Free quantum fields

- Field operators $\Rightarrow$ Hilbert space (Fock) $\Rightarrow$ particles

  Fixed Hilbert space
  Fields change with time } Heisenberg

- spin statistics and fermion fields

- covariant gauge fields and modified Hilbert space

Quantum mechanics

Classical hamiltonian:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \longrightarrow \dot{q}_i(q_j, p_j)$$

$$H(p,q) = \sum_j p_j \dot{q}_j(q_k, p_k) - L(q_k, \dot{q}_k(q,p))$$

$$\Rightarrow \quad \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases} \quad \Rightarrow \quad \frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$\equiv -\{f, H\}_P$$

Poisson bracket

$$\{a, b\}_P \equiv \sum_i \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial q_i} - \frac{\partial a}{\partial q_i}\frac{\partial b}{\partial p_i}$$

Quantisation for particles

$p, q \longrightarrow \hat{p}, \hat{q}$

Hilbert space: scalar product, positive norm, conserved current

$\{p, q\} \longrightarrow i[\hat{p}, \hat{q}]$

Example: $q^\mu = (t, \vec{x}) \Rightarrow p^\mu = (i\partial_t, -i\vec{\nabla}) = i\partial^\mu$

$$i[p^\mu, q^\nu] = \delta^{\mu\nu}$$

$$E = \frac{1}{2m} p^2 \longrightarrow i\partial_t \psi = -\frac{1}{2m} \Delta \psi$$

Hilbert space: $\langle \psi | \varphi \rangle \equiv \int d^3x \, \psi^* \varphi$

$\langle \psi | \psi \rangle > 0$

proba. $j_\mu = \left(|\psi|^2, -\frac{i}{2m} \psi^* \overleftrightarrow{\nabla} \psi + \text{c.c.} \right)$

can generate it from eigenstates of $\hat{H}$

Poisson brackets for a scalar field

$$\pi(x) = \int d^3y \, \frac{\partial \mathcal{L}(\varphi(y), \partial_\mu \phi(y))}{\partial \dot{\varphi}(x)} \equiv \frac{\delta L}{\delta \dot{\varphi}(x)}$$

$$H = \int d^3x \, \underbrace{\pi \dot{\varphi}(\varphi, \pi) - \mathcal{L}}_{\mathcal{H}\text{: hamiltonian density}}$$

$$\begin{cases} \dfrac{\delta H}{\delta \pi} = \dot{\varphi} = \{H, \varphi\}_P \\[2ex] -\dfrac{\delta H}{\delta \varphi} = \dot{\pi} = \{H, \pi\}_P \end{cases} \qquad \{A, B\} = \frac{\delta A}{\delta \pi}\frac{\delta B}{\delta \varphi} - \frac{\delta A}{\delta \varphi}\frac{\delta B}{\delta \pi}$$

$$\left.\begin{aligned} &\{\pi(x), \varphi(y)\}_{P, ET} = \delta^{(3)}(\vec{x} - \vec{y}) \\ &\{\pi, \pi\}_{P,ET} = \{\varphi, \varphi\}_{P,ET} = 0 \end{aligned}\right\} \text{equal times}$$

Quantisation for a scalar field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \hat{\varphi} \, \partial^\mu \hat{\varphi} - \frac{1}{2} m^2 \hat{\varphi}^2$$

$$\hat{\pi} = \dot{\hat{\varphi}} \qquad H = \frac{1}{2} \int d^3x \left(\hat{\pi}^2 + (\vec{\nabla}\hat{\varphi})^2 + m^2 \hat{\varphi}^2 \right)$$

positive definite

$$\begin{cases} \left[\hat{\varphi}(x), \hat{\pi}(y) \right]_{ET} = i \, \delta^{(3)}(\vec{x} - \vec{y}) \\ \left[\hat{\varphi}(x), \hat{\varphi}(y) \right]_{ET} = \left[\hat{\pi}(x), \hat{\pi}(y) \right]_{ET} = 0 \end{cases}$$

