Space Charge

ACCELERATOR PHYSICS

HT 2015

E. J. N. Wilson

Contents - Space Charge

Thanks to :Karlheinz SCHINDL - CERN/AB

Direct space charge (Self fields)	Fields and forces Defocusing effect of space charge Incoherent tune shift in a synchrotron
Image fields	Image effect on incoherent tune shift Coherent tune shift "Laslett" coefficients
Bunched beams	Effect of longitudinal motion Space-charge limited synchrotrons How to remove the space-charge limit

---

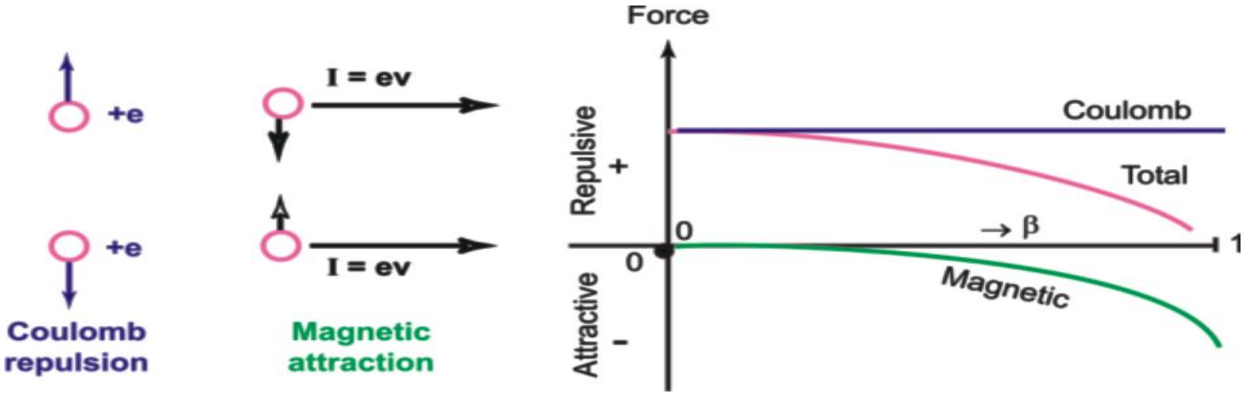
A. Hofmann, Tune shifts from self-fields and images, CAS Jyväskylä 1992, CERN 94-01, Vol. 1, p. 329

P.J. Bryant, Betatron frequency shifts due to self and image fields, CAS Aarhus 1986, CERN 87-10, p. 62

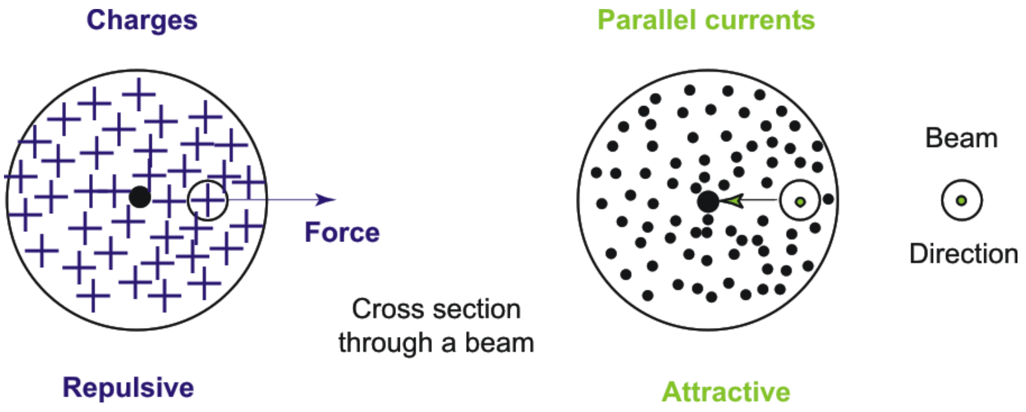
K. Schindl, Space charge, Proc. Joint US-CERN-Japan-Russia School on Part.Acc., "Beam Measurement", Montreux, May 1998, World Scientific, 1999, p. 127

