Entanglement Entropy and far-from-Equilibrium Energy Flow

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1 Motivation

2 Energy flow in strongly correlated systems
   • A universal regime of thermal transport
   • Holographic Model
   • Linearized solution

3 Information Flow
   • Entanglement Entropy
   • Mutual Information
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Far from equilibrium dynamics

- Compute real time dynamics.
- Transition to hydrodynamics at late times.
- In strongly correlated systems:
  - Organizing principles out of equilibrium?
  - Universality classes?

Examples:
- Thermalization of quark-gluon plasma.
- Quenches in condensed matter systems.
- Fluctuations in the fractional Hall effect.

Much simpler class of problems \(\rightarrow\) (non)equilibrium steady state.

Concrete setup: Two heat reservoirs at different temperatures are put in contact at time \(t = 0\).

Energy density \(\equiv \varepsilon(x, t = 0) = \frac{C\pi}{12} \left[ T_L^2 \Theta(-x) + T_R^2 \Theta(x) \right]\)
Issues

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A universal regime of thermal transport

[Bernard, Doyon ’12], [Bhaseen, Doyon, Lucas, Schalm ’13]

- Two systems initially at equilibrium.
- Independently thermalized.
- Conservation equations and tracelessness:

\[
\partial_x \langle T^{xx} \rangle = -\partial_t \langle T^{tx} \rangle = 0, \quad \langle T^{xx} \rangle = \langle T^{tt} \rangle
\]

- "Shockwaves" emanating from interface:

\[
\begin{align*}
\langle T^{tx} \rangle &= F(x - t) - F(x + t) \\
\langle T^{tt} \rangle &= F(x - t) + F(x + t)
\end{align*}
\]

- In \(d + 1\) dim: \(\langle T^{\mu\nu} \rangle = a_d T^{d+1} (\eta^{\mu\nu} + (d + 1)u^\mu u^\nu)\)
Hydrodynamics

[Bernard, Doyon ’12], [Bhaveen, Doyon, Lucas, Schalm ’13]

Prediction for the steady-state:

- Lorentz-boosted energy density with $u^\mu = (\cosh \theta, \sinh \theta, 0)$ and temperature
  
  \[ T_L = T \cdot e^\theta, \quad T_R = T \cdot e^{-\theta} \equiv \text{Doppler shift} \]

- Heat current: $J_E \equiv \langle T^{tx} \rangle_{st} = a_d \frac{T_L^{d+1} - T_R^{d+1}}{u_L + u_R}$ with $a_d \sim \frac{L^d}{G_N}$.

- Temperature: $T_{st} = \sqrt{T_L T_R}$

- Shockwave velocities: $c_s^2 = \frac{1}{d} = u_L u_R$. 

\[ T^{tt} - T^{tt}_{st}(T_L = 7, T_R = 1.9) \]
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Gravitational Dual

- Energy transport: Lorentz-boosted thermal distribution
  → Gravity dual: Boosted black brane

\[
S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left\{ R + 2\Lambda \right\}.
\]

\[
ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = \frac{L^2}{z^2} \left[ \frac{dz^2}{f(z)} - f(z) (\cosh \theta \, dt - \sinh \theta \, dx)^2 + (\cosh \theta \, dx - \sinh \theta \, dt)^2 + dx_{\perp}^2 \right]
\]

where \( f(z) = 1 - \left( \frac{z}{z_h} \right)^d \) and \( z_h = \frac{d}{4\pi T} \).

- \( g_{\mu\nu} \) is a solution of e.o.m. as long as \( z_h \neq z_h(t, x) \) and \( \theta \neq \theta(t, x) \).
- \( (t, x) \)-dependent solutions → linearization

\[
z_h(t, x) = z_h^{(0)} + \epsilon z_h^{(1)}(t, x) + \cdots, \quad \theta(t, x) = \epsilon \theta^{(1)}(t, x) + \cdots.
\]
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**Linearization**

\[
z_h(t, x) = z_h^{(0)} + \epsilon z_h^{(1)}(t, x) + \cdots, \quad \theta(t, x) = \epsilon \theta^{(1)}(t, x) + \cdots
\]

\[
g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon \begin{pmatrix}
\delta g_{tt}(t, x, z) & \delta g_{tx}(t, x, z) & 0 & 0 \\
\delta g_{tx}(t, x, z) & \delta g_{xx}(t, x, z) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta g_{zz}(t, x, z)
\end{pmatrix}
\]

**e.o.m.**

\[
\begin{cases}
0 = 2 \partial^2_t z_h^{(1)}(t, x) - \partial^2_x z_h^{(1)}(t, x) \\
0 = -z_h^{(0)} \partial_x \theta^{(1)}(t, x) + 2 \partial_t z_h^{(1)}(t, x)
\end{cases}
\]

**Solution:** Assuming some initial profile \( T_{ini}(x) \equiv T(t = 0, x) \)

\[
T(t, x) = \frac{3}{4\pi z_h(t, x)} = \frac{1}{2} \left[ T_{ini}(x + \frac{t}{\sqrt{2}}) + T_{ini}(x - \frac{t}{\sqrt{2}}) \right]
\]

\[
\theta(t, x) = -\epsilon \frac{2\sqrt{2}}{3} \pi z_h^{(0)} \left[ T_{ini}(x + \frac{t}{\sqrt{2}}) - T_{ini}(x - \frac{t}{\sqrt{2}}) \right]
\]

with \( z_h^{(0)} = \frac{3}{2\pi(T_L + T_R)} \). Note \( c_s = \frac{1}{\sqrt{2}} \).
Let us consider the initial profile:

\[ T_{\text{ini}}(x) = \frac{T_R + T_L}{2} + \frac{T_R - T_L}{2} \tanh(\alpha x) \]

