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I. Classical fields

1. How much is 1 Tesla in natural units?

- The Tesla is the unit of magnetic flux.

Definition: A particle of electric charge of 1 Coulomb that passes through a magnetic field of 1 Tesla at a speed of $1\ m.s^{-1}$ perpendicular to the field experiences a force of 1 Newton, according to the Lorentz force law.

- Finding units:

Lorentz force for $\vec{B}$ field ($\vec{E} = 0$):

$$\vec{F} = q\ \vec{v} \times \vec{B}$$

- $\vec{B}$: magnetic field
- $\vec{v}$: speed of the particle
- q: electric charge of the particle

And we know that $\vec{F} = m\,\vec{a}$

- $\vec{a}$: acceleration of the part.
- m: mass of the part.

In terms of units, the equalization of the 2 equations gives:

$$M.L.T^{-2} = Q.L.T^{-1}.\text{Tesla}$$

(Q: electric charge unit)

Thus: $\text{Tesla} \cdot Q = M\,T^{-1}$

Or, in other words:

$$1\ \text{Tesla} = \frac{kg}{C.s} = \frac{kg}{A.s^2}$$

(C: Coulomb, A: Ampère)

About natural units, we know:

$$E = mc^2 \longrightarrow [E] = M L^2 T^{-2}$$

$$E = h\nu \longrightarrow [E] = [h] T^{-1}$$

which gives: $[h] = [\hbar] = M L^2 T^{-1}$

and we search $\frac{kg}{s} = M T^{-1}$.

we have: $$\frac{[E]^2}{[\hbar c^2]} = \frac{M^2 L^4 T^{-4}}{M L^2 T^{-1} L^2 T^{-2}} = M T^{-1}$$

In particle physics, the energy is expressed in GeV.

By definition: $$1 \text{ GeV} = 10^9 \text{ eV}$$
$$= 10^9 \; 1.6 \; 10^{-19} \text{ joule}$$
$$= 10^9 \; 1.6 \; 10^{-19} \text{ kg m}^2 \text{ s}^{-2}$$

<u>Thus</u>: $$[1 \text{ Tesla?}] = \frac{(\text{GeV})^2}{(\hbar c^2)} \cdot \frac{1}{e}$$ where e = electric charge in Coulomb

$$= \frac{(10^9 \; 1.6 \; 10^{-19})^2}{1{,}05 \; 10^{-34} \; (3 . 10^8)^2} \cdot \frac{1}{1.6 \; 10^{-19}} \left[\frac{kg^2 \, m^4 \, s^{-4}}{C \; kg \, m^2 s^{-1} \, m^2 s^{-2}}\right]$$

$$= \frac{10^{18} \; 1.6 \; 10^{-19}}{1.05 \; 10^{-34} \; 9 \; 10^{16}} \left[\frac{kg}{C\,s}\right]$$

$$\simeq 1{,}77 \; 10^{16} \left[\frac{kg}{C\,s}\right]$$

$$\hookrightarrow 1 \text{ Tesla} = 1 \frac{kg}{C\,s} = \frac{1}{1{,}77 \; 10^{16}} \frac{\text{GeV}^2}{e \, (\hbar c^2)}$$

$(\hbar c^2)$: Not written in natural units $= 1$.

In natural units, the fine structure constante is $\alpha = \frac{e^2}{4\pi} = \frac{1}{137} \Longrightarrow e = \sqrt{\frac{4\pi}{137}}$.

Finally: $$\boxed{1 \text{ Tesla} \simeq 1.8 \; 10^{-16} \text{ GeV}^2}$$

I.2. Write the field-strength tensor $F^{\mu\nu}$ as a function of the electric (E_i) and magnetic (B_i) fields.

By definition: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

with $A^\mu = (\phi, \vec{A})$ the 4-potential vector.

