Heterotic String and Non-Kähler Calabi-Yau Manifolds

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The interplay between string theory and mathematics has brought significant advances in various branches of both mathematics and physics. Mirror symmetry is one of the most influential areas.

Today I will be speaking about a more recent developing area of string math collaboration. It is the study of non-Kähler manifolds with trivial canonical bundles, called non-Kähler Calabi-Yaus.

For string theory, non-Kähler Calabi-Yaus play an important role as they appear in supersymmetric flux compactifications. But let me begin by telling you why mathematicians were interested in non-Kähler Calabi-Yaus prior to string theory.

Non-Kähler Calabi-Yaus

After plenty of examples of projective Calabi-Yau manifolds were constructed, one of the most interesting problems is to understand its moduli space.

Unlike $K3$ surface, Calabi-Yau manifolds of dimension ≥ 3 have different homotopy types. So its moduli space is not connected. In 1984, I made the conjecture that there are only finitely many topological types of smooth projective Calabi-Yau manifolds for each dimension. (See e.g., my survey paper: *A review of complex differential geometry.* in Proc. Sympos. Pure Math., 52, Part 2.)

In addition, Calabi-Yaus on different components are not unrelated as seen from the following construction of Clemens and Friedman.

Suppose Y is a Calabi-Yau threefold containing a collection of mutually disjoint smooth rational curves. Assume that they have normal bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We can contract these rational curves and obtain a *singular* Calabi-Yau threefold X_0 with ordinary double-point singularities.

R. Friedman gave a condition to deform X_0 into a smooth complex manifold X_t with trivial canonical bundle. It is the compact version of local conifold transition which physicists are familiar with

$$Y \dashrightarrow X_0 \dashrightarrow X_t$$

The resulting Calabi-Yau X_t is in general non-Kähler.

To see this, we take a Calabi-Yau threefold Y with smooth rational curves C_i so that $\{C_i\}$ generate $H^{2,2}(Y)$. For instance, let Y be a quintic threefold in $\mathbb{P}^4$.

By contracting these rational curves, $H^{2,2}(Y)$ is killed. Then after smoothing, we end up with a non-Kähler complex manifold which is diffeomorphic to a k-connected sum of $S^3 \times S^3$, with $k \geq 2$. In this way, one can construct non-Kähler Calabi-Yaus of topological type $\#_k S^3 \times S^3$ for arbitrarily large k. Therefore, if we drop the Kähler condition, we get infinitely many families of non-Kähler Calabi-Yaus.

On topology of non-Kähler Calabi-Yaus : More generally, for a manifold $M_g = \#_g(S^n \times S^n)$, denote its group of orientation preserving C^∞-diffeomorphisms by $\mathrm{Diff}^\delta_+(M_g)$, when equipped with the discrete topology.

Conjecture (Morita '05)

For a fixed i, $H^i(\mathrm{BDiff}^\delta_+(M_g), \mathbb{Z})$ is stable in a range of values of the 'genus' g. In known cases, this means large enough g.

Viewed as a discrete group, $\mathrm{BDiff}^\delta_+(M_g)$ is an Eilenberg-MacLane space, in fact it is $K(MCG_g, 1)$ where MCG_g is the mapping class group of M_g. And there is a natural isomorphism:

$$H^*(\mathrm{BDiff}^\delta_+(M_g), \mathbb{Z}) = H^*(MCG_g, \mathbb{Z}).$$

Take $n = 3$ for threefolds. By example, we know the value g is unbounded for non-Kähler Calabi-Yaus.

Morita '05 is consistent with a corollary that follows from the theorem of Fu-Li-Yau that we shall prove shortly.

We conjecture that:
assuming that Morita '05 is correct, this cohomology should play a role that is universal to all three dimensional Calabi-Yau manifolds and presumably appears in conformal field theory as well.

hints: In the case of curves Σ_g or $\#_g(S^1 \times S^1)$, the above cohomology classes are related to Chern classes of the Hodge bundle over $\mathcal{M}_g$ defined by Mumford (1983). (Computable by Kodaira-Spencer theory.)

also: both the Siegel modular group $sp(2g, \mathbb{Z})$, and the Torelli group $\mathcal{I}_g$ which acts trivially on $H_3(M, \mathbb{Z})$, are $\subset MCG_g$. In fact, there is $0 \longrightarrow \mathcal{I}_g \longrightarrow MCG_g \longrightarrow sp(2g, \mathbb{Z}) \longrightarrow 0$.

In 1987, M. Reid put forth an interesting proposal, called Reid's fantasy. He speculated that all projective Calabi-Yau threefolds fit into a single universal moduli space in which families of smooth Calabi-Yaus of different homotopy types are connected to one another by the Clemens-Friedman conifold transitions that I just described.

