High Intensity Laser Matter Interaction

Z. Najmudin

January 14, 2015

Contents

1 References

P. Gibbon: *High intensity laser matter interaction*
E. Esarey et al, *Physics of laser-driven plasma-based electron accelerators*, Rev. Mod. Phys., vol. 81, no. 3, p. 1229
H. Daido et al, *Review of laser-driven ion sources and their applications.*, Reports Prog. Phys., vol. 75, no. 5, 056401

Also numerous IC thesis: **Underdense targets**: Z. Najmudin 1995, A. Modena 1996, B. Walton 2003, S. Mangles 2005, A. Thomas 2006, C. Kamperidis 2008. **Solid targets**: P. Lee 1996, M. Tatarakis 1997, E. Clark 2001, M. S. Wei 2003, L. Willingale 2007, S. Nagel 2009, C. Bellei 2009, S. Kneip 2010, C. Palmer 2011, N. Dover 2012, M. Streeter 2013

2 High intensity lasers and plasma

State of art e.g. Vulcan Petawatt (VPW), 400 J in 400 fs = 1 PW (compare UK power output $\sim$ 100 GW) focused to diameter spot $\phi = 5\mu$m, $\rightarrow$ intensity $I \sim 10^{21}$ Wcm^{-2} (= 10^{25} Wm^{-2}). The electric field of the laser can be found from the Poynting vector:

$$\mathbf{I} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \mathbf{B}/\mu_0 = \mathbf{E}^2\hat{\mathbf{k}}/\mathbf{c}\mu_0$$

For $E = E_0 \cos(kz - \omega t)$,

$$< \mathbf{I} >= \tfrac{1}{2}E_0^2/c\mu_0, \text{ so } E_0 = \sqrt{2c\mu_0 I} \approx 30\, I^{1/2} \text{ (SI)}$$

For VPW, $E \sim 9\times10^{11}$ Vcm^{-1}, *cf* $E_{Bohr} = 5\times10^7$ Vcm^{-1}.

The laser potential can easily exceed the atomic potential which means *all* materials are plasma at these intensities!

2.1 Field-ionisation

The regime of laser ionisation is set by the Keldysh Parameter. $\gamma_\kappa = (\epsilon_i/2\epsilon_p)^{1/2}$ - the ratio of ionisation to ponderomotive energies.
At low intensities $\gamma_\kappa \gg 1$, laser ionisation can happen by a multi-photon effect, over relatively long timescale. For $\gamma_\kappa \lesssim 1$, the tunnel regime, the Coulomb potential is reduced so the bound electrons can rapidly tunnel out of the atomic potential. In the extreme case, the potential is reduced such that the electron is no-longer bound - Barrier Suppression Ionisation (BSI). In this case, ionisation is near instantaneous. To calculate the BSI field, note for a laser field $E_l = E_0\cos(kx)$, the potential is $V_l = (E_0/k)\sin(kx) \approx E_0 x$ over atomic scales. So the potential energy of electron

$$U = -\frac{Ze^2}{(4\pi\epsilon_0)x} - eE_0 x.$$

The potential is a maximum for:

$$\frac{dU}{dx} = \frac{Ze^2}{4\pi\epsilon_0 x_{max}^2} - eE_0 = 0 \rightarrow x_{max} = \left(\frac{Ze}{4\pi\epsilon_0 E_0}\right)^{1/2}$$

$$U_m = \left(-\frac{Ze}{4\pi\epsilon_0 E_0}\frac{1}{x_{max}^2} - 1\right) eE_0 x_{max} = -2E_0 x_{max}$$

For immediate ionisation $U_m = \epsilon_i$, so

$$\epsilon_i = \left(2\frac{Ze^3 E_0}{4\pi\epsilon_0}\right)^{1/2} \rightarrow E_0 = \{\pi\epsilon_0/(Ze^3)\}\epsilon_i^2$$

and the threshold intensity is given by:

$$\begin{aligned} I_{BSI} = E_0^2/2\mu_0 c &= \{\pi^2\epsilon_0^2/(2Z^2e^6\mu_0 c)\}\epsilon_i^4 \\ &\approx 4\times 10^9(\epsilon_i/\mathrm{eV})^4 Z^{-2}\,\mathrm{Wcm}^{-2} \end{aligned}$$

where ϵ_i is in eV. Some values of I_{BSI}; H: 1.4×10^{14} Wcm^{-2}, He: 1.5×10^{15} Wcm^{-2}, He^+: 8.8×10^{15} Wcm^{-2}, Ne: 8.6×10^{14} Wcm^{-2}, Ne^{9+} : 1.4×10^{20} Wcm^{-2}.

3 Motion in laser field

Since $\dot{p_x} = -eE_0\cos\omega t, p_x = -eE_0\sin\omega t/\omega$,
Define $a_0 = eE_0/m_e c\omega$,
i.e. $\gamma m_e v_x = -a_0 m_e c\sin\omega t$,

a_0 is the "normalised momentum", or "normalised vector potential". Numerically:

$$a_0 \simeq 0.856\,(I\lambda^2)^{\frac{1}{2}},$$

for I [10^{18} Wcm^{-2}] and λ [μm].

a_0 is a figure of "laser strength", for $a_0 > 1 \rightarrow$ relativistic pulse (i.e. motion becomes relativistic). VPW has $a_0 \simeq 30$!