Space Charge Force

Two Particles



Many Particles

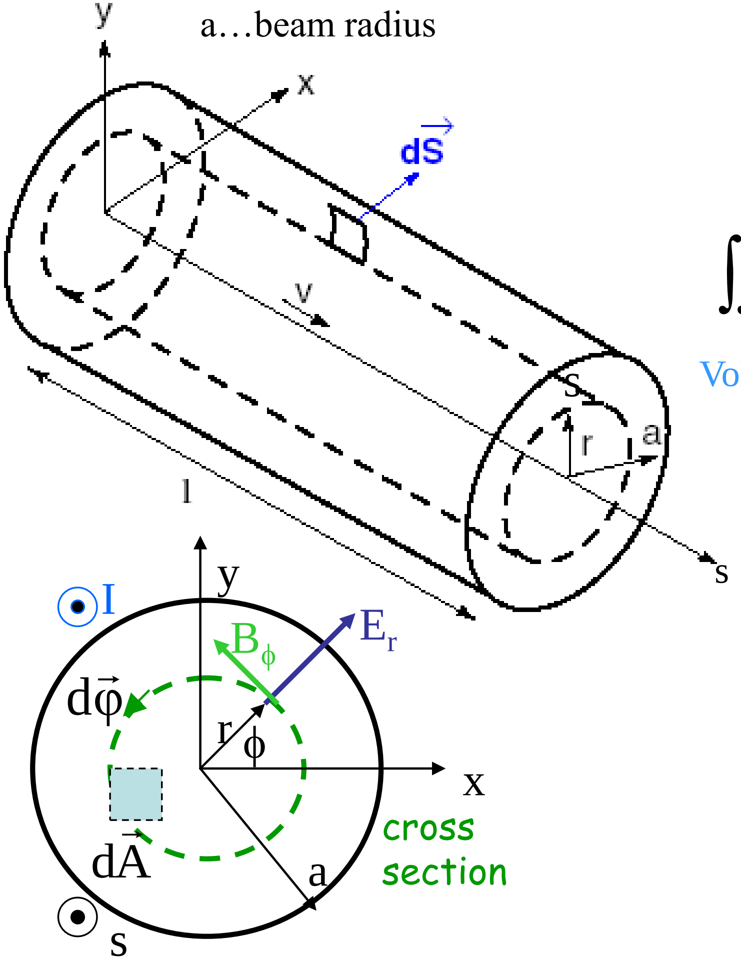


Force in beam centre = 0

Force larger near beam edge

Direct Space Charge - Fields

η…charge density in Cb/m^3
λ... constant line charge π a^{2}η
I…constant current λβc = π a^{2}ηβc
a…beam radius



Electric

$$\vec{E} = E_r$$

$$\operatorname{div}\vec{E} = \frac{\eta}{\varepsilon_0}$$

Magnetic

$$\vec{B} = B_\varphi$$

$$\operatorname{curl}\vec{B} = \mu_0 \vec{J}$$

Current density (βcη)

$$\iiint \operatorname{div}\vec{E}\,dV = \iint \vec{E}\,d\vec{S}$$

Volume element

$$\oint \vec{B}\,r\,d\vec{\varphi} = \iint \operatorname{curl}\vec{B}\,d\vec{A}$$

Apply these integrals over

cylinder radius r
length l

cross section
radius r

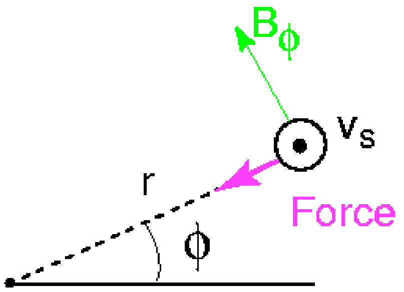
$$r^2\pi l\frac{\eta}{\varepsilon_0} = E_r\,2r\pi\,l$$

$$B_\varphi\,2r\pi = \mu_0 r^2\pi\beta c\eta$$

$$\boxed{E_r = \frac{I}{2\pi\varepsilon_0\beta c}\frac{r}{a^2}}$$

$$\boxed{B_\varphi = \frac{I}{2\pi\varepsilon_0 c^2}\frac{r}{a^2}}$$

Force on a Test Particle Inside the Beam



$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

$$F_r = e(E_r - v_s B_\varphi)$$

$$F_r = \frac{eI}{2\pi\varepsilon_0\beta c}(1-\beta^2)\frac{r}{a^2} = \frac{eI}{2\pi\varepsilon_0\beta c^2}\frac{1}{\gamma^2}\frac{r}{a^2}$$

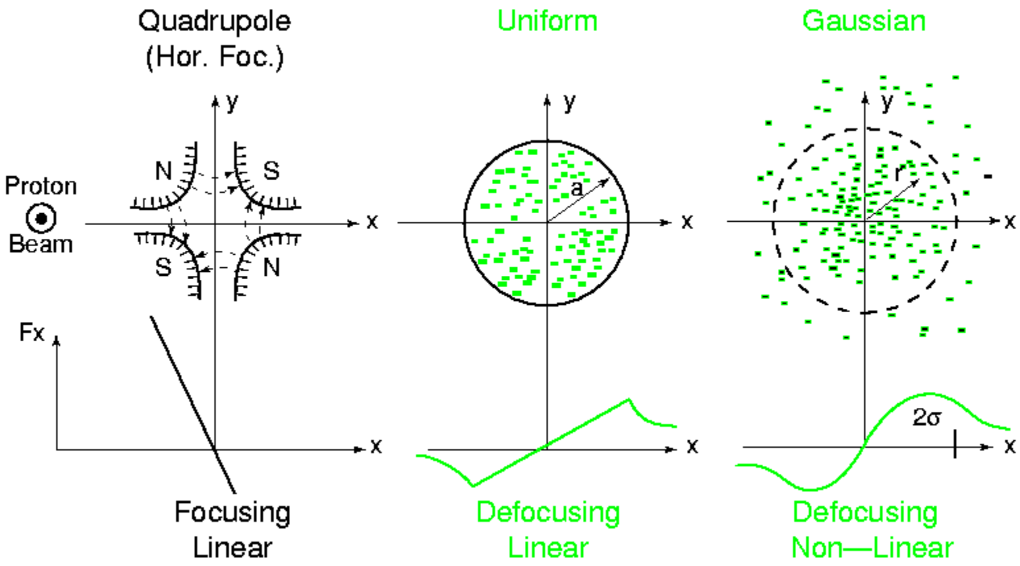
Electric

magnetic

$$\boxed{\begin{aligned} F_x &= \frac{eI}{2\pi\varepsilon_0\beta c\gamma^2 a^2}x \\ F_y &= \frac{eI}{2\pi\varepsilon_0\beta c\gamma^2 a^2}y \end{aligned}}$$

Space charge force

- circular beam
- uniform charge density
- F_x, F_y linear in x, y
- force → 0 for γ » 1 (β→1)
- defocusing lens in either plane



