A short course on

Quantum Field Theory

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Goal:
- understand Feynman diagrams
- be able to calculate elementary cross sections

Lectures:

1) Classical relativistic field theory
2) Free quantum fields
3) Interacting fields
4) Elementary processes

References I used:

M.E. PESKIN - D.V. SCHROEDER, An introduction to quantum field theory, 2nd ed., Perseus

C. ITZYKSON - J.B. ZUBER, Quantum field theory, McGraw-Hill

N.N. BOGOLYUBOV - D.K. SHIRKOV, Introduction to the theory of quantized fields, Interscience

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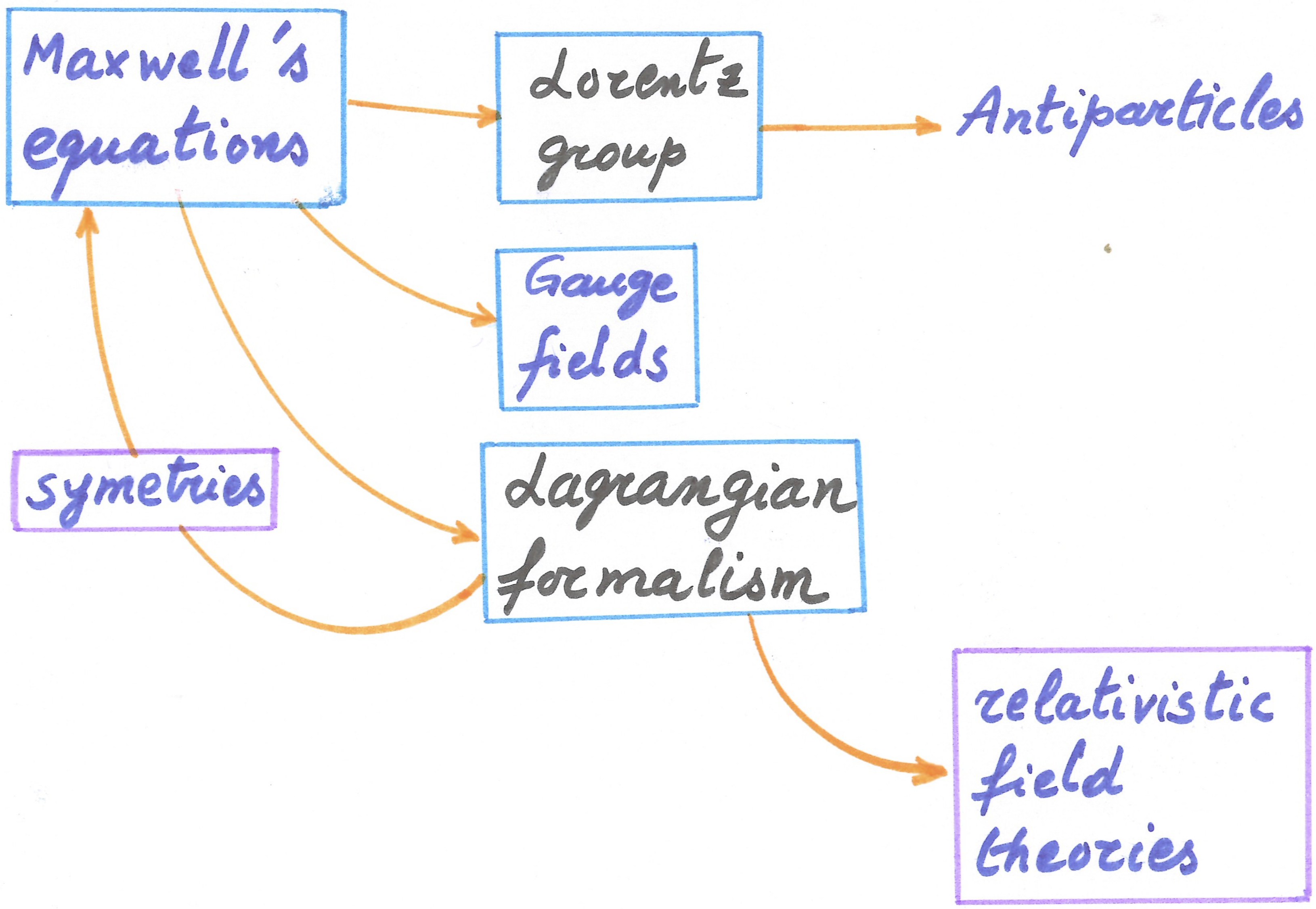
J. COSTELLA et Al, "Classical antiparticles", hep-ph / 9704210

