 \prod_{i=1}^{N} G(q_i) = -2\pi i \sum \text{Res}_{\text{Im} \ q_0 < 0} \left[ \prod_{i=1}^{N} G(q_i) \right]
\]

- Select the residue of the poles with negative imaginary part:

\[
\text{Res}_{\{i-\text{th pole}\}} \left[ \prod_{j=1}^{N} G(q_j) \right] = \text{Res}_{\{i-\text{th pole}\}} G(q_i) \prod_{j=1}^{N} G(q_j)_{\{i-\text{th pole}\}}
\]

Supplementary formulas:

\[
\left[ \text{Res}_{\{i-\text{th pole}\}} \frac{1}{q_i^2 + i0} \right] = \int dq_0 \delta_+(q_i^2) \quad \left[ \prod_{j \neq i} G(q_j)_{\{i-\text{th pole}\}} \right] = \prod_{j \neq i} \frac{1}{q_j^2 - i0 \eta(q_j - q_i)}
\]

- Set internal propagators on-shell

- Introduction of «dual propagators» (\eta prescription, a future- or light-like vector)

**Derivation**

Catani et al, JHEP 09 (2008) 065