$|cp_x| = a_0 mc^2$, so using $\mathcal{E}^2 = m^2c^4 + c^2p^2$,

$$\begin{aligned} \mathcal{E}^2 = (\gamma mc^2)^2 &= (mc^2)^2 + (a_0 mc^2)^2 \\ \rightarrow \gamma &= \sqrt{1+a_0^2} \end{aligned}$$

For circular polarisation this is exact. For linear polarisation time averaging gives,

$$\gamma = \sqrt{1+a_0^2/2}.$$

(NB we'll find this is wrong later!)

3.1 Pondermotive Force

The energy of the electron is intensity dependent,

$$U = \gamma mc^2 = \{\sqrt{1+a_0^2/2}\}mc^2.$$

For low-energy ($a_0 \ll 1$); the kinetic energy is,

$$T = U - mc^2 \sim (a_0^2/4)mc^2 = e^2E^2/4m_e\omega^2.$$

which is the non-relativistic ponderomotive potential energy.

As a particle moves away from regions of high-intensity it can retain some of this energy, how much depends on the steepness of the gradient. Thus, we can say that there was a force acting on the particle: $F = -\nabla T$,

$\rightarrow F = (e^2/4m_e\omega^2)\nabla E^2$ the ponderomotive force.

The relativistic form can be found by taking $F = -\nabla U$,

i.e.
$$\begin{aligned} F &= (mc^2)\nabla\gamma \\ &= (mc^2)\nabla(1+a_0^2/2)^{1/2} \\ &= (mc^2/2)(1+a_0^2/2)^{-1/2}\nabla(a_0^2/2) \\ &= (mc^2/4\gamma)\nabla a_0^2 \end{aligned}$$

as above but with a γ thrown in for relativity.

4 Single Particle Motion

4.1 Effect of magnetic field of laser

We neglected the B field so far;

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = -e(\mathbf{E} + (\mathbf{v}\times\mathbf{B}))$$

For laser propagating in the z direction in an E-field, $\mathbf{E} = E_x\cos(kz-\omega t)\hat{\mathbf{i}}$, Faraday's law $(\nabla\times\mathbf{E} = -\dot{\mathbf{B}})$ gives, $\mathbf{B} = \hat{\mathbf{j}}(kE_x/\omega)\cos(kz-\omega t) = \hat{\mathbf{j}}B_y\cos(kz-\omega t)$, where $E_x = cB_y$ in vacuum. So we can write coupled equations for the particles motion in the em field:

$$\begin{aligned} m\dot{v}_x &= -eE_x\cos(kz-\omega t) \\ m\dot{v}_y &= 0 \\ m\dot{v}_z &= -ev_xB_y\cos(kz-\omega t) \end{aligned}$$

(NB we have ignored the $B_y\times v_z$ force on the v_x component, assuming that it is of 2nd order of smallness).
As before, solving for the x componen:

$$v_x = a_0 c\sin(kz-\omega t),$$

and so

$$\begin{aligned} \dot{v}_z &= (ea_0cE_x/mc)\sin(kz-\omega t)\cos(kz-\omega t) \\ &= (a_0^2 c\omega/2)\sin 2(kz-\omega t) \\ \rightarrow v_z &= -(a_0^2c/4)\cos 2(kz-\omega t) + k \end{aligned}$$

Initially $v_x = 0, v_z = 0$, then $k = (a_0^2c/4)$, so

$$\begin{aligned} v_z &= (a_0^2c/4)(1-\cos 2(kz-\omega t)) \\ &= (a_0^2c/2)(\sin^2(kz-\omega t)) \end{aligned}$$

These are parabolic tracks with $v_z = \frac{1}{2}v_x^2$. Note that the longitudinal motion is $\propto a_0^2$. It is the source of the longitudinal ponderomotive force. The motion of the electron is then:

$$\begin{aligned} x &= a_0 c\omega(\cos(kz-\omega t) - 1) \\ y &= 0 \\ z &= ({a_0}^2 c/4)t - ({a_0}^2 c/8)\sin 2(kz-\omega t)) \end{aligned}$$

Note that this implies a constant drift with $v_d = \frac{1}{4}{a_0}^2 c$ on top of a 'figure of eight' motion.

4.2 Full relativistic treatment (Advanced)

For $v \to c$, we should have written our equation of motion as:

$$\frac{d\mathbf{p}}{dt} = -e(\mathbf{E} + (\mathbf{v}\times\mathbf{B}))$$

Since $\mathbf{p}$ depends on γ too. For convenience, rewrite

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{dt} - \nabla\Phi \quad \text{and} \quad \mathbf{B} = \nabla\times\mathbf{A}$$

A is the vector potential. So,

$$\frac{d\mathbf{p}}{dt} = -e\left(-\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi + (\mathbf{v}\times\nabla\times\mathbf{A})\right)$$