Classical fields



Maxwell's equations

$\partial_t \equiv \frac{\partial}{\partial t}$

Electrical engeneering version:

$$\begin{cases} \varepsilon_0 \vec{\nabla}\cdot\vec{E}_e = \rho_e \\ \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}_e - \varepsilon_0 \partial_t \vec{E}_e = \vec{j}_e \\ \vec{\nabla}\times\vec{E}_e + \partial_t \vec{B}_e = 0 \\ \vec{\nabla}\cdot\vec{B}_e = 0 \end{cases}$$

$\longrightarrow$

$$\vec{\nabla}\cdot\left(\sqrt{\varepsilon_0}\,\vec{E}_e\right) = \frac{\rho_e}{\sqrt{\varepsilon_0}} = \rho$$

$$\frac{1}{\sqrt{\mu_0}}\vec{\nabla}\times\underbrace{\left(\frac{\vec{B}_e}{\sqrt{\mu_0}}\right)}_{\equiv \vec{B}} - \sqrt{\varepsilon_0}\,\partial_t \underbrace{\left(\sqrt{\varepsilon_0}\,\vec{E}_e\right)}_{\equiv \vec{E}} = \sqrt{\varepsilon_0}\,\frac{j_e}{\sqrt{\varepsilon_0}}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$\downarrow$ Heavyside Lorentz

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ c\,\vec{\nabla}\times\vec{B} - \partial_t \vec{E} = \vec{j} \\ c\,\vec{\nabla}\times\vec{E} + \partial_t \vec{B} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

$$F_c = \frac{1}{4\pi}\frac{qq'}{r^2}$$

$$[q] = M^{1/2} L^{3/2} T^{-1}$$

$\longleftarrow$ Natural units

$$\boxed{\begin{gathered} \begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \partial_t \vec{E} = \vec{j} \\ \vec{\nabla}\times\vec{E} + \partial_t \vec{B} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases} \\ [q] = 1\,\hbar^{1/2} c^{1/2} \\ [\vec{E}] = [\vec{B}] = \mathrm{GeV}^2 \end{gathered}}$$

Simplification:

$$\vec{\nabla}\cdot\vec{B} = 0 \iff \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\nabla\times(\vec{E}+\partial_t\vec{B}) = 0 \iff \vec{E} = -\vec{\nabla}\phi - \partial_t\vec{A}$$

"scalar" potential

vector potential

Notation:

$\mu = 0, 1, 2$ or 3

$$x^\mu \equiv (t, x, y, z) \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

$$x_\mu \equiv (t, -x, -y, -z) \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

$$j^\mu \equiv (\rho, j_x, j_y, j_z) \qquad A^\mu = (\phi, A_x, A_y, A_z)$$

$$k\cdot l = \Big(\sum_\mu\Big) k_\mu l^\mu = \Big(\sum_\mu\Big) k^\mu l_\mu \qquad x_\mu = g_{\mu\nu}x^\nu, \quad x^\mu = g^{\mu\nu}x_\nu$$

implicit

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix}$$

Maxwell's equations 2

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} \cdot & E_x & E_y & E_z \\ -E_x & \cdot & -B_z & B_y \\ -E_y & B_z & \cdot & -B_x \\ -E_z & -B_y & B_x & \cdot \end{pmatrix}$$