The Maxwell equations gives the definition of the electric and magnetic fields from the potential vector:

$$\begin{cases} \vec{E} = -\partial_t \vec{A} - \vec{\nabla}\phi \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

which, in terms of vector components, reads

$$\begin{cases} E^i = -\partial_0 A^i - \partial_i \phi \\ B^i = \epsilon_{ijk}\,\partial_j A^k \quad (= \partial_j A^k - \partial_k A^j \text{ with } j \neq k \neq i) \end{cases}$$

when ϵ_{ijk} is the Levi-Civita tensor defined such that:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = \begin{cases} (1,2,3) \\ (2,3,1) \\ (3,1,2) \end{cases} \\ -1 & \text{if } (i,j,k) = \begin{cases} (1,3,2) \\ (2,1,3) \\ (3,2,1) \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

Property of ϵ_{ijk}:

$$\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$$

where $\delta^{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

The field-strenght tensor is antisymmetric : $F^{\mu\nu} = -F^{\nu\mu}$

→ zeros on diagonal → $F^{\mu\mu} = 0$; $\mu = 0,1,2,3$

→ Only 6 independent parameters

• $\mu = 0$; $\nu = j$ $(= 1,2,3)$

$$F^{0j} = \partial^0 A^j - \partial^j A^0$$
$$= g^{0\sigma}\partial_\sigma A^j - g^{j\rho}\partial_\rho A^0$$
$$= \partial_0 A^j - (-1)\partial_j A^0 \qquad \text{[Minkowski space]}$$
$$= \partial_0 A^j + \partial_j A^0$$
$$= -E^j$$

• $\mu = i$; $\nu = j$

$$F^{ij} = \partial^i A^j - \partial^j A^i$$
$$= -\partial_i A^j + \partial_j A^i$$

→ looks like $B^k = \epsilon^{klm}\partial_l A^m$

$$\Leftrightarrow \epsilon^{ijk} B^k = \epsilon^{ijk}\epsilon^{klm}\partial_l A^m$$
$$= \epsilon^{kij}\epsilon^{klm}\partial_l A^m$$
$$= (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})\partial_l A^m$$
$$= \partial_i A^j - \partial_j A^i$$

↳ $F^{ij} = -\epsilon^{ijk} B^k$

* Finally: $$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & +B^3 & 0 & -B^1 \\ E^3 & -B^2 & +B^1 & 0 \end{pmatrix}$$

Show that : $$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

I.3. What is the field of a static charge in each of the following gauges:

- Lorenz : $\partial_\mu A^\mu = 0$
- Coulomb : $\vec{\nabla} \cdot \vec{A} = 0$
- Axial : $n_\mu A^\mu = 0$

The gauge invariance is the fact that we can add a term to the 4-potential vector without modifying the physics, i.e. what is measured during an experiment; here, the fields $\vec{E}$ and $\vec{B}$.

Formally, a gauge change is the transformation of the field A^μ to $A^{\mu'}$ such like: $A'^\mu = A^\mu + \partial^\mu \Lambda(x^\mu)$

for a certain scalar function Λ that depends on the 4 space-time coordinates.

Under this type of transformation, the action is not changed (gauge invariant) which implies that the equation of motion remain the same.

The action for a charged particle in EM field is:

$$S = \int \{ -m\,ds - e A_\mu dx^\mu \}$$

$$S \rightarrow S' = \int \{ -m\,ds - e(A_\mu + \partial_\mu \Lambda) dx^\mu$$

$$= S - \int e\, \partial_\mu \Lambda\, dx^\mu = S - \underbrace{\int \partial_\mu (e \Lambda)\, dx^\mu}_{\text{4-divergence term}}$$

$$= S$$

4-divergence term ↳ = 0

↳ Does not change the equation of motion.

> Note: This is true because the charge is assumed to be invariant in space-time

The gauge invariance thus tells us that we can add a term without changing the physics.

We can thus choose the gauge which mathematically helps at solving the problem.

However, a gauge choice does not always completely fix the potential vector. We need, sometimes, to additionally impose boundary conditions such as asking for vanishing waves/fields at infinity.