Now to test this proposal, understanding non-Kähler Calabi-Yau manifolds becomes essential. The first question one can ask is: what constraint one should put on hermitian metrics on such non-Kähler Calabi-Yau manifolds, so that these metrics reflect their geometric and topological structures.

Balanced Metrics

In 1982, Michelsohn considered a generalization of Kähler metrics in the following way. Recall that a hermitian metric ω is Kähler if

$$d\omega = 0.$$

This is equivalent to the vanishing of torsion tensor.

Michelsohn analyzed a weaker condition, called balanced, which is the vanishing of the *trace* of the torsion tensor. For threefolds, it is equivalent to

$$d(\omega \wedge \omega) = 2\,\omega \wedge d\omega = 0 \qquad (\textit{balanced})\ .$$

Every Kähler metric is clearly balanced.

Balanced condition has good properties. It is preserved under proper holomorphic submersions and also under birational transformations (Alessandrini-Bassanelli).

Examples of non-Kähler compact balanced manifolds include:

- T^2-bundles over Kähler manifolds constructed by Goldstein and Prokushkin;
- Natural metrics on twistor spaces of self-dual compact four manifolds;
- Moishezon spaces.

With Jixiang Fu and Jun Li (2008), we proved the following

Theorem (Fu-Li-Yau)

Let Y be a smooth Kähler Calabi-Yau threefold and let $Y \to X_0$ be a contraction of mutually disjoint rational curves. Suppose X_0 can be smoothed to a family of complex manifolds X_t. Then for sufficiently small t, X_t admit smooth balanced metrics.

Our construction provides balanced metrics on a large class of threefolds. In particular,

Corollary

There exists a balanced metric on $\#_k(S^3 \times S^3)$ for any $k \geq 2$.

To really understand Reid's proposal for Calabi-Yau moduli space, it is important to define some *canonical* balanced metric which would satisfy an additional condition, like the Ricci-flatness condition for the Kähler Calabi-Yau case.

We would like to have a natural condition, and string theory gives some suggestions. Physicists have been interested in non-Kähler manifolds in the context of compactifications with fluxes and model building.

For heterotic string, the conditions for preserving $N = 1$ supersymmetry with H-fluxes was written down by Strominger in 1986. Strominger's system of equations specifies the geometry of a complex threefold M (with a holomorphic three-form Ω) and in addition a holomorphic vector bundle V over M.

Strominger's System

The hermitian metric ω on the manifold M and the metric h on the bundle V satisfy the system of differential equations:

(1) $d(\| \Omega \|_\omega \, \omega \wedge \omega) = 0;$

(2) $F_h^{2,0} = F_h^{0,2} = 0, \quad F_h \wedge \omega^2 = 0;$

(3) $\sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}\left[\mathrm{tr}\big(R_\omega \wedge R_\omega\big) - \mathrm{tr}\big(F_h \wedge F_h\big)\right].$

The first equation is equivalent to the existence of a (conformally) balanced metric. The second is the Hermitian-Yang-Mills equations which is equivalent to V being a poly-stable bundle. The third equation is the anomaly equation.

When M is a Kähler manifold and V is the tangent bundle T_M, the system is solved with $h = \omega$, the Kähler Calabi-Yau metric.

Using a perturbation method, Jun Li and I have constructed smooth solutions on a class of Kähler Calabi-Yau manifolds with *irreducible* solutions for vector bundles with gauge group $SU(4)$ and $SU(5)$.

Andreas and Garcia-Fernandez have generalized our construction on Kähler Calabi-Yau manifolds for any stable bundle V satisfying $c_2(V) = c_2(M)$.

Jixiang Fu and I constructed solutions of Strominger's system on a class of non-Kähler Calabi-Yau threefolds.

These manifolds are T^2-bundles over $K3$ surfaces constructed by Goldstein and Prokushkin. As I mentioned earlier, they admit a balanced metric. In addition, the following metric

$$\omega_u = e^u \omega_{K3} + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}$$

satisfy the conformally balanced equation in the Strominger's system. Here u is any function on $K3$ surface, θ is the connect form on the T^2-bundle. Similar ansatz were also considered by Dasgupta-Rajesh-Sethi and Becker-Becker-Dasgupta-Green earlier.

Now it is clear that the first two equations in Strominger's system can be solved. The last one which couples the two metrics is the most demanding.

Fu and I analyzed carefully the anomaly equation in this case and reduced it to the following Monge-Ampère equation:

$$\triangle(e^u - \frac{\alpha'}{2} f e^{-u}) + 4\alpha' \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0,$$

where f and μ are functions on $K3$ surface satisfying $f \geq 0$ and $\int_S \mu\omega_{K3}^2 = 0$.

We obtained some crucial a priori estimates up to third order in derivatives and then used the continuity method to solve the equation.