field strength tensor

inhomogeneous:

$$\boxed{\partial^\mu F_{\mu\nu} = j_\nu}$$

$$\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = \begin{pmatrix} \cdot & -B_x & -B_y & -B_z \\ B_x & \cdot & -E_z & E_y \\ B_y & E_z & \cdot & -E_x \\ B_z & -E_y & E_x & \cdot \end{pmatrix}$$

dual tensor

$\varepsilon^{0123} = 1$

homogeneous:

$$\boxed{\partial^\mu \tilde{F}_{\mu\nu} = 0}$$

current conservation: $\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0$

Maxwell's equations 3 : gauge field

$$\Box A^\nu - \partial^\nu(\partial\cdot A) = j^\nu \qquad \Box \equiv \partial_\mu\partial^\mu$$

gauge invariance : A^ν equivalent to
$A'^\nu = A^\nu + \partial^\nu\Lambda$

gauge condition :
$$\begin{cases} \partial_\mu A^\mu = 0 & \text{Lorenz} \\ n_\mu A^\mu = 0 & \text{axial} \\ \vec{\nabla}\cdot\vec{A} = 0 & \text{Coulomb} \end{cases}$$

$\Rightarrow$ the equations become:

$$\boxed{\begin{aligned} \Box A^\nu &= j^\nu \\ \partial\cdot A &= 0 \end{aligned}}$$

Invariance properties (2-d)

What are the tranformations $(t,x) \to (t',x')$ that leave the equations invariant?

- $\partial_\mu x^\mu = 4$ invariant $\to$ $\partial_\mu A^\mu$ invariant if A^μ transforms like x^μ
- $\Box A^\mu = j^\mu$ OK if $\begin{cases} A^\mu \text{ transforms like } j^\mu \\ \Box \text{ is invariant} \end{cases}$

$$\Box = \partial_t^2 - \partial_x^2$$

start with $\Delta = \partial_x^2 + \partial_y^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = M^{-1} \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} \qquad \Delta' = (\partial_{x'}, \partial_{y'}) \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix}$$

$$= (\partial_x \ \partial_y) M^T M \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

$\Rightarrow M^T M = \mathbb{1}$ orthogonal SO(2)

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Invariance properties 2

$t = i\xi \Rightarrow \Box = -(\partial_\xi^2 + \partial_x^2) \Rightarrow \begin{cases} x' = \cos\theta\, x + \sin\theta\, \xi \\ \xi' = -\sin\theta\, x + \cos\theta\, \xi \end{cases}$

real if $\theta = i\eta \qquad \sin\theta = i \sinh\eta \qquad \cos\theta = \cosh\eta$

$\begin{cases} x' = \cosh\eta\; x + \sinh\eta\; t \\ t' = \sinh\eta\; t + \cosh\eta\; x \end{cases} \qquad SO(1,1)$

Interpretation: $x' = 0 \Rightarrow x = -\underbrace{\tanh\eta}_{1 \leq v \leq 1}\, t$

$\left.\begin{array}{l} \cosh\eta = \dfrac{1}{\sqrt{1-v^2}} \\ \sinh\eta = \dfrac{v}{\sqrt{1-v^2}} \end{array}\right\} \Rightarrow \begin{cases} x' = \dfrac{1}{\sqrt{1-v^2}}(x + vt) \\ t' = \dfrac{1}{\sqrt{1-v^2}}(t + vx) \end{cases} \qquad \boxed{\text{boost!}}$

η is the rapidity (usually y): additive

$\eta = \frac{1}{2}\log\frac{1+v}{1-v} = \frac{1}{2}\log\frac{p+E}{E-p}$ for 1 particle

Lorentz groups

proper : $SO(3,1)$ 3 rotations, 3 boosts Λ

improper : add
$$\begin{cases} \text{parity } P \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix} \\ \text{time reversal } T \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} -t \\ \vec{x} \end{pmatrix} \end{cases}$$

representations :