For example, if we consider the Lorenz gauge $\partial_\mu A^\mu = 0$, then the physics will be unchanged under the transformation $A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu \Lambda$ for certain forms of Λ.

Indeed: $\partial_\mu A'^\mu = 0 \iff \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda = 0$

Thus, if $\Box \Lambda = 0$ $\quad$ ($\Box \equiv \partial_\mu \partial^\mu \equiv$ d'Alembertian)

then the physics is unchanged.

Solutions of $\Box \Lambda = 0$ are $\Lambda = 0$ (not interesting)

$\Lambda = C\, e^{i k_\mu x^\mu} \equiv$ Plane Waves

In this example, we see that the Lorenz gauge fix the 4-potential vector up to a plane wave term.

Back to the exercise.

- The field equations are: $\partial_\mu F^{\mu\nu} = j^\nu$

$$\Leftrightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu$$

$$\Leftrightarrow \Box A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$$

$(\partial_\mu \partial^\nu = \partial^\nu \partial_\mu$ in Minkowski space$)$

- j^ν is the 4-current.

For static charge: $j^\nu = (q\delta^{(3)}, \vec{0})$

(a) Coulomb gauge: $\vec{\nabla}.\vec{A} = 0 \quad (\Rightarrow) \quad \partial_i A^i = 0$ [summation!]

Field equation become:

$$\Box A^\nu - \partial^\nu(\partial_0 A^0 + \cancel{\partial_i A^i}^{=0}) = j^\nu$$

$\nu = 0$: $\Box A^0 - \partial^0 \partial_0 A^0 = q\delta^{(3)}$

$(\Rightarrow) (\partial_0 \partial^0 + \partial_i \partial^i) A^0 - \partial_0 \partial^0 A^0 = q\delta^{(3)}$

$(\Rightarrow) \partial_i \partial^i A^0 = +q\delta^{(3)}$

$(\Rightarrow) \Delta A^0 = -q\delta^{(3)} \implies \boxed{A^0 = \frac{+q}{4\pi r}}$

$\left(\begin{array}{l}\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \\ = \sum_i \partial_i \partial_i = -\partial^i \partial_i\end{array}\right)$

Because $\Delta\left(\frac{1}{r}\right) = -4\pi\delta^{(3)}$ is a result from distribution analysis.

$\nu = i$: $\Box A^i - \partial^i \partial_0 A^0 = 0$

$(\Rightarrow) \Box A^i = 0 \quad (\partial_0 A^0 = 0$ from above$)$

$A^i = \mathcal{E} e^{ik_\mu x^\mu} \equiv$ Plane waves.

⊠ Lorenz gauge: $\partial_\mu A^\mu = 0$

Field equations: $\Box A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$

$$\Leftrightarrow \Box A^\nu = j^\nu$$

(1) $\underline{\nu = 0}$: $\Box A^0 = q\, \delta^{(3)}$

(2) $\underline{\nu = i}$: $\Box A^i = 0$ $\Longrightarrow$ Plane waves. $A^i = G\, e^{i k_\mu x^\mu}$.

(1) Is an inhomogeneouse equation.
It's solution is the superposition of solution of the homogeneouse equation + another (damping) term.

But, we also see that we can add a further constraint in Lorenz gauge in order to fix the 4-potential A^μ.
If we ask $\partial_0 A^0 = 0$, then we come back to the Coulomb gauge for which we already write the solution.

↳ Can we ask this?
That means, can we find a function Λ such that $A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu \Lambda$ and have $\partial_0 A'^0 = 0$?