Space Charge in a Transport Line

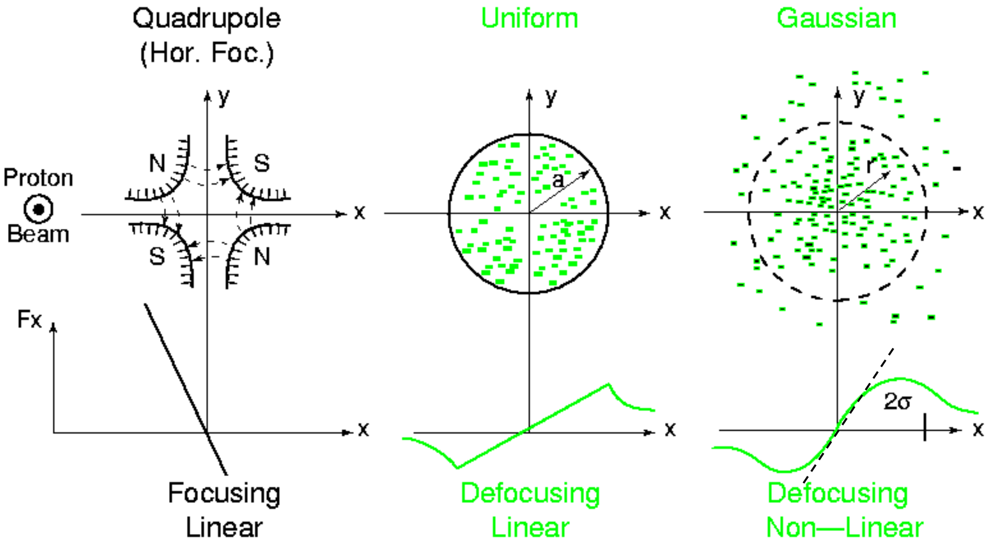
$x'' + K(s)x = 0$ Transport line with quadrupoles

$x'' + (K(s) + \underline{K_{SC}(s)})x = 0$ Transport line with quadrupoles and space charge

$$x'' = \frac{d^2x}{ds^2} = \frac{1}{\beta^2c^2}\frac{d^2x}{dt^2} = \frac{1}{\beta^2c^2}\frac{F_x}{m_0\gamma} = \frac{2r_0I}{ea^2\beta^3\gamma^3c}x \qquad \text{where} \qquad r_0 = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$

$$\boxed{x'' + \left(K(s) - \underbrace{\frac{2r_0I}{ea^2\beta^3\gamma^3c}}_{\mathbf{K_{SC}}}\right)x = 0}$$

In a **transport line**, the focusing by quadrupoles is counteracted by **space charge**, making **focusing weaker**



Incoherent Tune Shift in a Synchrotron

- Beam not bunched (so no acceleration)
- Uniform density in the circular x-y cross section (not very realistic)

$x'' + \left(K(s) + \underline{K_{SC}(s)}\right)x = 0$ ➔ Q_{x0} (external) + ΔQ_x (space charge)

For small "gradient errors" k_x $\underline{\Delta Q_x} = \frac{1}{4\pi}\int_0^{2R\pi} k_x(s)\beta_x(s)ds = \frac{1}{4\pi}\int_0^{2R\pi} \underline{K_{SC}(s)}\beta_x(s)ds$

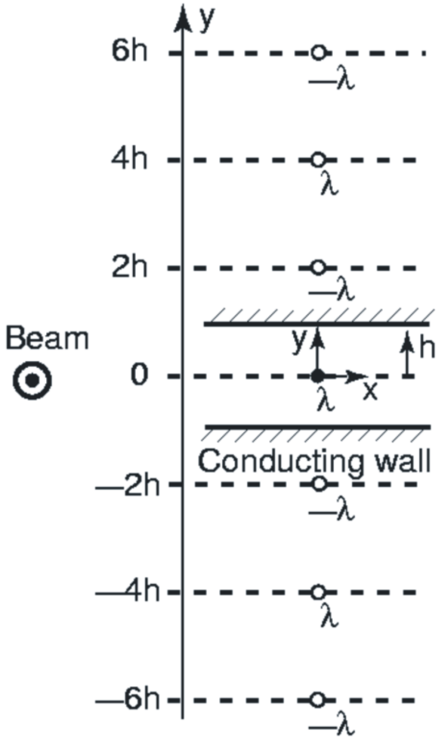
$$\Delta Q_x = -\frac{1}{4\pi}\int_0^{2R\pi}\frac{2r_0 I}{e\beta^3\gamma^3 c}\frac{\beta_x(s)}{a^2}ds = -\frac{r_0 RI}{e\beta^3\gamma^3 c}\left\langle\frac{\beta_x(s)}{a^2(s)}\right\rangle = -\frac{r_0 RI}{e\beta^3\gamma^3 cE_x}$$

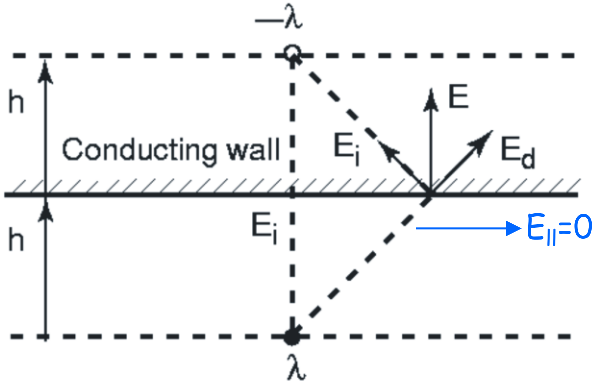
$$\boxed{\Delta Q_{x,y} = -\frac{r_0 N}{2\pi E_{x,y}\beta^2\gamma^3}}$$

using $I = (Ne\beta c)/(2R\pi)$ with
N…number of particles in ring
$E_{x,y}$….emittance containing 100% of particles

- "Direct" space charge, unbunched beam in a synchrotron
- Vanishes for γ » 1
- Important for low-energy machines
- Independent of machine size 2πR for a given N