As a conclusion, we proved the following existence theorem:

Theorem (Fu-Yau)

Let S be a $K3$ surface with Calabi-Yau metric ω_S. Let ω_1 and ω_2 be anti-self-dual (1,1)-forms on S such that $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^2(S, \mathbb{Z})$. Let M be a T^2-bundle over S constructed (twisted) by ω_1 and ω_2. Let E be a stable bundle over S with the gauge group $SU(r)$. Suppose ω_1, ω_2 and $c_2(E)$ satisfy the topological constraint

$$\alpha'(24 - c_2(E)) = -\left(Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right)\right).$$

Then there exists a smooth function u on $K3$ surface and a hermitian-Yang-Mills metric h on E such that $(M, \pi^* E, \omega_u, \pi^* h)$ is a solution of Strominger's system.

Recently, Teng Fei and I found symmetric solutions to the Strominger system on $SL(2, \mathbb{C})$ (the smoothed conifold) with either trivial or non-trivial F_h term. Potentially, they may serve as local models in understanding Strominger system under conifold transitions. In our work, more general hermitian connections (Strominger-Bismut connection for instance) other than the Chern connection on tangent bundle is used to compute the curvature R_ω.

As is clear in heterotic string theory, understanding stable bundles on Calabi-Yau threefolds is important.

Donagi, Pantev, Bouchard and others have done nice works on constructing stable bundles on Kähler Calabi-Yau threefolds to obtain realistic heterotic models of nature.

Andreas and Curio have done analysis on the Chern classes of stable bundles on Calabi-Yau threefolds, verifying in a number of cases a proposal of Douglas-Reinbacher-Yau. Recently, Baosen Wu and I used different construction of stable bundles to obtain refined results.

Stability and DRY

Stability is an important concept in both mathematics and physics.

In the heterotic string, Hermitian-Yang-Mills equation is equivalent to the conformal invariance of the $(0, 2)$-nonlinear σ-model. (as Ricci-flow governs that of the $(2, 2)$ σ-model.)

Often the discussion of stability is accompanied by a discussion of the moduli space of nice objects.

In the heterotic string, we are interested in holomorphic vector bundles. Many of these are isolated points of moduli space of stable sheaves.
Hence, they are special points, 'stuck' in some sense.

In physics, there are also objects stuck at special points of moduli space. BPS blackholes 'fix' the moduli of Calabi-Yau threefolds at the horizon to attractor points, via an attractor flow.

In (DRY 2006), we use this observation to motivate new inequalities for Chern-classes of vector bundles on Calabi-Yau threefolds. These are not obvious from mathematics.

More concretely, we use the 'mirror' attractor equation:

$$2\mathrm{Re}(\overline{C}\hat{\Omega}) = \vec{v}(E)$$

Here $\hat{\Omega} = e^{B+i\omega} \in H^{2*}(X)$ and BPS charge is given by Mukai vector $\vec{v}(E) = \mathrm{Tr}(e^{F(\nabla_E)}\sqrt{\mathrm{Td}(T_M)})$. The constant $\overline{C}$ is an 'integration constant' for the flow.

We can always twist the bundles so that $c_1(E) = 0$. Then the 'mirror' attractor equations can be solved up to a constant parameter.

Crucially, we introduce an ample class H (a divisor class).

$$H^2 = \frac{1}{r}\left(c_2(E) - \frac{r}{24}c_2(T_M)\right)$$

The DRY inequality is a consequence of the Hodge-type inequality for ample classes (for M irreducible and complete), curve classes from decomposing H^2.

For example, for not necessarily distinct ample classes H_i, $i = 1, 2, 3$, we have $(H_1 H_2 H_3)^3 \geq H_1^3 H_2^3 H_3^3$

Then taking $H_1 = H_2 = H$ and an arbitrary ample class $H_3 = J$,

$$|c_3(E)| \leq (H)^3 \Longrightarrow (c_3(E))^2 \sim (H^3)^2(J)^3 < (H^2 \cdot J)^3$$

To undo the twist on E, in the definition of the ample class H just replace $c_2(E)$ with the 'discriminant': $\Delta = \frac{1}{2r}(2rc_2 - (r-1)c_1^2)$. By the usual inequality $\Delta \cdot \omega^{n-2} > 0$ for stable bundle.

Finally, the DRY inequality can be stated for the above data as:

Conjecture (Douglas-Reinbacher-Yau)

If the stated data satisfy: $\left| c_1^3 + 3r(r\, ch_3 - ch_2\, c_1)(E) \right| < r^3 H^3$, *then there exists a reflexive sheaf with the specified integral classes as Chern classes, stable with respect to some ample class* J. *Especially, there exist such stable sheaves for any given* J.