This implies $\partial_0 A^0 + \partial_0 \partial^0 \Lambda = 0$

$$\Leftrightarrow \partial_0 \partial^0 \Lambda = -\partial_0 A^0$$

$$\Leftrightarrow \partial_0 \Lambda = -A^0 + \text{Cst}$$

$$\Leftrightarrow \Lambda = -A^0 t + \text{Cst}$$

Thus, we can find such a function Λ in order to impose "$\partial_0 A^0 = 0$".
The solutions of the field are thus exactly the same as for the Coulomb gauge:

$$\begin{cases} A^0 = \dfrac{\pm q}{4\pi r} \\ A^i = G\, e^{i k_\mu x^\mu} \end{cases}$$

⊛ Axial gauge: $n_\mu A^\mu = 0$

There are 3 types of axial gauge:

- time-like: $n = (1, \vec{0}) \Longrightarrow A^0 = 0$
- space-like: $n = (0,0,0,1) \Rightarrow A^3 = 0$
- light-cone: $n = (1,0,0,1) \Rightarrow A^0 - A^3 = 0$

Say time-like axial gauge: $A^0 = 0$

From field equation: $\Box A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu$

$$\underline{\nu = 0} \Rightarrow \Box A^0 - \partial^0 \partial_\mu A^\mu = q\,\delta^{(3)}$$

$$\Leftrightarrow \quad -\partial^0 \partial_i A^i = q\,\delta^{(3)} \qquad (A^0 = 0)$$

$$\Leftrightarrow \quad \partial_i A^i = -q.t\,\delta^{(3)} + Cst(t)$$

In spherical coordinates

$$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\partial_r[r^2 A_r]$$

when { centrosymmetric / spherical symmetry

Thus:

$$\frac{1}{r^2}\partial_r[r^2 A_r] = -q.t$$

$$\Leftrightarrow \quad \partial_r[r^2 A_r] = -q.t\, r^2$$

$$\Leftrightarrow \quad r^2 A_r = -\int dr\, q t r^2$$

$$\Leftrightarrow \quad r^2 A_r = -q.t\,\frac{r^3}{3}$$

$$\Rightarrow \quad A_r = -\frac{q.t\,r}{3}$$

In cartesian coordinates: $\vec{A} = -\frac{qt}{3}(x, y, z)$ if you like.

I.4. Show that the rapidity y is additive with respect to boosts and that $y \sim v$ when $c \to \infty$

I.5. Derive the canonical equation of motion from the Hamiltonian: $H(p,q) = \sum_i \{ p_i \dot{q}_i - L(q, \dot{q}(p,q)) \}$

Take the total derivative of the Hamiltonian

$$dH = \sum_i \left\{ \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right\}$$

q_i and p_i are canonical conjugate variables

So: $p_i = \frac{\partial L}{\partial \dot{q}_i}$ where $\dot{q}_i = \frac{dq_i}{dt}$

and Euler-Lagrange equation says:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad \dot{p}_i = \frac{\partial L}{\partial q_i}$$

Let's compute $\frac{\partial H}{\partial p_i}$ and $\frac{\partial H}{\partial q_i}$

- $$\frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i}\left\{ \sum_j \left[p_j \dot{q}_j(p,q) - L(q, \dot{q}(p,q)) \right] \right\}$$

$$= \dot{q}_i + \sum_j \left\{ p_j \frac{\partial \dot{q}_j(p,q)}{\partial p_i} - \frac{\partial L(q, \dot{q}(p,q))}{\partial p_i} \right\}$$

$$= \dot{q}_i + \sum_j \underbrace{\left\{ \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial L}{\partial p_i} \right\}}_{=0}$$

since q does not depend on p

$\hookrightarrow \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i} = \frac{\partial L}{\partial p_i}$

$$= \dot{q}_i$$

$$\cdot \frac{\partial H}{\partial q_i} = \frac{\partial}{\partial q_i} \left\{ \sum_j \left(p_j \dot{q}_j(p,q) \right) - L(q, \dot{q}(p,q)) \right\}$$

$$= \sum_j \Big\{ p_j \frac{\partial \dot{q}_j}{\partial p} \frac{\partial p}{\partial q_i} + p_j \frac{\partial \dot{q}_j}{\partial q} \frac{\partial q}{\partial q_i}$$

$$- \frac{\partial L}{\partial q} \frac{\partial q}{\partial q_i} - \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial p} \frac{\partial p}{\partial q_i} - \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q} \frac{\partial q}{\partial q_i} \Big\}$$

and $p_j = \frac{\partial L}{\partial \dot{q}_j}$; $\frac{\partial p}{\partial q_i} = 0$; $\frac{\partial q}{\partial q_i} \rightsquigarrow \sum_j \frac{\partial q_j}{\partial q_i} = \delta_{ij}$