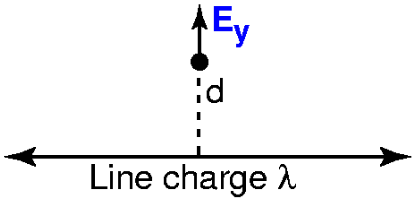
Incoherent Tune Shift: Image Effects





“Image charge” −λ to render $E_{||} = 0$ on conductive wall

Electric field around a line charge



$$E_y = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{d}$$

Image (line) charges created by two **parallel conducting plates**, distance 2h

$$E_{i1y} = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2h-y}-\frac{1}{2h+y}\right), \quad E_{i2y} = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{4h+y}-\frac{1}{4h\text{ - }y}\right)$$

$$E_{iny} = \frac{(-1)^{n+1}\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2nh-y}-\frac{1}{2nh+y}\right) = (-1)^{n+1}\frac{\lambda}{4\pi\varepsilon_0}\frac{y}{n^2h^2}$$

Image Field E_{iny} generated by the n-th pair of line charges

Image Effect of Parallel Conducting Plates ctd.

$$E_{iy} = \sum_{n=1}^{\infty} E_{iny} = \frac{\lambda}{4\pi\varepsilon_0 h^2} \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}}\, y = \frac{\lambda}{4\pi\varepsilon_0 h^2}\boxed{\frac{\pi^2}{12}}\, y$$

Vertical image field E_{iy}:

- ☐ vanishes at y=0
- ☐ linear in y
- ☐ vertical defocusing
- ☐ large if vacuum chamber small (small h)

$$\mathrm{div}\vec{E}_i = 0 = \frac{\partial E_{ix}}{\partial x} + \frac{\partial E_{iy}}{\partial y} \Rightarrow E_{ix} = -\frac{\lambda}{4\pi\varepsilon_0 h^2}\frac{\pi^2}{12}x$$

because between the conducting plates no **image** charges

$$F_{ix} = -\frac{e\lambda}{\pi\varepsilon_0 h^2}\frac{\pi^2}{48}x$$

$$F_{iy} = \frac{e\lambda}{\pi\varepsilon_0 h^2}\frac{\pi^2}{48}y$$

From these image forces F_{ix} and F_{iy} ⇨ K_{SC} ⇨ $\Delta Q_{x,y}$

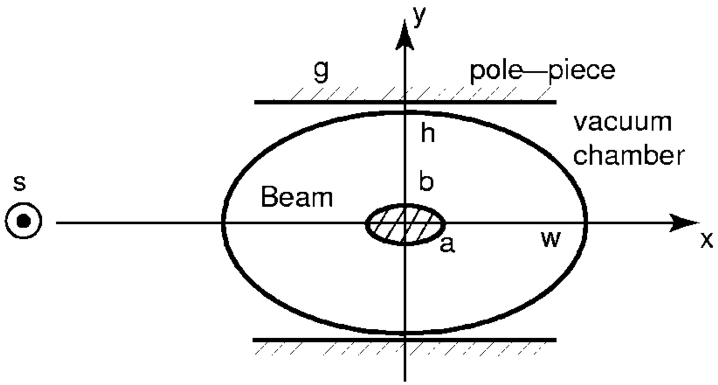
$$\Delta Q_x = -\frac{2r_0 IR\langle\beta_x\rangle}{ec\beta^3\gamma}\left(\frac{1}{2\langle a^2\rangle\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

tune shift **direct** **image**

$$\Delta Q_y = -\frac{2r_0 IR\langle\beta_y\rangle}{ec\beta^3\gamma}\left(\frac{1}{2\langle b^2\rangle\gamma^2} + \frac{\pi^2}{48h^2}\right)$$

- ☐ Image effects do not vanish for large γ, thus not negligible for electron machines
- ☐ Electrical image effects normally focusing in horizontal, defocusing in vertical plane
- ☐ Image effects also due to ferromagnetic boundary (e.g. synchrotron magnets)

The "Laslett"* Coefficients



$$\Delta Q_{y,inc} = -\frac{Nr_0\langle\beta_y\rangle}{\beta^2\gamma\pi}\left(\frac{\varepsilon_0^y}{b^2\gamma^2} + \frac{\varepsilon_1^y}{h^2} + \beta^2\frac{\varepsilon_2^y}{g^2}\right)$$

direct electr. image magnet. image

$$\Delta Q_{y,coh} = -\frac{Nr_0\langle\beta_y\rangle}{\beta^2\gamma\pi}\left(\frac{\xi_1^y}{h^2} + \beta^2\frac{\xi_2^y}{g^2}\right)$$

Laslett coefficients	Circular ($a = b,\ w = h$)	Elliptical (e.g. $w = 2h$)	Parallel plates ($h/w = 0$)
ε_0^x	$1/2$	$\frac{b^2}{a(a+b)}$	
ε_0^y	$1/2$	$\frac{b}{a+b}$	
ε_1^x	0	-0.172	-0.206
ε_1^y	0	0.172	0.206
ξ_1^x	$1/2$	0.083	0
ξ_1^y	$1/2$	0.55	$0.617(\pi^2/16)$
ε_2^x	$-0.411(-\pi^2/24)$	-0.411	-0.411
ε_2^y	$0.411(\pi^2/24)$	0.411	0.411
ξ_2^x	0	0	0
ξ_2^y	$0.617(\pi^2/16)$	0.617	0.617