The Hodge type inequality allows us to remove the class H from the above inequality, obtaining an inequality for the Chern classes of E alone, which involves the polarization J.

Bundle Chern Classes

Some results on the Chern classes of both existing bundle constructions and some new constructions for bundles were reported in joint work with Peng Gao and Yang-Hui He (arXiv:1403.1268).

Using a generalized Hartshorne-Serre construction, we can construct higher rank ($r \geq 3$) bundles (reflexive sheaves) whose second Chern class equals $c_2(T_M)$. Satisfying anomaly condition automatically.

For this to work, $c_2(T_M)$'s dual class needs to be an effective curve class [C]. Satisfied for elliptically fibered CY 3-folds, but not generally since Calabi-Yau 3-folds are not uniruled.

This bundle is a deformation of the semi-stable sheaf $\mathcal{O}_M^{\oplus \mathrm{r}-1} \oplus \mathfrak{I}_C$. Similar in construction to Fu-Li-Yau. Stability can be proved using the recent work of Wu-Yau.

By construction, $c_1(E) = 0$, and $c_2(E) = [C]$. The third Chern class is allowed to vary, as is the genus of a space curve in $\mathbb{P}^3$. In the simplest construction, $c_3(E) = 2g - 2$.

In comparison, the spectral cover construction does not give bundles with $c_2(E) = c_2(T_M)$. Our construction satisfies this constraint, but often yields reflexive sheaves. This depends very much on the curve C.

If $c_2(E) = c_2(T_M)$ is relaxed, multiple curve classes can be used to refine this construction. Also we have constructed a similar class of bundles for non-Kähler Calabi-Yaus with Wu (later in this talk).

Our interest in the Chern classes were partly motivated by the Douglas-Reinbacher-Yau (DRY) proposal.

Using existing bundles (Monads, spectral cover, polystable etc.) data, we find their Chern classes satisfy the DRY inequalities. Said differently, a stable bundle exists whenever DRY claims it should based on Chern classes numerical relations. This is a nontrivial check, but limited in rank of E (focus on $r = 3, 4, 5$).

Desirable to understand relation with similar inequalities motivated by Bridgeland stability. There are additional structures in type II string, compared to Heterotic. But we can invoke duality. This is on going work with P. Gao.

Interesting numerical patterns emerge (for $r = 3, 4, 5$) from the data of spectral cover bundles on elliptic fibered Calabi-Yau threefolds.
We graph the Chern numbers $(|c_2(T_M) - c_2(E)|, c_3(E))$ for spectral cover bundles.

Vertical axis is $c_3(E)$.
Horizontal axis is $|c_2(T_M) - c_2(E)| = \sqrt{a_F^2 + W_B \cdot W_B} > 0$, where the integer a_F and class W_B are defined w.r.t. elliptic fibration as follows, $c_2(TX) - c_2(V) = (W_B, a_F)$.

$W_B := 12c_1(T_B) - \eta$, $a_F := c_2(T_B) + 11c_1(T_B)^2 - c_F(F \cdot \sigma)$.
Here F and σ are respectively the fiber class and the zero-section.
If the spectral cover divisor is of degree n over B, for E to be stable, it is required that $\eta - nc_1(T_B)$ is effective curve in B and the linear system $|\eta|$ is base point free.

For the graph, a total 42352 points are used, each corresponding to a distinct spectral cover bundle.

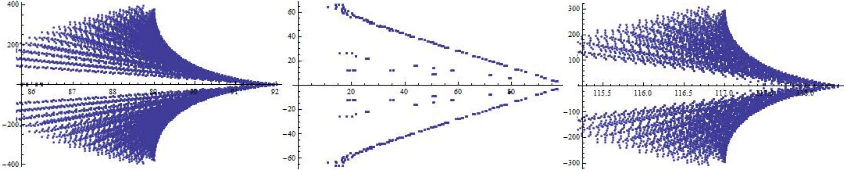

Figure: Rank $r = 3, 4, 5$ respectively.

Base are blow-ups of Hirzebruch surfaces, $B = \widehat{\mathbb{F}}_{r=0,\dots,3}$, **data all mixed together**. Regularity of the pattern suggests these bundles possibly form a web connected by extremal-like transitions involving both the bundle E and the CY threefold M. It is desirable to further investigate this class of bundles.

New Bundles for String Compactification

In string compactification, the number of "generations" of particles is $\frac{1}{2}|\chi|$, where χ is the Euler number of the Calabi-Yau threefold. As 3 generations are observed in nature, we were particularly interested in searching for Calabi-Yau threefolds with $\chi = 6$.