We use the convective derivative for the momentum (since it is a function of time and space), and also the triple product identity:

$$\frac{d\mathbf{p}}{dt} = \frac{\partial\mathbf{p}}{\partial t} + v\cdot\nabla\mathbf{p} = \frac{\partial\mathbf{p}}{\partial t} + (\nabla\mathbf{p})\cdot\mathbf{v} - (\mathbf{v}\times\nabla\times\mathbf{p})$$

The equation of motion becomes,

$$\begin{aligned} &\frac{\partial\mathbf{p}}{\partial t} + (\nabla\mathbf{p})\cdot\mathbf{v} - (\mathbf{v}\times\nabla\times\mathbf{p}) \\ &= -e\left(-\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi + (\mathbf{v}\times\nabla\times\mathbf{A})\right) \\ \to &\frac{\partial\mathbf{u}}{\partial t} + (\nabla\mathbf{p})\cdot\mathbf{v} - (\mathbf{v}\times\nabla\times\mathbf{u}) = -e(-\nabla\Phi) \end{aligned}$$

where we have introduced the canonical momentum $\mathbf{u} = \mathbf{p} - e\mathbf{A}$.
Note that $(\nabla\mathbf{p})\cdot\mathbf{v} = (\mathbf{p}\cdot\nabla\mathbf{p}/m\gamma) = \nabla p^2/2m\gamma = m^2c^2\nabla\gamma^2/2m\gamma = mc^2\nabla\gamma$,
So

$$\to \quad \frac{\partial\mathbf{u}}{\partial t} = (\mathbf{v}\times\nabla\times\mathbf{u}) + \nabla(e\Phi - \gamma mc^2)$$

Taking the curl of the above equation:

$$\frac{\partial(\nabla\times\mathbf{u})}{\partial t}\cdot\mathbf{v} = (\mathbf{u}\times(\mathbf{v}\times\nabla\times\mathbf{u}))$$

which means if $\nabla\times\mathbf{u} = 0$ initially, then it is always thus,

$$\frac{\partial\mathbf{u}}{\partial t} = \nabla(e\Phi - \gamma mc^2)$$

Assuming we are in (near) vacuum or unperturbed plasma, we can take $\Phi = 0$. Also assuming infinite plane wave means $\nabla_\perp = 0$, and we can write out the components:

$$\begin{aligned} \frac{\partial\mathbf{u}_\perp}{\partial t} &= 0 \\ \text{and} \quad \frac{\partial\mathbf{u}_\parallel}{\partial t} &= \nabla_\parallel\gamma mc^2 \end{aligned}$$

The first tells us that transverse canonical momentum is conserved ($p_\perp - eA = 0$). Hence if the vector potential is given by $eA = a$, then $p_\perp = a$. The second equation can be dealt with by transforming to the wave frame i.e. use the quasistatic approximation, so that $\frac{\partial}{\partial t} \simeq c\frac{\partial}{\partial\xi}$ and of course $\nabla_\parallel = \frac{\partial}{\partial\xi}$, so

$$\frac{\partial(cu_\parallel - \gamma mc^2)}{\partial\xi} = \frac{\partial(cp_\parallel - \gamma mc^2)}{\partial\xi} = 0$$

Integrating and noting that $\gamma = 1$ and $p_\parallel = 0$ initially,

$$cp_\parallel - \gamma mc^2 = -mc^2 \quad \to \quad cp_\parallel + mc^2 = \gamma mc^2$$

squaring both sides and using $\gamma^2m^2c^4 = m^2c^4 + c^2p_\parallel^2 + c^2p_\perp^2$,

$$2cp_\parallel mc^2 = c^2a^2$$

so $p_\parallel = a^2/2mc$.
So finally we have $p_x = a_x$, $p_y = a_y$ and $p_z = \frac{1}{2}a_z^2$ (where we have normalised all the momenta to mc) and is true for arbitary polarisation. For linear polarisation, this is the parabolic dependence obtained before, but more strictly between transverse and longitudinal momentum *not* velocities. For circular polarisation, this means that there is an arbitrary *constant* longitudinal drift, which is $\frac{1}{2}a_z^2$ if the particle started at rest, since a is now constant.

5 Relativistic Thomson Scattering

One implication is that for $a \gg 1$ the motion becomes far from sinusoidal. Hence the current due to the electron motion becomes non-linear. This causes harmonics to be produced in the emission spectrum of the transmitted beam, (see for example Umstadter, Nature 1998).

$$\begin{aligned} \gamma\beta_x &= a \\ \gamma\beta_z &= a^2/2 \\ \& \quad \gamma = \sqrt{1+\gamma^2\beta_x^2+\gamma\beta_z} &= \sqrt{1+a^2+a^4/4} \\ &\to \gamma = 1 + a^2/2 \end{aligned}$$

(NB how we wrote the value of γ incorrectly before when we ignored the longitudinal component) so:

$$\begin{aligned} \beta_x &= a/(1+a^2/2) \\ \beta_z &= a^2/(2+a^2) \end{aligned}$$