- SCALAR : $\Lambda s' = s'$ $PS = S$ $TS = S$
- PSEUDOSCALAR: $\Lambda \tilde{S} = \tilde{S}$ $P\tilde{S} = -\tilde{S}$ $T\tilde{S} = -\tilde{S}$
- VECTOR x^μ
- PSEUDOVECTOR $p^\mu = (E, \vec{p})$
- TENSOR $g^{\mu\nu}$, $F^{\mu\nu}$
- SPINORS $SO(3,1) \sim SU(2) \otimes SU(2)$

$\Rightarrow$ ψ_R , ψ_L 2 components

Antiparticles

worldline of a particle : $x^\mu = (t, \vec{x}(t))$

$j^\mu \sim x^\mu$ and $j^\mu \doteq \frac{dx^\mu}{d\tau} q$

$\Rightarrow \tau$ must be invariant

$(d\tau)^2 = (dt)^2 - (d\vec{x})^2$ invariant

$\tau = \int_0^t d\tau = \pm \int_0^t dt' \sqrt{1 - (\frac{dx}{dt'})^2}$ proper time

τ grows with t $\longrightarrow$ particle

τ decreases with t :

$\left.\begin{array}{l}\text{same } x(t) \\ \text{opposite } j\end{array}\right\}$ antiparticle

$q(-\sqrt{1-v^2}) = -q\sqrt{1-v^2}$

NB: gravity $\sim T_{\mu\nu} \sim mv^2 \Rightarrow \left\{ m \frac{dx^\mu dx^\nu}{d\tau^2} \right.$ unchanged

CPT theorem

$$\left.\begin{array}{l}\text{locality / causality}\\ \text{unitarity}\\ \text{Lorentz proper}\end{array}\right\}\ \text{CPT conserved}$$

$t<0$ $\tau<0$

antiparticle
at $(-t,-\vec{x})$

$t>0$ $\tau>0$

particle
at $(t,\vec{x})$

C, P, T conserved separately by QED, QCD, GR

$$\left.\begin{array}{l}\text{C, P, T}\\ \text{CP, PT, CT}\end{array}\right\}\ \text{not conserved by weak interactions}$$

Maxwell's equations 4

Principle of extremal action for fields

mechanics

$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t))\, dt$$

$\frac{\delta S}{\delta Q_i} = 0 \Rightarrow Q_i(t)$ physical trajectories

$$S = S(Q_i) + \int_{t_1}^{t_2} \sum_i \frac{\delta L}{\delta q_i}\Big|_{Q(t)} dq_i(t)$$

$$\delta S = \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

$$= \sum_i \int \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) + \underbrace{\text{surface}}_{\text{assume } 0}$$

$$\Rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

Euler Lagrange

fields

$$S = \underbrace{\int dt \int d^3y}_{d^4x}\, \mathcal{L}(q(y,t), \underbrace{\dot{q}(y,t), \vec{\nabla} q}_{\partial_\mu q})$$

$\delta S = 0$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial q} - \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu q} = 0$$

y = space-time coordinates $(t, \vec{x})$

q = field $= \varphi(t, \vec{x})$

$[\mathcal{L}] = GeV^4$

Symmetries : Noether's theorem

continuous symmetry $\varphi \to \varphi' = U(\alpha)\varphi$

such that $S(\varphi') = S(\varphi)$

i.e. $\mathcal{L}(\varphi') = \mathcal{L}(\varphi) + \partial_\mu J^\mu(x)$

$\Rightarrow$ conserved current: j^μ

$$\begin{cases} \partial_\mu j^\mu = 0 \\ Q = \int d^3x \, j_0(x) \quad \text{charge} \quad \dfrac{dQ}{dt} = 0 \end{cases}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \Delta\varphi - J^\mu$$

$$U(\alpha) = \mathbb{1} + \alpha \Delta\varphi$$

Maxwell

gauge invariance $\Rightarrow$ $F_{\mu\nu}\,,\ \tilde{F}_{\mu\nu}$

Lorentz invariance $\Rightarrow$ $\underbrace{F_{\mu\nu}F^{\mu\nu}}_{2(-E^2+B^2)}\,,\ \tilde{F}_{\mu\nu}F^{\mu\nu}$