In 1985, I constructed the first explicit example of projective Calabi-Yau threefold with Euler number $|\chi| = 6$. (Symposium on anomalies, geometry, topology, 395–406, World Sci., Singapore, 1985)

Now I describe a new construction of Calabi-Yau threefolds M together with $SU(5)$ stable bundles V satisfying constraints:

(1) $c_1(V) = 0$;

(2) $c_2(V) = c_2(M)$;

(3) $c_3(V) = 6$.

The manifold M is constructed as a $K3$-fibered Calabi-Yau, Kähler or non-Kähler. This pair of Calabi-Yau and stable bundle satisfies three generation requirement in a realistic Heterotic superstring compactification model. Apparently this is the first example of such bundles which are not related to the tangent bundle of the Calabi-Yau manifold in an obvious way.

We construct Calabi-Yau threefolds M as double covers of twistor spaces. For this, we recall several basic facts on twistor spaces.

For a self-dual four-manifold M^4, there is a twistor space $\mathrm{Tw}(M^4)$. It is an S^2-fibration over M^4 with a natural complex structure. In the compact case, the twistor space is Kähler if and only if M^4 is a standard sphere or $\mathbb{P}^2$ with Fubini-Study metric.

When $M^4 = n\mathbb{P}^2$, the connected sum of n copies of $\mathbb{P}^2$s, Lebrun wrote down an explicit conformal structure. The corresponding twistor space $\mathrm{Tw}(n\mathbb{P}^2)$ is Moishezon. For $n > 4$ and a general self-dual structure on $n\mathbb{P}^2$, the associated twistor space has algebraic dimension 0.

To construct M, we choose two smooth anti-canonical divisors X_1 and X_2 in $\mathrm{Tw}(n\mathbb{P}^2)$ intersecting along a smooth curve C. The double cover of $\mathrm{Tw}(n\mathbb{P}^2)$ branched along the union $X_1 \cup X_2$ is a singular Calabi-Yau threefolds. A crepant resolution gives a smooth Calabi-Yau threefold which is non-Kähler when n is large.

It is convenient to use an equivalent point of view. We first blowup the curve C to get a $K3$-fibration over $\mathbb{P}^1$. Then we take double cover branched over two disjoint smooth fibers. We arrive at the same manifold constructed above. It is a $K3$-fibered Calabi-Yau threefold. (See figures below for the example of $\mathbb{P}^3 = \mathrm{Tw}(S^4)$.)

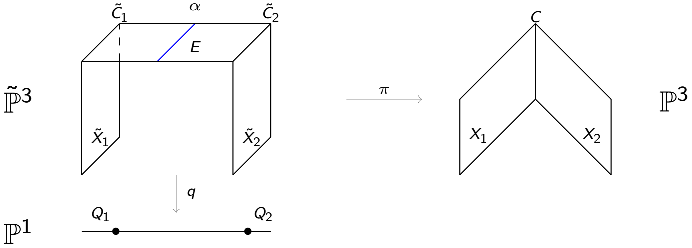

Figure: Blowingup along C

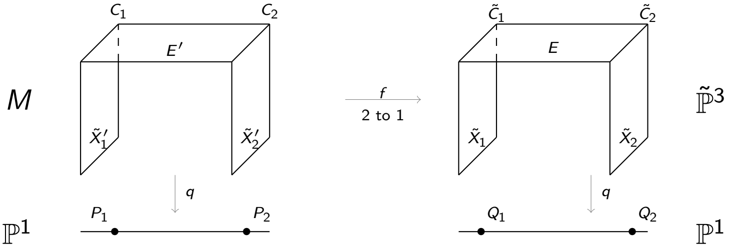

Figure: Double cover along fibers

From the construction of M

$$M \xrightarrow{f} \widetilde{\mathrm{Tw}(n\mathbb{P}^2)} \xrightarrow{\pi} \mathrm{Tw}(n\mathbb{P}^2)$$

we can write down an explicit balanced metric on M.

For the natural balanced metric ω_0 on $\mathrm{Tw}(n\mathbb{P}^2)$, we can find positive $(1,1)$-form ω on M so that

$$\omega^2 = C \cdot (f^*\pi^*\omega_0^2) + c_1(L,h)^2$$

for a sufficiently large constant $C > 0$. Here L is a suitable line bundle over M with a hermitian metric h which is positive on the ramification divisor of $\pi \circ f : M \to \mathrm{Tw}(n\mathbb{P}^2)$.

Now we describe the construction of stable bundle V over M. To illustrate, we consider the simplest case of Calabi-Yau constructed above for $n = 0$, i.e., M is a smooth model of the double cover of $\mathbb{P}^3 = \mathrm{Tw}(S^4)$.

We compute the Chern classes of M as follows:

1. $c_1(M) = 0$;
2. $c_2(M) = f^*\pi^*\ell + f^*\alpha$;
3. $c_3(M) = -168$.