6 Relativistic Self-Focusing

The relativistic motion has an effect on the refractive index of the plasma,

$$\eta = \left(1 - \frac{\omega_p^2}{\gamma\omega^2}\right)^{1/2}$$

where we used the 'relativistic mass increase' to modify the plasma frequency. For a simple picture of self-focusing, consider two rays, one travelling along the axis of the laser beam, and the other travelling the natural divergence angle of the beam $\theta = \lambda/2w_0$, where w_0 is the beam radius at focus. The path difference between the two rays over a Rayleigh length z_R is,

$$\delta L = z_r(1 - \cos\theta) \approx \frac{1}{2}z_r\theta^2 = \frac{z_r\lambda^2}{8w_0^2}$$

Now the difference in optical path lengths due to the intensity dependent refractive index, can be written as

$$\begin{aligned} \delta z &= \left(\int \eta \cdot dl\right)_{inner} - \left(\int \eta \cdot dl\right)_{outer} \\ &\approx z_r(\eta_{inner} - \eta_{outer}) \end{aligned}$$

Assuming that the outer path is unaffected, and that intensities are sufficiently small that we can write $\eta = \left(1 - \frac{\omega_p^2}{(1+\alpha)^{1/2}\omega^2}\right)^{1/2}$, and for sufficiently low densities, we expand the brackets to get $\delta z = z_r(\omega_p^2/2\omega^2)\alpha = z_r(\omega_p^2/4\omega^2)(a^2/2)$, where we have used the value of γ calculated in the last section. Equating this to the geometric path difference found above;

$$\frac{z_r\lambda^2}{8w_0^2} = z_r\left(\frac{\omega_p^2}{4\omega^2}\right)\left(\frac{a^2}{2}\right)$$

where after some cancelling: $a^2w_0^2 = \left(\frac{2\pi c}{\omega_p}\right)^2 = \lambda_p^2$. Hence relativistic self-focusing leads to a spot size $w_0 \sim \lambda_p$.
Rewriting a in terms of E, we get

$$E^2w_0^2 = \frac{4\pi^2m^2c^4}{e^2}\left(\frac{\omega^2}{\omega_p^2}\right)$$

Noting that $E^2 = 2\mu_0 cI$ the RHS is thus $\propto \mathcal{P}$ - the laser power. So,

$$\mathcal{P} = I\pi w_0^2 = \frac{2\pi^3m^2c^4}{c\mu_0 e^2}\left(\frac{\omega^2}{\omega_p^2}\right) = \mathcal{P}_{cr}\left(\frac{n_{cr}}{n_e}\right)$$

where $P_{cr} = 21$ GW. Hence there is a power (*not intensity*) threshold for relativistic self-focusing to occur. More rigourous calculations finds $\mathcal{P}_{cr} \simeq 17$ GW

7 Laser wakefield

Remember to derive electron plasma wave generation we start with the equation of motion, the continuity equation and Gauss's law

$$m\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -e(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$
$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = 0$$
$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0 = -e(n_e - n_i)/\epsilon_0$$

The non-linear force terms can be grouped by noting that $\mathbf{B} = \nabla\times\mathbf{A}$, and that to first order $\mathbf{u} = (e/m)\mathbf{A}$ (which is the conservation of canonical momentum). So,

$$\begin{aligned} &-m(\mathbf{u}\cdot\nabla)\mathbf{u} - e(\mathbf{u}\times\mathbf{B}) \\ &\simeq -(e^2/m)\left[(\mathbf{A}\cdot\nabla)\mathbf{A} + (\mathbf{A}\times(\nabla\times\mathbf{A})\right] \\ &= (e^2/2m)\nabla A^2 \\ &= \frac{1}{2}mc^2\nabla a^2 \end{aligned}$$

Thus the non-linear terms together combine to make the ponderomotive force. Considering variations from eqm. values, i.e. $n_e = n_0 + n_1$, $n_i = n_0$ (and noting that eqm. value of u and E are 0). So linearising the longitudinal components become,

$$m\frac{\partial u}{\partial t} = -eE + \frac{1}{2}mc^2\nabla a^2$$
$$\frac{\partial n_1}{\partial t} + n_0\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial E}{\partial x} = -en_1/\epsilon_0$$

We take the derivative wrt x for the first eqn. and wrt t for the 2nd,

$$m\frac{\partial}{\partial x}\frac{\partial u}{\partial t} = -e\frac{\partial E}{\partial x} + \frac{1}{2}mc^2\frac{\partial^2 a^2}{\partial x^2}$$
$$-\frac{1}{n_0}\frac{\partial^2 n_1}{\partial t^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial t}$$
$$\frac{\partial E}{\partial x} = -en_1/\epsilon_0$$

so that we can substitute eqn.2 and 3 for the first two terms in eqn.1:

$$\frac{\partial^2 n_1}{\partial t^2} + (e^2n_0/\epsilon_0 m)n_1 = \frac{1}{2}n_0c^2\frac{\partial^2 a^2}{\partial x^2}$$
$$\rightarrow \quad \frac{\partial^2 n_1}{\partial t^2} + \omega_p^2 n_1 = \frac{1}{2}n_0c^2\frac{\partial^2 a^2}{\partial x^2}$$

A wave equation driven by the gradient of the ponderomotive force ($\propto \partial a^2/\partial x$).