$\uparrow$ total derivative

$$\mathcal{L} = \underbrace{-\frac{1}{4}}_{\substack{\text{- to get } H>0 \\ \frac{1}{4} \text{ from quantization}}} F^{\mu\nu}F_{\mu\nu} \quad \underbrace{-\,j^\mu A_\mu}_{\substack{\partial_\mu j^\mu = 0 \text{ from} \\ \text{gauge invariance}}}$$

Scalar fields

real:
$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

$$\Rightarrow (\Box + m^2)\varphi = 0$$

complex:
$$\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi^*\phi$$

$$(\Box + m^2)\phi = (\Box + m^2)\phi^* = 0$$

symmetry: $\phi \to e^{i\alpha}\phi$

$$\Rightarrow \quad j^\mu = i[(\partial_\mu\phi^*)\phi - \phi^*\partial_\mu\phi]$$

Spin ½ fields

generators of the Lorentz group $\qquad J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

rotations $\qquad L_1 = J_{23} \quad L_2 = J_{31} \quad L_3 = J_{12}$

$$[L_i, L_j] = i\varepsilon_{ijk} L_k$$

boosts $\qquad B_1 = J_{01} \quad B_2 = J_{02} \quad B_3 = J_{03}$

$$[B_i, B_j] = -i\varepsilon_{ijk} L_k$$

$$[L_i, B_j] = i\varepsilon_{ijk} B_k$$

Factorize: $\qquad K_i^{\pm} = \frac{1}{2}(L_i \pm i B_i) \qquad [K_i^{\pm}, K_j^{\pm}] = i\varepsilon_{ijk} K_k^{\pm}$

$$[K_i^{\mp}, K_j^{\pm}] = 0$$

$$SO(3,1)$$

$$SU(2) \otimes SU(2)$$

$$K_i^+ \qquad\qquad K_i^-$$

spin ½ $\quad \frac{\sigma_i}{2} = K_i^+ \qquad\qquad K_i^- = 0$

ψ_L

spin ½ $\quad K_i^+ = 0 \qquad\qquad \frac{\sigma_i}{2} = K_i^-$

ψ_R

as $\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = \varepsilon_{ijk}\,\sigma_k/2$

$P\,\psi_R = \psi_L$; $\quad \psi_R \sim i\sigma_2 \psi_L^*$: Majorana

Spinorial invariants

$$\psi_L^\dagger \sigma_\mu \partial^\mu \psi_L \quad , \quad \psi_R^\dagger \bar{\sigma}_\mu \partial^\mu \psi_R \qquad \text{kinetic}$$

$(1, \vec{\sigma})$ $\qquad$ $(1, -\vec{\sigma})$

$$\psi_R^\dagger \psi_L \quad , \quad \psi_L^\dagger \psi_R \qquad \text{mass}$$

$$\mathcal{L} = \psi_L^\dagger \sigma_\mu i\partial^\mu \psi_L + \psi_R^\dagger \bar{\sigma}_\mu i\partial^\mu \psi_R + m\left(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R\right)$$

Bi-spinors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \bar{\psi} = \left(\psi_R^\dagger, \psi_L^\dagger\right) \qquad \gamma_\mu = \begin{pmatrix} \cdot & \sigma_\mu \\ \bar{\sigma}_\mu & \cdot \end{pmatrix}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu}$$

$$\mathcal{L} = \bar{\psi}\left(\gamma_\mu i\partial^\mu - m\right)\psi$$

Dirac Lagrangian