Here ℓ is the line class of $\mathbb{P}^3$, α is the curves class in $\tilde{\mathbb{P}}^3$ of proper transform of C under blowup. $\alpha = [\tilde{C}_1] = [\tilde{C}_2]$. (See figure.)

We first construct a polystable V_0 as a direct sum

$$V_0 = V_2 \oplus V_3$$

so that V_2 is a rank 2 stable bundle with

1. $\wedge^2 V_2 \cong \mathcal{O}_M$;
2. $c_2(V_2) = 6 f^* \pi^* \ell$;
3. $c_3(V_2) = 0$,

and V_3 is a rank 3 stable bundle with

1. $\wedge^3 V_3 \cong \mathcal{O}_M$;
2. $c_2(V_3) = f^* \alpha$;
3. $c_3(V_3) = 6$.

We define V_2 as the pull back of an instanton bundle over $\mathbb{P}^3$ with $c_2 = 6\ell$.

Recall that for any collection of mutually disjoint $k+1$ lines in $\mathbb{P}^3$, there is a rank 2 stable bundle $V_2(k)$ fitting into

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow V_2(k) \longrightarrow \mathcal{I}(1) \longrightarrow 0,$$

where $\mathcal{I}$ is the ideal sheaf of the union of these $k+1$ lines.

It is easy to verify that $c_1(V_2(k)) = 0$ and $c_2(V_2(k)) = k\ell$. In fact, $V_2(k)$ correspond to the 't Hooft instantons on S^4 by twistor construction.

For V_3, we need to use the following construction:

Theorem (Wu-Yau)

Let $M \to \mathbb{P}^1$ be a $K3$-fibered Calabi-Yau threefold. Let $\{Y_i\}$ be disjoint reduced and irreducible curves in distinct fibers of M. Suppose $g(Y_i) \geq 1$. Then there exists a rank 3 stable bundle W over M with

1. $\wedge^3 W \cong \mathcal{O}_M$;
2. $c_2(W) = \sum [Y_i]$;
3. $c_3(W) = \sum (2g(Y_i) - 2)$.

To apply this theorem to our situation, we need to find curves $\{Y_i\} \subset M$ satisfying conditions

1. $\sum [Y_i] = f^*\alpha$;
2. $\sum (2g(Y_i) - 2) = 6$.

This is achieved by applying the following theorem of Mori:

Theorem (Mori)

There exists a non-singular curve of degree $d > 0$ and genus $g \geq 0$ on a non-singular quartic surface in $\mathbb{P}^3$ if and only if (1) $g = \frac{d^2}{8} + 1$ or (2) $g < \frac{d^2}{8}$ and $(d, g) \neq (5, 3)$.

Back to the construction of Calabi-Yau M, we choose particular $K3$ surfaces X_1 and X_2 in $\mathbb{P}^3$ according to this theorem.

Explicitly, the pairs $(d, g) = (4, 1)$ and $(d, g) = (8, 4)$ satisfy the assumption therein, there exists X_1 and X_2 which contains curves Y_1' and Y_2' of such pairs of degree and genus respectively.

Notice that $\mathrm{Pic}\,\mathbb{P}^3 \cong \mathbb{Z}$, f is a double cover, and α is the proper transform of C, a degree 16 curve in $\mathbb{P}^3$. We let Y_1 be the proper transform of $6Y_1'$ and Y_2 be the proper transform of Y_2'. Then $Y_1 \cup Y_2$ satisfies the requirement.

With this polystable bundle V_0, we apply some sophisticated deformation theory of vector bundles to deduce that it can be deformed to a stable bundle V.

It is easy to see from the argument that we can also construct stable bundles with other values of c_3, say $c_3 = 12$ for instance. In that case, we hope to find a free $\mathbb{Z}_2$ action on certain M so that the resulting quotient is a Calabi-Yau with fundamental group $\mathbb{Z}_2$, and with a stable bundle with $c_3 = 6$. By choosing a Wilson line, we can break down to the standard model gauge group.

Having talked about complex balanced manifolds and Strominger's system, now we ask the following question: is there a mirror dual of a complex balanced manifold in string theory that is symplectic and generally non-Kähler?

Such a symplectic mirror will not be found in heterotic string. All supersymmetric solutions satisfy the Strominger system in heterotic string. So the mirror dual of a complex balanced manifold with bundle should be another complex balanced manifold with bundle.

It turns out the answer can be found in type II string theories. As I will describe shortly, the equations for non-Kähler Calabi-Yau in type II string also give new insights into the natural cohomologies on non-Kähler manifolds.

Type II Strings: Non-Kähler Calabi-Yau Mirrors

In type II string theory, supersymmetric compactifications preserving a $SU(3)$ structure have been studied by many people in the last ten years. Since we are interested in non-Kähler geometries of compact manifolds, any supersymmetric solution will have orientifold sources. The type of sources help determine the type of non-Kähler manifolds. I will describe the supersymmetric equations written in a form very similar to that in Grana-Minasian-Petrini-Tomasiello (2005) and Tomasiello (2007). My description below is from joint work with Li-Sheng Tseng.