$\hookrightarrow \frac{\partial H}{\partial q_i} = - \frac{\partial L}{\partial q_i} = - \dot{p}_i$ (from Euler-Lagrange)

<u>Thus</u>: $dH = \sum_i \left\{ \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right\}$

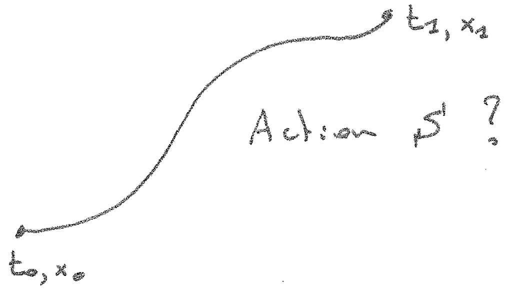
$$= \sum_i \left\{ \dot{q}_i \, dp_i - \dot{p}_i \, dq_i \right\}$$

The canonical equations of motion are:

$$\boxed{\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i \end{aligned}}$$

I.6. Find, for a single particle, the action which is invariant with respect to boosts and obtain the Euler-Lagrange equation, the momentum and the energy.

Boost-invariant means invariant under Lorentz transformation.



By definition, the proper-time is a (special) relativistic invariant.

$$\tau = \int_{t_0}^{t_1} d\tau \quad \text{with } d\tau = dt^2 - dx^2$$

The action S is a dimensionless quantity.

In natural units, we have $[d\tau] = GeV^{-1}$.

Therefore, the simplest action that we can build will have the form : $S = \kappa \int d\tau$ with $[\kappa] = GeV$

A good idea is to say that κ is the proper-mass of the particle (m_0) which has such units.

Thus, the simplest action which is relativitic invariant is $S = m_0 \int_{t_0}^{t} d\tau$.

From this action, we now search for the Euler-Lagrange equations, the momentum and the energy of the particle.

- In general, the action is written in the form: $S = \int dt\, L$ ← Lagrangian.

We have:
$$S = m_0 \int d\tau$$
$$= m_0 \int \sqrt{dt^2 - dx^2}$$
$$= m_0 \int dt \sqrt{1 - (dx/dt)^2}$$
$$= m_0 \int dt \sqrt{1 - v^2} \quad \text{with } v = \dot{x} = \frac{dx}{dt}$$
$$= m_0 \int \frac{dt}{\gamma} \quad \text{where } \gamma = \frac{1}{\sqrt{1-v^2}} \text{ (relativity)}$$

The Lagrangian is thus: $L = \frac{m_0}{\gamma} = m_0 \sqrt{1 - \dot{x}^2}$.

Euler-Lagrange eq:
$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0}$$

So:
$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{x}}\left(m_0\sqrt{1-\dot{x}^2}\right)\right\} - \frac{\partial}{\partial x}\left(m_0\sqrt{1-\dot{x}^2}\right) = 0$$

$$\Leftrightarrow \frac{d}{dt}\left(m_0 \frac{-\dot{x}}{\sqrt{1-\dot{x}^2}}\right) + 0 = 0$$

$$\Leftrightarrow -m_0\left[\ddot{x}\gamma - \frac{1}{2}\frac{\dot{x}(-2\dot{x})\ddot{x}}{(1-\dot{x}^2)^{3/2}}\right] = 0$$

$$\Leftrightarrow \frac{m_0}{(\sqrt{1-\dot{x}^2})^3}\left[\ddot{x}(1-\dot{x}^2) + \dot{x}^2\ddot{x}\right] = 0$$

$$\Leftrightarrow m_0 \gamma^3 \ddot{x} = 0 \quad \Longrightarrow \quad \boxed{\ddot{x} = 0}$$

A free particle does not undergo any acceleration!