7.1 The quasistatic approximation

To solve the above equation transform to a frame in which everything looks stationary, by using the transformations of $\xi = x - ct$ and $\tau = t$. The first variable is just the phase term of the moving driver for the wakefield - assuming it is travelling at the speed c which is the case we are most interested in. In this case,

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -c\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} \simeq -c\frac{\partial}{\partial \xi}$$

$$\text{and} \quad \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}$$

where the time derivative in the first term is ignored, since time variations are assumed to be small in the frame in which the pulse is stationary (compared to the spatial motions). And so

$$\frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2} \quad \text{and} \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}$$

Using the quasistatic approximation, our wave equation becomes:

$$c^2\frac{\partial^2 n_1}{\partial \xi^2} + \omega_p^2 n_1 = \frac{1}{2}n_0 c^2 \frac{\partial^2 a^2}{\partial \xi^2}$$

$$\frac{\partial^2 n_1}{\partial \xi^2} + k_p^2 n_1 = \frac{1}{2}n_0 \frac{\partial^2 a^2}{\partial \xi^2}$$

By making use of the quasistatic form of Poisson's (Gauss's Law);

$$\frac{\partial^2 \phi}{\partial \xi^2} = -\frac{\partial E}{\partial \xi} = \frac{e}{\epsilon_0}n$$

The wave equation can also be written in terms of E and ϕ.

$$\frac{\partial^2 E}{\partial \xi^2} + k_p^2 E = -\frac{1}{2}k_p^2(mc^2/e)\frac{\partial a^2}{\partial \xi}$$

$$\frac{\partial^2 \phi}{\partial \xi^2} + k_p^2 \phi = \frac{1}{2}k_p^2(mc^2/e)a^2$$

Where in all of the 'fast' variations have been ignored by implicit use of the ponderomotive force. A second order ODE, which has a simple solution if we have a laser envelope $a(\xi) = a_0 \sin(b\xi)$ (for $0 < \xi < \pi/b$ and zero otherwise), where $1/b$ is the pulse length. The ponderomotive force then goes like $\partial(a_0^2 \sin^2(b\xi))/\partial\xi = 2ba_0^2 \cos(b\xi)\sin(b\xi) = ba_0^2 \sin(2b\xi)$, which is resonant if $2b = k_p$, and means the pulse is on for only $0 < \xi < 2\pi/k_p$, after that the ponderomotive term becomes zero, and a non-dissipating wakefield is left behind.
From the expressions above we use the one for the E-field since it contains the expression for the ponderomotive force. Hence at resonance,

$$\begin{aligned}\frac{\partial^2 E}{\partial \xi^2} + k_p^2 E &= -\frac{1}{2}k_p^2(mc^2/e)(k_p/2)a_0^2 \sin(k_p\xi) \\ &= \kappa \sin(k_p\xi)\end{aligned}$$

where κ is the constant term. The solution is (taking the plasma initially at rest),

$$E(\xi) = -\frac{\kappa\left(\sin(k_p\xi) - \cos(k_p\xi)\,k_p\xi\right)}{{k_p}^2}$$

which when the laser pulse passes ($\xi = 2\pi/k_p$) has the value,

$$E = k_p\left(\frac{mc^2}{e}\right)\frac{\pi a_0^2}{4} = \left(\frac{mc\omega_p}{e}\right)\frac{\pi a_0^2}{4} = E_0\frac{\pi a_0^2}{4}$$

A full relativistic treatment for square pulses gives $E = E_0 a_0^2/(1+a_0^2)^{1/2}$. Hence for very relativistic intensities, the E field only increases $\propto a_0$.

7.2 Wavebreaking

Note that the maximum electric field that a wakefield can support is when $n = n_0$, and from Gauss' law, then the maximum electric field is given by;

$$E_{max} = -\left(\frac{e}{\epsilon_0}\right)\int n_0 sin(k_p\xi)\,d\xi$$

$$\rightarrow \quad E_{max} = n_0 e/k_p\epsilon_0 = \left(\frac{mc\omega_p}{e}\right)$$

Hence the wakefield generated by a laser pulse of $a_0 \approx 1$ is as big as it gets. At higher intensity, the wavebreaks, charge sheets can cross, and electrons can overshoot their oscillation and are accelerated unidirectionally.

7.3 Acceleration

For a field of $E = (mc\omega_p/e)\sin(k_p x - w_p t)$, we can remove the time dependence by boosting into the stationary frame of the plasma wave ($v_\phi = \omega_p/k_p$). Remembering to include the relativistic length increase in the stationary frame $E' = (mc\omega_p/e)\sin(k_p x'/\gamma)$. The potential is then given by $\phi' = (\gamma mc\omega_p/k_p e)\cos(k_p x'/\gamma) = (\gamma mc^2/e)\cos(k_p x'/\gamma)$. Note that due to the shape of the potential, (hills and troughs) it is most likely that the particle gains energy only in a trough. So there is only energy gain in ¼ of the wave. Hence the maximum energy an electron can gain falling down this potential (from $x = 0$ to $x = \pi\gamma/2k_p$) $\rightarrow W' = [e\phi]_{x=0}^{x=\pi\gamma/2k_p} = \gamma mc^2$, and momentum is $p' = \gamma mc\beta$. But the Lorentz transform