$\pi^2/4$

8

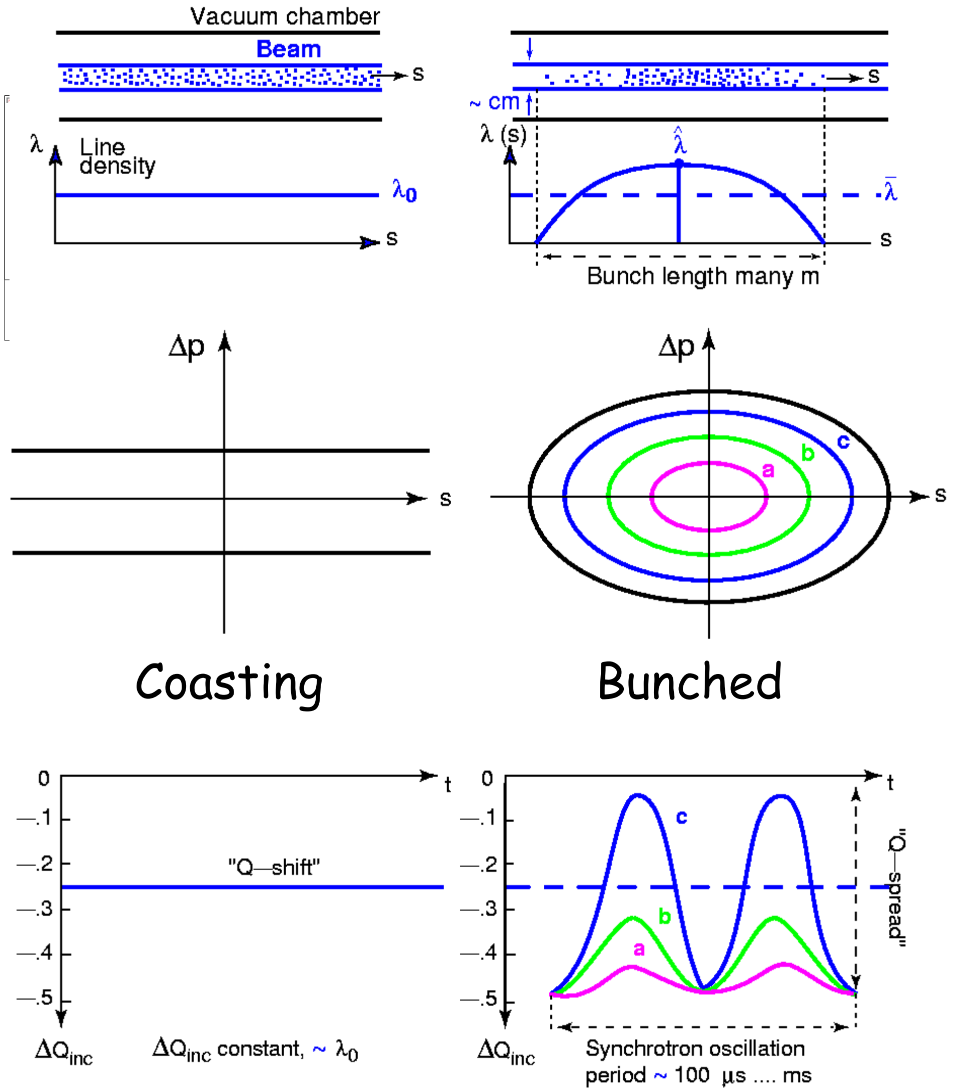
Uniform, elliptical beam in an elliptical beam pipe.

Similar formulae for $\Delta\mathbf{Q_x}$

In general, $\Delta Q_y > \Delta Q_x$

*L.J. Laslett, 1963

Bunched Beam in a Synchrotron



line density

longitudinal phase plane

Q-shift over a synchrotron period

What's different with bunched beams?

- ☐ Q-shift much larger in bunch centre than in tails
- ☐ Q-shift changes periodically with ω_s
- ☐ peak Q-shift much larger than for unbunched beam with same N (number of particles in the ring)
- ☐ Q-shift ⇨ Q-spread over the bunch

Incoherent ΔQ: A Practical Formula

$$\langle\beta\rangle = \frac{R}{Q}$$

$$\Delta Q_y = -\frac{r_0}{\pi}\left(\frac{q^2}{A}\right)\frac{N}{\beta^2\gamma^3}\frac{F_y G_y}{B_f}\left\langle\frac{\beta_y}{b(a+b)}\right\rangle \qquad \left\langle\frac{\beta_y}{b(a+b)}\right\rangle = \left\langle\frac{\beta_y}{b^2\left(1+\frac{a}{b}\right)}\right\rangle \approx \frac{1}{E_y\left(1+\sqrt{\frac{E_x Q_y}{E_y Q_x}}\right)}$$

$$\boxed{\Delta Q_{x,y} = -\frac{r_0}{\pi}\left(\frac{q^2}{A}\right)\frac{N}{\beta^2\gamma^3}\frac{F_{x,y} G_{x,y}}{B_f}\frac{1}{E_{x,y}\left(1+\sqrt{\frac{E_{y,x} Q_{x,y}}{E_{x,y} Q_{y,x}}}\right)}}$$