Then one gets the \( t \)-dependent solution:

\[ T(t, x) = \frac{T_R + T_L}{2} \left[ 1 + \epsilon \frac{T_R - T_L}{T_R + T_L} \left( \tanh(\alpha (x + t/\sqrt{2})) + \tanh(\alpha (x - t/\sqrt{2})) \right) \right] \]

\[ \theta(t, x) = -\epsilon \frac{1}{\sqrt{2}} \frac{T_R - T_L}{T_R + T_L} \left[ \tanh(\alpha (x + t/\sqrt{2})) - \tanh(\alpha (x - t/\sqrt{2})) \right] \]

\( \mathcal{O}(\epsilon) \) are corrections in \( \frac{|T_R - T_L|}{T_R + T_L} \ll 1 \) \( \iff \) Small gradients in \( T(t, x) \).

Example: \( T_{\text{steady state}} \equiv T(t \to +\infty, x) = \sqrt{T_L T_R} = \frac{T_L + T_R}{2} + \mathcal{O}(\epsilon^2) \).
Linearization

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Entanglement Entropy

How does information get exchanged between the systems which are isolated at $t = 0$?

- **Entanglement entropy**: Holographic measure $\implies$ Generalization of Bekenstein-Hawking entropy formula

  $\left[\text{Ryu, Takayanagi '06}, \text{Hubeny, Rangamani, Takayanagi '07}\right]$

  $$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{d+2}}; \quad \text{Area with boundary } \partial A$$

- **Regularization**: $\text{Area}(\gamma_A^\text{div}) \sim \frac{1}{z_{uv}}$.
- **Spatial induced metric on surface**: $ds^2 = h_{\sigma\sigma}d\sigma^2 + g_{yy}dy^2$.
- **Action**: $S = L \int ds \sqrt{\frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \sigma}} \tilde{g}_{ab}$
- **Minimal surface $\implies$ Geodesic equations** for metric $\tilde{g}_{ab} = g_{yy}g_{ab}$:

  $$\frac{d^2 x^a}{ds^2} + \tilde{\Gamma}^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0, \quad a = t, x, z$$

  $$s \equiv \text{affine parameter} \implies \frac{\partial x^a}{\partial s} \frac{\partial x^b}{\partial s} \tilde{g}_{ab} = 1.$$
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\[ S^\text{reg}_A = \frac{1}{4G_N^{d+2}} \left( \text{Area}(\gamma_A) - \text{Area}(\gamma_A^{\text{div}}) \right) \]
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**Entanglement Entropy**

\[ S = L \int_{-s_{uv}}^{s_{uv}} ds = 2L \cdot s_{uv} \]

\[ S^{\text{reg}} = S - S^{\text{div}} = 2L \cdot \left( s_{uv} - \frac{1}{Z(s_{uv})} \right) \]

- **Geodesic equations** are 3 coupled equations of 2nd order \( \Rightarrow \) solutions:

\[ t = t(s), \quad x = x(s), \quad z = z(s) \]

- 6 boundary conditions:

\[
\begin{align*}
  t(s_{uv}) &= t(-s_{uv}) = t_0 \\
  x(s_{uv}) &= -x(-s_{uv}) = \ell/2 \\
  z(s_{uv}) &= z(-s_{uv}) = z_{uv}
\end{align*}
\]
Geodesics

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Entanglement Entropy

Contour plot of $T(t, x)$ with $T_L = 0.5$ and $T_R = 0.6$. 
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Entanglement Entropy

\[ T_{R} = 0.6 \]
\[ T_{L} = 0.5 \]
\[ T_{\text{steady state}} = 0.55 \]
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Information Flow

What do we learn about A by looking at B?

\[ I(A, B) = S(A) + S(B) - S(A \cup B) \]

Basic features:
- Shockwaves transport information.
- When does system A find out about the existence of system B?
  \( I(A, B) = \text{finite} \rightarrow \text{no need to regularize!!!} \)

We find:
\[ \partial_t I(A, B) \geq 0 \]
Mutual Information

$S_A + S_B = 0.5, \quad T_R = 0.6$

$I_{A,B} = 0.5, \quad T_R = 0.6$
Conclusions and future directions

- We have studied a holographic model for time-dep energy flow.
- **Linearization** → analytical solution valid for $|T_L - T_R| \ll T_L + T_R$ (linear response regime).
- We have computed the time evolution of **Entanglement Entropies** → Mutual information grows with time.

What next?

- Complete analysis beyond linear response regime: $0 < T_L/T_R < 1$ → Full numerical solution of Einstein e.o.m. [Chesler, Yaffe ’09], [Amado, Yarom ’15], [Ecker, Grumiller, Stricker ’15], [Erdmenger, Fernandez, Flory, EM, Straub ’15].
- **Other possible solutions?** [Chang, Karch, Yarom ’14].
- Higher dimensions: $(2 + 1)\text{dim} \rightarrow (3 + 1)\text{dim}$. What changes? Preliminary analysis confirms the change in shockwave velocities: $c_s^2 = 1/d$. 

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Thank You!