* Momentum

The momentum p is given from the Lagrangian

by: $p = \frac{\partial L}{\partial \dot{x}}$

So: $p = \frac{\partial}{\partial \dot{x}} \left\{ m_0 \sqrt{1-\dot{x}^2} \right\}$

$= - \frac{m_0 \dot{x}}{\sqrt{1-\dot{x}^2}}$ $\Rightarrow$ $p = -m v$

with $m = m_0 \gamma$

$= \frac{m_0}{\sqrt{1-\dot{x}^2}}$.

* Energy

The energy is given by the Hamiltonian of the system.

Thus: $E = H = p\dot{x} - L$

$= - \frac{m_0}{\sqrt{1-\dot{x}^2}} \cdot \dot{x}^2 - m_0 \sqrt{1-\dot{x}^2}$

$= - \frac{m_0}{\sqrt{1-\dot{x}^2}} \left(\dot{x}^2 + 1 - \dot{x}^2 \right)$

$= - \frac{m_0}{\sqrt{1-\dot{x}^2}} = - m$

↳ $\boxed{E = m}$ → we should have say $K = -m_0$ instead of $K = m_0$.

which is $E = mc^2$ in natural units where $c = 1$

↳ Einstein was right!

$E = m c^2$

$= \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} c^2$

I. 10. The Lagrangian density for a massive vectorfield is given by: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$

Prove that the equation $\partial_\mu A^\mu = 0$ is a consequence of the equations of motion.

Eq. of motion ≡ Euler-Lagrange eq.:

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial^\mu A^\alpha} = 0$$

We have: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \longrightarrow \frac{\partial F^{\mu\nu}}{\partial A^\alpha} = 0$

$$\rightarrow \frac{\partial F^{\mu\nu}}{\partial \partial^\alpha A^\beta} = \delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\nu\alpha}\delta^{\mu\beta}$$

Thus:

- $\frac{\partial \mathcal{L}}{\partial A^\beta} = m^2 A^\beta$
- $\frac{\partial \mathcal{L}}{\partial \partial^\alpha A^\beta} = -\frac{1}{4} \cdot 2 F^{\mu\nu} \left(\delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\nu\alpha}\delta^{\mu\beta}\right)$
  $= -\frac{1}{2}\left(F^{\alpha\beta} - F^{\beta\alpha}\right) = -F^{\alpha\beta}$

E-L eq. give: $m^2 A^\beta + \partial_\alpha F^{\alpha\beta} = 0$

And $\partial_\beta (E\text{-}L) \longrightarrow m^2 \partial_\beta A^\beta + \partial_\beta (\partial_\alpha F^{\alpha\beta}) = 0$

And $\partial_\beta \partial_\alpha F^{\alpha\beta} = \partial_\beta \partial_\alpha \left[\frac{1}{2} F^{\alpha\beta} + \frac{1}{2} F^{\alpha\beta}\right]$

$= \frac{1}{2} \partial_\beta \partial_\alpha \left[F^{\alpha\beta} - F^{\beta\alpha}\right]$

$= \frac{1}{2} \left\{\partial_\beta \partial_\alpha F^{\alpha\beta} - \partial_\beta \partial_\alpha F^{\beta\alpha}\right\}$

$= \frac{1}{2} \left\{\partial_\beta \partial_\alpha F^{\alpha\beta} - \partial_\alpha \partial_\beta F^{\beta\alpha}\right\}$ (with $\alpha \to \beta$, $\beta \to \alpha$, $\beta \to \alpha$, $\alpha \to \beta$) ⟹ Rename indices that are contracted ⟹ ALLOWED!

$= 0$

This is trivial since contraction of symmetric thing with antisymm!

We finally have $\partial_\beta (E\text{-}L) \Longrightarrow m^2 \partial_\beta A^\beta = 0$

$$\Longleftrightarrow \boxed{\partial_\mu A^\mu = 0}$$