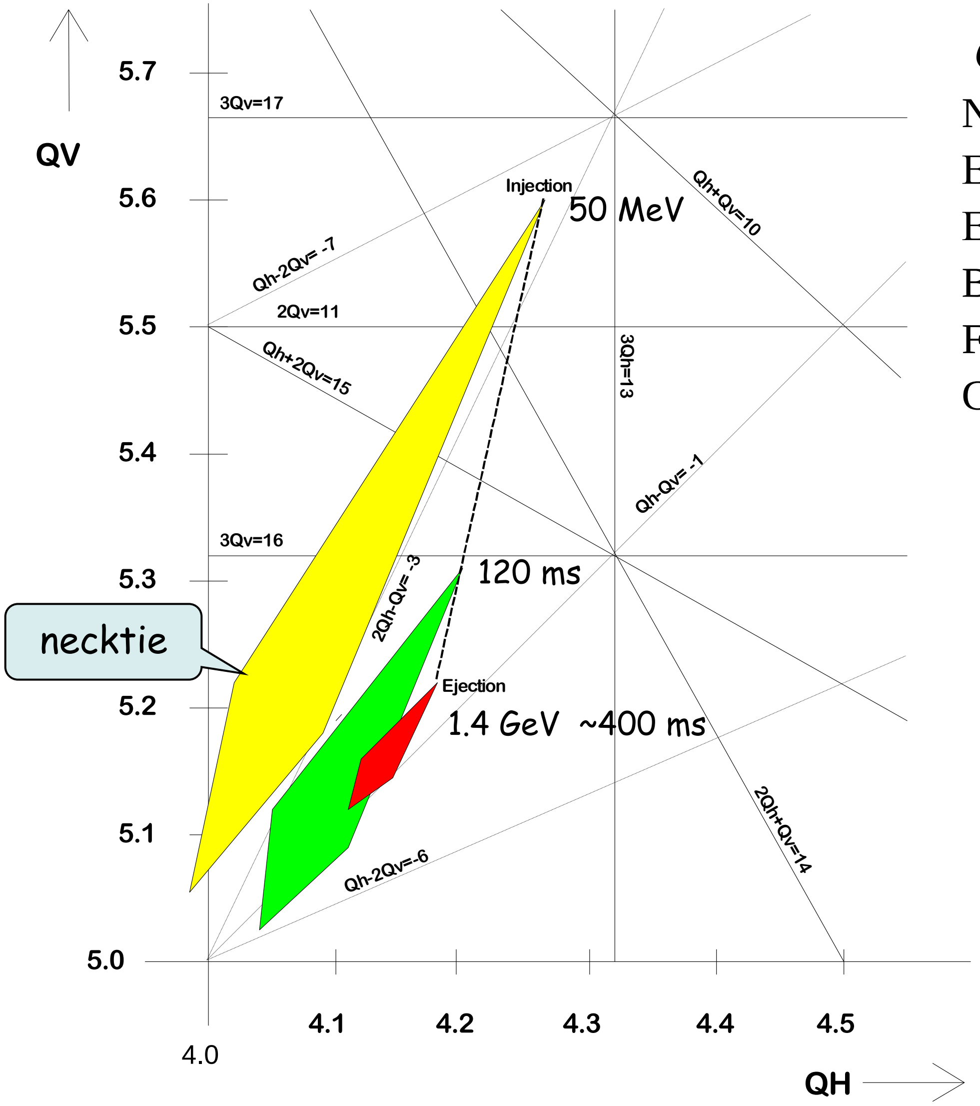
q/A…… charge/mass number of ions (1 for protons, e.g. 6/16 for ${}_{16}O^{6+}$)

$F_{x,y}$……"Form factor" derived from Laslett's image coefficients ε_1^x, ε_1^y, ε_2^x, ε_2^y ($F \approx 1$ if dominated by direct space charge)

$G_{x,y}$……Form factor depending on particle distribution in x,y. In general, $1 < G \leq 2$
Uniform G=1 ($E_{x,y}$ 100% emittance)
Gaussian G=2 ($E_{x,y}$ 95% emittance)

B_f…… "Bunching Factor": average/peak line density $B_f = \frac{\bar{\lambda}}{\hat{\lambda}} = \frac{\bar{I}}{\hat{I}}$

A Space-Charge Limited Accelerator



CERN PS Booster Synchrotron

$N = 10^{13}$ protons

$E_x^* = 80$ μrad m $[4\,\beta\gamma\,\sigma_x^2/\beta_x]$ hor. emittance

$E_y^* = 27$ μrad m vertical emittance

$B_f = 0.58$

$F_{x,y} = 1$

$G_x/G_y = 1.3/1.5$

- Direct space charge tune spread **~0.55 at injection**, covering 2nd and 3rd order stop-bands
- **“necktie”-shaped tune spread shrinks rapidly** due to the $1/\beta^2\gamma^3$ dependence
- Enables the working point to be moved **rapidly** to an area clear of strong stop-bands

How to Remove the Space-Charge Limit?