Complex Balanced Geometry in Type IIB

The supersymmetric equations that involve complex balanced threefolds is found in type IIB theory in the presence of orientifold 5-branes (and possibly also D5-branes). These branes are wrapped over holomorphic curves. In this case, the conditions on the hermitian $(1,1)$-form ω and $(3,0)$-form Ω can be written as

$$d\Omega = 0 \qquad (\textit{complex integrability})$$

$$d(\omega \wedge \omega) = 0 \qquad (\textit{balanced})$$

$$2i\,\partial\bar{\partial}(e^f \omega) = \rho_B \qquad (\textit{source})$$

where ρ_B is the sum of Poincaré dual currents of the holomorphic curves that the five-brane sources wrap around, and f is a distribution that satisfies

$$i\,\Omega \wedge \bar{\Omega} = 8\,e^{-f}\,\omega^3/3!\ .$$

The balanced and source equations are interesting in that they look somewhat similar to the Maxwell equations. If one notes that $*\omega = \omega^2/2$ (where the $*$ is with respect to the compatible hermitian metric), then the equations can be expressed up to a conformal factor as

$$d(\omega^2/2) = 0$$

$$2i\,\partial\bar{\partial} * (\omega^2/2) = \rho_B$$

Now this is somewhat expected as the five-brane sources are associated with a three-form field strength F_3 which is hidden in the source equation. These two equations however do tell us something more.

Recall the Maxwell case. The equations in four-dimensions are

$$d\,F_2 = 0\,,$$
$$d*\,F_2 = \rho_e\,,$$

where ρ_e is the Poincaré dual current of some electric charge configuration.

Now, if we consider the deformation $F_2 \to F_2 + \delta F_2$ with the source fixed, that is $\delta\rho_e = 0$, this leads to

$$d(\delta F_2) = d*(\delta F_2) = 0\ ,$$

which is the harmonic condition for a degree two form in de Rham cohomology. So clearly, the de Rham cohomology is naturally associated with Maxwell's equations.

For type IIB complex balanced equations, we can also deform $\omega^2 \to \omega^2 + \delta\omega^2$. Now if we impose that the source currents and the conformal factor remains fixed, then we have the conditions

$$d(\delta\omega^2) = \partial\bar{\partial} * (\delta\omega^2) = 0 \ ,$$

which turn out to be the harmonic condition for a (2,2)-element of the Bott-Chern cohomology:

$$H^{p,q}_{BC} = \frac{\ker d \cap \mathcal{A}^{p,q}}{\mathrm{im}\ \partial\bar{\partial} \cap \mathcal{A}^{p,q}} \ .$$

This cohomology was introduced by Bott-Chern and Aeppli in the mid-1960s.

The string equations thus points to the Bott-Chern cohomology as the natural one to use for studying complex balanced manifolds. Note when the manifold is Kähler, the $\partial\bar{\partial}$-lemma holds and the Bott-Chern and Dolbeault cohomology are in fact isomorphic.

Symplectic Mirror Dual Equations in Type IIA

The mirror dual to the complex balanced manifold is found in type IIA string. Roughly, the type IIA equations can be obtained from the IIB equations, by first replacing $\omega^2/2$ with $(\mathrm{Re}\, e^{i\,\omega})$ and then exchanging $e^{i\,\omega}$ with Ω.

$$d(\omega^2/2) = 0 \quad \Leftrightarrow \quad d(\mathrm{Re}\, e^{i\,\omega}) = 0 \qquad \longleftrightarrow \qquad d\, \mathrm{Re}\ \Omega = 0$$

Thus, $d\, \mathrm{Re}\, \Omega = 0$ is the condition that is suggested by string for symplectic conifold transition.

This condition is part of the type IIA supersymmetric conditions in the presence of orientifold (and D-) six-branes wrapping special Lagrangian submanifolds:

The type IIA equations that are mirrored to the IIB complex balanced system are

$$d\omega = 0\,, \qquad (\textit{symplectic})$$
$$d\,\mathrm{Re}\;\Omega = 0\,, \qquad (\textit{almost complex})$$
$$\partial_+\partial_- * (e^{-f}\,\mathrm{Re}\,\Omega) = \rho_A\,, \qquad (\textit{source})$$

where ρ_A is the Poincaré dual of the wrapped special Lagrangian submanifolds. ∂_+ and ∂_- are linear symplectic operators that can be thought of as the symplectic analogues of the Dolbeault operators, ∂ and $\bar{\partial}$. Tseng and I introduced them recently, so let me describe them a little bit more.