$$W = \gamma(E' + \beta cp') = \gamma^2 mc^2(1+\beta^2)$$

which for $\beta \rightarrow 1$ gives

$$W_{max} = 2\gamma^2 mc^2$$

Since the phase velocity of a laser driven wave is equal to the laser group velocity, so $\gamma_p \approx (\omega/\omega_p)$, a sufficiently underdense plasma ($\omega \ll \omega_p$) ensures high gain.
Note also that the acceleration length in the wave frame is $L' \approx \gamma\lambda_p/4$ where we have remembered the pulse lengthening in the waves rest frame. The time for a particle of $v' \sim c$ to travel this distance is given by $t' \approx \gamma\lambda_p/4c$, which in the lab frame would be longer by γ due to time dilation, i.e. $t' \approx \gamma^2\lambda_p/4c$, which corresponds to a lab distance of $L \approx \gamma^2\lambda_p/4$. These distances are typically small due to the huge fields supported by plasmas, for example an electron can be accelerated to about 1 GeV at a density of $1 \cdot 10^{18}$ cm^{-3} in a distance of around 1 cm.

8 Solid interaction

8.1 Critical density

Consider n electrons quivering with $\gamma = 1 + a^2/2$, hence the total (density of) kinetic energy of the quivering electrons is $T = n(a^2/2)mc^2$. Equating this to the energy density in the laser field

$$(1/2)\epsilon_0 E^2 = (1/2)\epsilon_0 (mc\omega/e)^2 a^2 = na^2mc^2/2$$

$$n_{cr} = \frac{\epsilon_0 m\omega^2}{e}$$

This tells us that above this density there is not enough energy in the laser beam to keep the electrons quivering, and the field strength is quickly attenuated. This is the critical density that you will have seen derived usually from the dispersion relation. At relativistic intensities the relativistic mass dependence means that n_{cr} is given by:

$$n_{cr} = \frac{\gamma m\epsilon_0\omega^2}{e}$$

and so one might be able to reach to higher densities within a target due to relativistic effects. In any case for a typical laser ($\lambda = 1$ μm), the critical density is $\approx 1 \cdot 10^{21}$ cm^{-3}. Since solid densities are typically $\gtrsim 1 \cdot 10^{23}$ cm^{-3} a laser will always be stopped by normal solid targets.

8.2 Ionisation

Modelling ionisation of solids is more difficult than for atoms, due to the effect of the surface finish, and different resistivities of materials, but occurs at lower intensity. Metals will be ionised when the (non-relativistic) ponderomotive potential equals the work function (typically 4 eV as for Al and Cu) $\rightarrow \Phi = e^2E^2/(2m\omega^2)$, which corresponds to $I \sim 10^{13}$ Wcm^{-2}. In practise ionisation can happen at much lower intensities, sometimes below 10^{10} Wcm^{-2}. This can be a problem for investigating high intensity interactions with solid densities, since if there is any prepulse, then the high intensity laser interacts first with the underdense plasma blown off by the prepulse.

8.3 High intensity absorption

In the coronal plasma around a solid target, the absorption mechanisms present at lower intensities can still be effective. In particular resonance absorption and Raman instabilities have a greater growth rate, and so can produce hot electrons effectively. Collisional absorption though tends to be less important since at high quiver velocities, the collision frequency ($\propto v^{-4}$) drops drastically. However for sufficiently clean and short, or intense and long pulses, the interaction can almost be with a step function (of density). In the former case, this is due to the lack of time for the plasma to expand, whereas in the latter case, it is due to steepening of the density profile by the light pressure. This can lead to new absorption mechanisms.

8.3.1 Brunel acceleration

Consider a laser incident at angle on a step function density with P-polarisation (i.e. some of the laser field directed into the target). The laser can pull out an electron sheet, until the E-field of the sheet equals the normal component of the E field of the laser. i.e.

$$E_d = E_L \sin\theta$$

where θ is the angle made by the incoming beam to the normal to the target. This arrangement is like a capacitor, and so the number of electrons required to make this field is $E = \sigma/\epsilon_0$. Equating the two fields gives, $\sigma = \epsilon_0 E_L \sin\theta$. The areal density of the electrons is thus $\Sigma = \epsilon_0 E_L \sin\theta/e$. Once the field of the laser decreases, this induced field, causes the electrons to return to the target with the energy that they obtained from the quiver of the laser field $\therefore$ the return velocity is,

$$v_d = v_L \sin\theta = \frac{eE}{m\omega}\sin\theta$$

The energy absorbed into the target per laser cycle is $\Sigma m v_d^2/2$, and the rate of energy absorption (power) is

$$P_a = \Sigma m v_d^2/2\tau = \frac{\epsilon_0 E_L^3 \sin^3\theta}{4\pi}\left(\frac{e}{m\omega}\right)$$

Since the power per unit area $= I = E_L^2 \cos\theta/2\mu_0 c$, the fractional absorption is:

$$P_a/I = \frac{\sin^3\theta}{2\pi\cos\theta}\left(\frac{eE_L}{mc\omega}\right) = \frac{\sin^3\theta}{2\pi\cos\theta}a_0$$

For large angles and high intensities, this mechanism becomes appreciable (and the above expression needs to add relativistic corrections). Also reflected light can increase the maximum electric field that expels electrons.