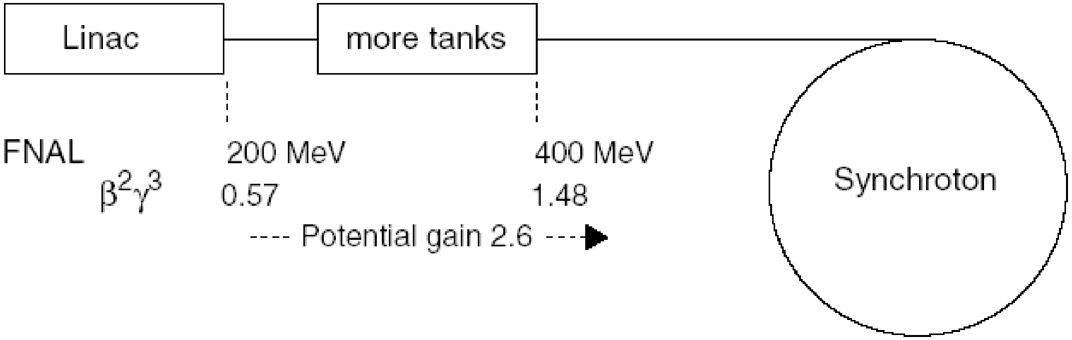
Direct space charge

$$\Delta Q_y \approx \frac{N}{E_y \beta^2 \gamma^3} \frac{\hat{I}}{\bar{I}}$$

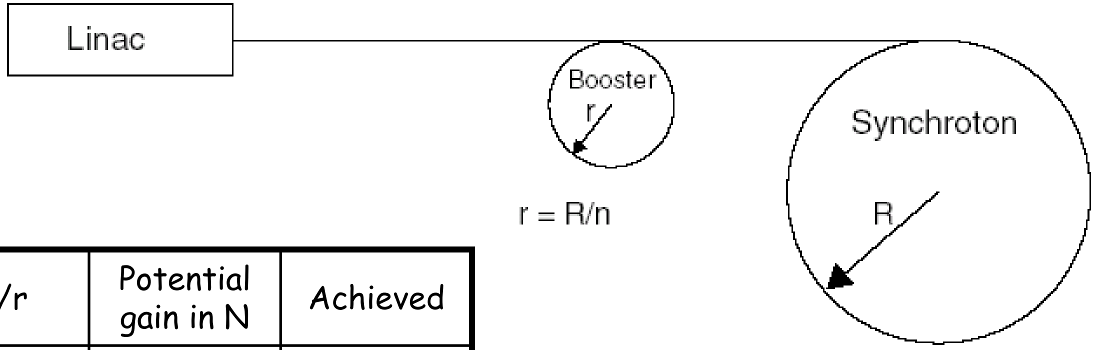
Problem: A **large proton synchrotron is limited in N** because ΔQ_y reaches 0.3 … 0.5 when filling the (vertical) acceptance.

Solution: Increase N by raising the injection energy and thus $\beta^2\gamma^3$ while keeping to the same ΔQ. Ways to do this:

Make a **longer** (higher-energy) **Linac** (by adding tanks as has been done in Fermilab)



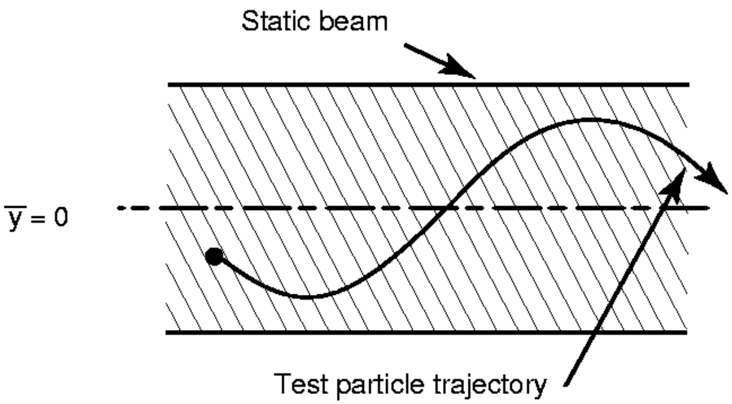
Add a **small "Booster" synchrotron** of radius r = R/n with n the number of batches (BNL) or rings (CERN)



	Linac (MeV)	Booster (GeV)	n=R/r	Potential gain in N	Achieved
CERN PS	50	1	4(rings)	59	~15
BNL AGS	200	1.5	4(batches)	26	~8

Incoherent and Coherent Motion

Incoherent motion

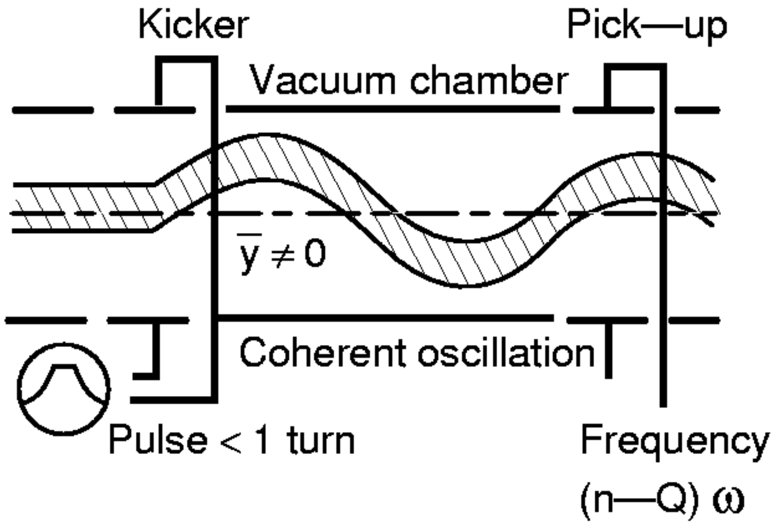


Test particle in a beam whose **centre of mass does not move**

The **beam environment does not "see"** any motion

Each particle features its **individual amplitude and phase**

Coherent motion

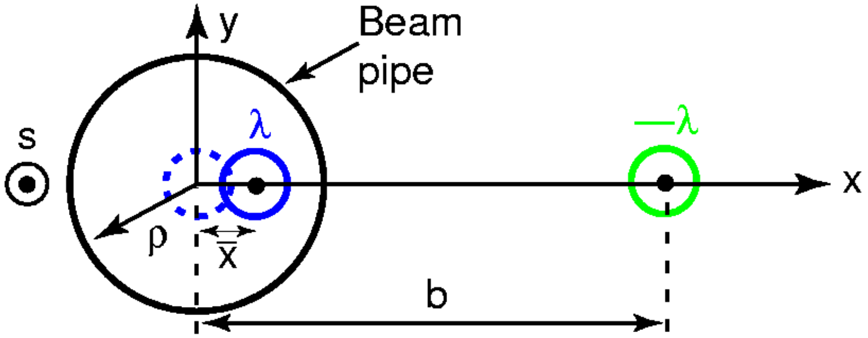


The **centre of mass moves** doing betatron oscillation as a whole

The **beam environment** (e.g. a position monitor "**sees**" the "**coherent** motion")

On top of the coherent motion, each particles has still **its individual** one

Coherent Tune Shift, Round Beam Pipe



$\bar{x}$..hor. beam position (centre of mass)
a...beam radius
ρ...beam pipe radius (ρ » a)