(∂_+, ∂_-) appear from a symplectic decomposition of the exterior derivative

$$d = \partial_+ + \omega \wedge \partial_- \ .$$

∂_+ raises the degree of a differential form by one, and ∂_- lowers the degree by one. They are defined with the property

$$\partial_\pm : \mathcal{P}^k \to \mathcal{P}^{k\pm 1} \ ,$$

where $\mathcal{P}^k$ is the space of primitive k-form. (A primitive form is one that vanishes after being contracted with ω^{-1}.) And like their complex counterparts,

$$(\partial_+)^2 = (\partial_-)^2 = 0 \ ,$$

and effectively, they also anticommute with each other.

With the linear symplectic operators, (∂_+, ∂_-), we can write down an interesting elliptic complex.

Proposition (Tseng-Yau)

On a symplectic manifold of dimension $d = 2n$, the following differential complex is elliptic.

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\partial_+} & \mathcal{P}^0 & \xrightarrow{\partial_+} & \dots & \xrightarrow{\partial_+} & \mathcal{P}^{n-1} & \xrightarrow{\partial_+} & \mathcal{P}^n \\
 & & & & & & & & \downarrow \partial_+\partial_- \\
0 & \xleftarrow{\partial_-} & \mathcal{P}^0 & \xleftarrow{\partial_-} & \dots & \xleftarrow{\partial_-} & \mathcal{P}^{n-1} & \xleftarrow{\partial_-} & \mathcal{P}^n
\end{array}$$

Associated with this elliptic complex are four different finite-dimensional cohomologies which gives new symplectic invariants for non-Kähler manifolds.

Symplectic cohomologies (Tseng-Yau):

Symplectic (X, ω)	Complex (X, J)	
$PH^s_{\partial_\pm} = \dfrac{\ker \partial_\pm \cap \mathcal{P}^s}{\mathrm{im}\ \partial_\pm \cap \mathcal{P}^s}$	$\dfrac{\ker \bar\partial \cap \mathcal{A}^{p,q}}{\mathrm{im}\ \bar\partial \cap \mathcal{A}^{p,q}}$	(Dolbeault)
$PH^s_{\partial_+\partial_-} = \dfrac{\ker \partial_+\partial_- \cap \mathcal{P}^s}{(\mathrm{im}\ \partial_+ + \mathrm{im}\ \partial_-) \cap \mathcal{P}^s}$	$\dfrac{\ker \partial\bar\partial \cap \mathcal{A}^{p,q}}{(\mathrm{im}\ \partial + \mathrm{im}\ \bar\partial) \cap \mathcal{A}^{p,q}}$	(Aeppli)
$PH^s_{\partial_+ + \partial_-} = \dfrac{\ker d \cap \mathcal{P}^s}{\mathrm{im}\ \partial_+\partial_- \cap \mathcal{P}^s}$	$\dfrac{\ker d \cap \mathcal{A}^{p,q}}{\mathrm{im}\ \partial\bar\partial \cap \mathcal{A}^{p,q}}$	(Bott-Chern, Aeppli)

The middle-degree cohomology

$$PH^n_{\partial_+ + \partial_-} = \frac{\ker d \cap \mathcal{P}^n}{\mathrm{im}\ \partial_+\partial_- \cap \mathcal{P}^n}$$

turns out to appear in type IIA string.

For consider the deformation: $\Omega \longrightarrow \mathrm{Re}\,\Omega + \delta\mathrm{Re}\,\Omega$ with $\delta\rho_A = 0$ and conformal factor remaining invariant. Then the $\delta\mathrm{Re}\,\Omega$ satisfy

$$d(\delta\mathrm{Re}\,\Omega) = 0 \ , \qquad\qquad \partial_+\partial_- * (\delta\mathrm{Re}\,\Omega) = 0 \ ,$$

which is the harmonic condition of the primitive $PH^n_{\partial_+ + \partial_-}$ cohomology.

In fact, a subspace of the linearized deformation of the type IIA symplectic system can be parametrized by the cohomology

$$\delta\Omega \in P\mathcal{H}^3_{\partial_+ + \partial_-} \cap \mathcal{A}^{2,1} \ .$$

Non-Kähler geometry on six-dimensional manifolds will have a lot activities in the near future. These geometries can have relations with four- and three-dimensional manifolds. One can construct non-Kähler six-manifolds by the twistor construction. The twistor space of a self dual four-manifolds has a complex structure, and the twistor space of a hyperbolic four-manifold has a symplectic structure. The S^3 bundle over a hyperbolic three-manifold is also complex. (Fine-Panov have given examples of the hyperbolic constructions.) There should also be interesting dualities relating complex and symplectic structures on non-Kähler six-manifolds. The major guiding influence will be string theory.