8.3.2 J × B heating

In the preceding discussion we neglected the forward motion of the electrons, which is important at high intensity. Forward going electrons can also be injected into a target at sufficient depth to be liberated from the influence of the laser field. The $\mathbf{j} \times \mathbf{B}$ mechanism is otherwise the same as Brunel heating, and both are sometimes termed as vacuum heating (since it is hot particles from outside the surface that are ejected into the target at high velocity). Since the $\mathbf{j} \times \mathbf{B}$ force propels electrons into the target twice in any laser cycle, electron bunches spaced by $c/2\omega$ are generated. The existence of these bunches can be inferred by the detection of transition radiation when these bunches exit the rear of a thin solid target, at a characteristic frequency of 2ω (along with the ω signature of Brunel heated electrons). Unlike Brunel heating, $\mathbf{j} \times \mathbf{B}$ heating is equally effective for either S or P polarisation. Hence the lack of sensitivity on the polarisation in high intensity laser solid interaction due to the increasing effect of $\mathbf{j} \times \mathbf{B}$ heating.

In any case at high intensity, a tail of electrons is observed whose energy varies as $(I\lambda^2)^{1/2}$ not $I\lambda^2$ suggesting that though absorption is high, the energy is quickly redistributed to other electrons.

8.4 Light pressure

The pressure due to a light beam of N photons each of momentum p being absorbed on a surface is given by,

$$P_L = F/A = N\Delta p/A\Delta t = n_p cpA\Delta t/A\Delta t = n_p cp$$

where n_p is the density of the photons. But we can also write the intensity (energy per unit time per unit area), as,

$$I = n_p \hbar\omega c = n_p \hbar k = n_p p$$

Hence $P_L = I/c$. Note for reflected photons it would be twice this value. So for example for Vulcan PW, $I = 1 \cdot 10^{21}$ Wcm^{-2}, $P = 3 \cdot 10^{16}$ Pa = 300 Gbar.

8.5 Hole boring shock

We can include this light pressure in the fluid equations of motion, rewriting continuity and force equations,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$
$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial}{\partial x}(ZP_e + P_L) = 0$$

In a frame in which the shock is stationary, then the time derivatives become zero, so the equations become,

$$\rho u = \text{const.}$$
$$\rho u^2 = P_L = I/c$$

From the second equation we get directly,

$$u = (I/\rho c)^{1/2}$$

for Vulcan PW, $I = 1 \cdot 10^{21}$ Wcm^{-2}, and an Al target of $2.7 \cdot 10^3$ kgm^{-3} gives $u \approx 0.01c$ or about 3 μm/ps.
The energy of the protons associated with this shock is 60 keV. By 'bouncing' stationary ions of the shock front, it is possible to gain ion energies in excess of $2u_{shock}$, i.e. $\approx 4\times$ the energy. However this is still much smaller than the energies found in these interaction energies (protons $\sim$ 50 MeV cf next section). Of course reducing density could lead to more energetic ions.

8.6 Radiation pressure acceleration

However if whole foil is thin enough ($d < c/\omega_p$), then all electrons can be pushed forward and then foil can move as a single body;

$$\rho dA\frac{dv}{dt} = IA/c$$

Assuming constant acceleration $a = I/\rho cd$, and so $v = I\tau/\rho cd$. For VulcanPW ($I = 1 \times 10^{25}$ Wcm^{-2}) and then v can become relativistic in only $\approx$ 100's fs. However this requires that the foil is not strongly heated (and so exploding) before main pulse arrives. Circular polarisation can help turn off $j \times B$ heating.

9 Ion acceleration

The quiver velocity of ions in a laser field is given by $v_{osc}/c = ZeE/Mc\omega = (Zm/M)a_0$. Even for protons this is $\sim 2000\times$ less than for electrons, hence $v_{osc}/c \to 1$ only for intensities $I > 1 \cdot 10^{24}$ Wcm^{-2}.
However ions can be accelerated more efficiently by the space charge forces produced by energetic electrons. The hot electrons produced in high intensity interaction have a temperature on the order of the ponderomotive potential T_h. A Debye sheath forms on the back surface which has a potential $U \propto k_B T_h$. As the sheath expands, so ions can experience this field for longer, and so gain energies greater than $k_B T_h$. The sheath field can be calculated by balance with the hot electron pressure gradient,

$$n_h eE = -\frac{\partial P_h}{\partial x} = -k_B T_h \frac{\partial n_h}{\partial x}$$

So the field is $E \sim T_h/L_h$ where $L_h = n_h/(\partial n_h/\partial x)$. The continuity and force equation for the ions are

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0$$
$$\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_i}{\partial x} = \frac{Ze}{M}E$$

Putting in the E field generated by the hot electrons and assuming quasi-neutrality $n_h \simeq Zn_i$,

$$\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_i}{\partial x} = -\frac{ZT_h}{M}\frac{1}{n_h}\frac{\partial n_h}{\partial x} = -c_s^2\frac{1}{n_i}\frac{\partial n_i}{\partial x}$$

where we denote the ion expansion speed as $c_s = (ZT_h/M)^{1/2}$. There exists a self-similar solution,

$$v_i = c_s + x/t$$
$$n_i = n_0 \exp(-x/c_s t)$$

The solution does has a set-back of being infinite in extent. This can be resolved by truncating the density profile at the point that the density scale length equals the local Debye length, (where the sheath prevents further electron escape).