$b\bar{x} = \rho^2$ (mirror charge on a circle)

$$E_{ix}(\bar{x}) = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{b-\bar{x}} \approx \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{b} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\rho^2}\bar{x}$$

$$F_{ix}(\bar{x}) = \frac{e\lambda}{2\pi\varepsilon_0}\frac{1}{\rho^2}\bar{x}$$

☐ same in vertical plane (y) due to symmetry
☐ force linear in $\bar{x}$
☐ force positive hence defocusing in both planes

$$\boxed{\Delta Q_{x,y\,coh} = -\frac{r_0 R\langle\beta_{x,y}\rangle I}{ec\beta^3\gamma\rho^2} = -\frac{r_0\langle\beta_{x,y}\rangle}{2\pi\beta^2}\frac{N}{\gamma\rho^2}}$$

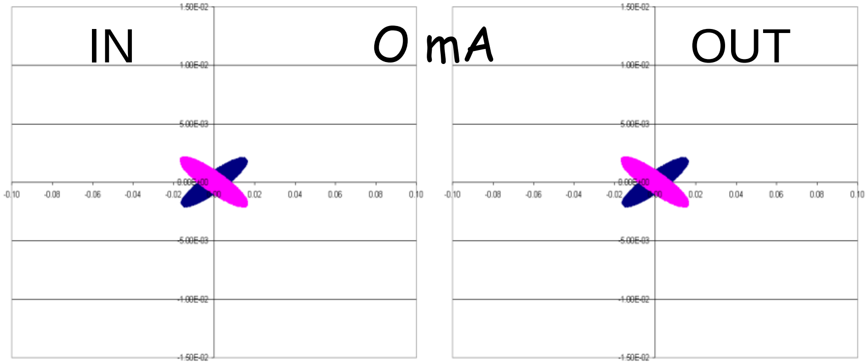
Coherent tune shift, round pipe
☐ negative (defocusing) both planes
☐ only weak dependence on γ
☐ ΔQ_{coh} always negative

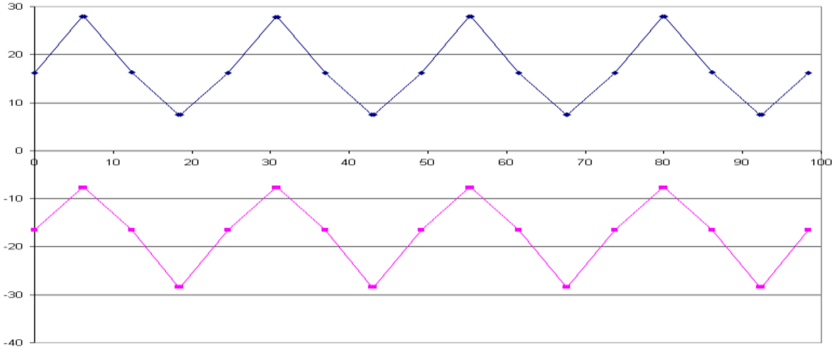
High Intensity Proton Beam in a FODO Line

Transverse phase planes
rad vs. m

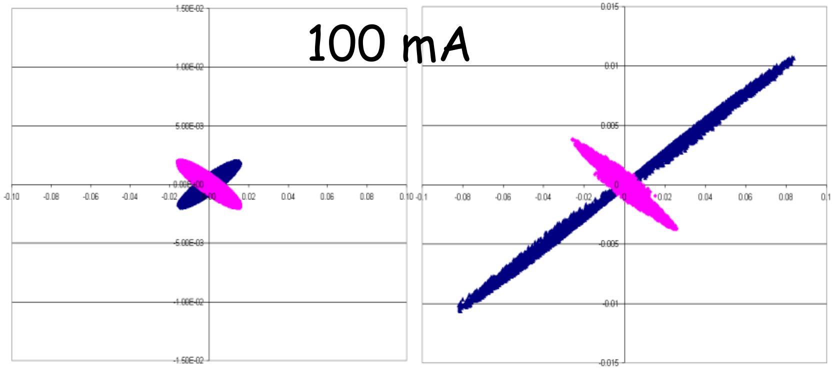
horizontal
vertical

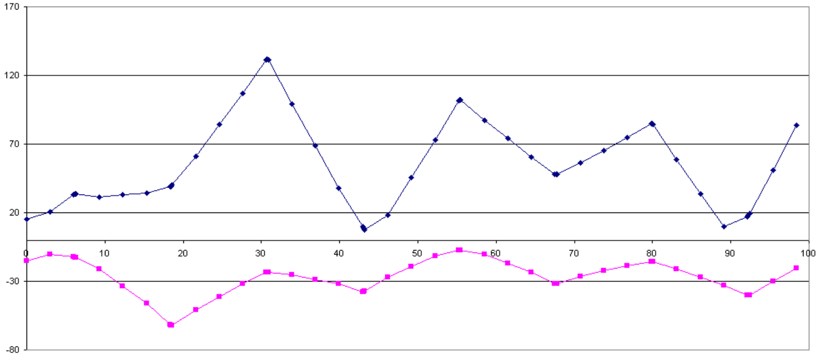
Transverse envelopes
mm vs. m





50 MeV





Courtesy of Alessandra Lombardi/ CERN, 8/04

Summary

"Direct" space charge generated by the self-field of the beam

- ➢ acts on incoherent motion but has no effect on coherent (dipolar) motion
- ➢ proportional to beam intensity
- ➢ defocusing in both transverse planes
- ➢ scales with $1/\gamma^3$ ⇨ barely noticeable in high-energy hadron and low-energy lepton machines

Image effects due to mirror charges induced in the vacuum envelope

- ➢ proportional to beam intensity
- ➢ scales with $1/\gamma$ ⇨ not negligible for high-γ beams and machines
- ➢ give rise to a further change in the incoherent motion, but focusing in one plane, defocusing in the other plane
- ➢ modify the transverse coherent motion (coherent Q-change)

Bunched beams: Space-charge defocusing depends on the particle's position in the bunch leading to a Q-spread (rather than a shift)

- ➢ Direct space charge is a hard limit on intensity/emittance ratio
- ➢ can be overcome by a higher-energy injector → M€