$$L_n = \left|\frac{n_i}{\partial n_i/\partial x}\right| = c_s t_f = \lambda_d(x_f) \qquad \text{where}$$
$$\lambda_d(x_f) \equiv \left(\frac{\epsilon_0 k_B T}{Zne^2}\right)^{1/2} = \frac{c_s}{\omega_{pi0}}\left(\frac{n_0}{n_f(x_f)}\right)^{1/2}$$

Using the self-similar solutions,

$$\begin{aligned} v_{max} &= c_s\left(1+\ln\left(\frac{n_0}{n_f(x_f)}\right)\right) \\ &= c_s\left(1+\ln\left(\omega_{pi0}\frac{\lambda_D(x_f)}{c_s}\right)^2\right) \\ &= c_s[1+2\ln(\omega_{pi0}t_f)] = \alpha c_s \end{aligned}$$

where α is thus a multiplier accounting for the expansion of the front. The energy of the ions is given by $\frac{1}{2}Mv_{max}^2$ (since these energies are still non-relativistic for ions). So,

$$W_{max} = \alpha^2 k_B T/2$$

Typically α is only of the order of a few, and so the energies are on the order of T_{hot}

Note that energetic ions can be found on both the rear as well as the front surface of solid targets. This is because the hot electrons generated on the front surface of a high intensity interaction will be transported (almost collisionlessly) to the back surface, where the same sheath expansion occurs. Indeed the ions from the rear surface can actually have better qualities, since the extra distance can allow smoothing of the hot electron flux from the rear surface (resulting in a more uniform acceleration). These ion beams, being quasineutral, have very high quality compared to other sources of energetic ions (though large energy spread usually).

10 Electron transport

If one assumes that all of the absorbed intensity from a short pulse laser is converted into a beam of energetic electrons travelling into the target, then one can calculate the current due to this beam. Unfortunately the magnetic energy of the beam would be several orders of magnitude greater than the energy in the electron beam itself. Hence one cannot transport an unneutralised current in a solid. Instead a current from the cold background must oppose the hot current of electrons. Though the hot electrons are not particularly collisional, the return current is, and this can severely restrict the energy transport of hot electrons into a solid target.

The transport of electrons is given by $n_h = A\exp(\phi/k_bT)$ where ϕ is the inhibiting potential. So $E = -T_h(\nabla n_h/n_h)$. Using the continuity equation and noting that the hot electron current is determined by the cold resisitivity $j_c = -j_h = \sigma E$. This gives a non-linear diffusion equation,

$$\frac{\partial n_h}{\partial t} + \nabla\cdot\left[\frac{\sigma T_h}{e n_h}\right]\nabla n_h = 0$$

which has a solution,

$$n_h = n_{h0}\frac{t}{\tau_L}\left(\frac{x}{x+R_h}\right)^2$$

where the constants were given as,

$$n_{h0} = I_{18}^2\frac{\tau}{100\ \mathrm{fs}}\left(\frac{T_h}{100\ \mathrm{keV}}\right)^{-3}\left(\frac{\sigma}{10^6\ \Omega\mathrm{m}^{-1}}\right)^{-1} \times\ 1.4\cdot 10^{23}\ \mathrm{cm}^{-3}$$

$$R_h = \left(\frac{T_h}{100\ \mathrm{keV}}\right)^2\left(\frac{\sigma}{10^6\ \Omega\mathrm{m}^{-1}}\right) I_{18}^{-1}\ 3\ \mu\mathrm{m}$$

which means even for modest intensities e.g. $I_{18} = 1$, $T_h = 100$ keV, the inhibition distance is $\approx 3\ \mu$m. Normal collisional stopping distance for the same energy $> 70\ \mu$m, hence the importance of collective stopping.

11 Magnetic fields

Generation of magnetic fields in blow-off plasmas has been observed for a long time, and is generally due to the electrothermal term. i.e. for a quasistatic plasma

$$eE = \nabla P = \nabla n k_b T$$

$$\nabla\times\mathbf{E} = \frac{\nabla\times\nabla n k_b T}{ne}$$

$$\frac{\partial B}{\partial t} = \frac{\nabla n\times\nabla k_b T}{ne}$$

Fields as large as 30 MG have been seen in the blow-off plasma from high intensity laser matter interaction. However the very large currents produced at the surface of a high intensity laser solid interaction can produce even larger magnetic fields. For electrons which are ponderomotively accelerated into a solid target (by $\mathbf{j}\times\mathbf{B}\propto\nabla I$ heating). So $j = nev_p \propto n\nabla I$, and so using Ampere's law, $\nabla^2 B\propto\nabla n\times\nabla I$, which is in the opposite direction to the thermoelectric magnetic field. These fields can be huge, and fields in excess of 700 MG have been observed by cut-off of harmonics generated at the surface of the laser-solid interaction.