$\hat{\varphi}$ obeys the Euler-Lagrange equation

$$(\Box + m^2) \hat{\varphi} = 0 \implies \hat{\varphi}_p = \hat{a}_p e^{-i p_\mu x^\mu}$$

$\Rightarrow$ $\hat{\varphi}$ is a superposition of these

$$E_p = \sqrt{\vec{p}^2 + m^2} \qquad p^0 = \pm E_p$$

$$\begin{cases} \hat{\varphi}(\vec{x}) = \displaystyle\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\hat{a}_{\vec{p}}\, e^{-ip_\mu x^\mu} + \hat{a}^+_{\vec{p}}\, e^{ip_\mu x^\mu} \right)\Big|_{t=0} \\ \hat{\pi}(\vec{x}) = \dot{\hat{\varphi}}(\vec{x}, t=0) = -i \displaystyle\int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(\hat{a}_{\vec{p}}\, e^{-ip\cdot x} - \hat{a}^+_{\vec{p}}\, e^{ip\cdot x} \right)\Big|_{t=0} \end{cases}$$

$$[\,,\,] \Rightarrow \begin{cases} [\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\ [\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}] = [\hat{a}^+_{\vec{p}}, \hat{a}^+_{\vec{q}}] = 0 \end{cases}$$

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} E_p \left(\hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} + \underbrace{\frac{1}{2} [\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{p}}]}_{(2\pi)^3 \delta^{(3)}(0)} \right)$$

"zero point energy"

This looks like a collection of harmonic oscillators $\rightarrow$ we can find the spectrum

The zero-point energy

- free to choose the order of operators
  → normal order everything
  ≡ put $\hat{a}$ to the right, $\hat{a}^+$ to the left
  → zero-point energy disappears

- Work in the Heisenberg picture

$$\begin{cases} |\psi\rangle \text{ fixed} \\ i\dot{\hat{O}} = [\hat{O}, \hat{H}] \end{cases} \rightarrow \hat{H}+c \Leftrightarrow \hat{H} \text{ if number } c$$

$$|\psi\rangle_S = e^{-i(\hat{H}+c)t}|\psi\rangle_H \text{ does not exist if } c \to \infty$$

The solution

$|0\rangle$ $\begin{cases} \text{state of minimum energy } E=0 \\ \hat{a}_{\vec{p}}\,|0\rangle = 0 \qquad\qquad \langle 0|0\rangle = 1 \end{cases}$

If all operators are made of $\hat{\varphi}$, $\hat{\pi}$,
$|0\rangle$ is unique

spectrum: $\quad (a_p^{+})^m \cdots (a_q^{+})^n\,|0\rangle$

$a_p^{+} a_p = \hat{N}_p$ counts the excitations of p^{th} oscillator

particles $\begin{cases} E = m E_p + n E_q + \cdots \\ \vec{P} = -\int d^3x\, \pi \vec{\nabla}\varphi = m\vec{p} + n\vec{q} \cdots \end{cases}$

$\hat{a}^{+}$ creates particles, $\hat{a}$ destroys particles

$$\hat{a}^\dagger_{\vec{p}}\,\hat{a}^\dagger_{\vec{q}} = \hat{a}^\dagger_{\vec{q}}\,\hat{a}^\dagger_{\vec{p}} \quad \Rightarrow \quad \text{bosons}$$

isomorphism:

state vectors of free bosons $\longleftrightarrow$ solutions to the harmonic oscillator

Hilbert space : Fock space

$$\mathcal{F} = \underset{\substack{\uparrow \\ |0\rangle}}{\mathcal{H}_0} + \underset{\substack{\uparrow \\ |p\rangle}}{\mathcal{H}_1} + \mathcal{H}_2 + \ldots\; \mathcal{H}_n + \ldots$$

$\hat{\varphi}(x) \quad \Rightarrow$ Hilbert space

$\Rightarrow$ particle content

All operators are made of fields

Relativistic normalisation

make $\langle \vec{p} | \vec{q} \rangle$ Lorentz invariant

$$|\vec{p}\rangle = N(\vec{p})\, a_p^{\dagger} |0\rangle$$

$$\langle \vec{p} | \vec{q} \rangle = |N|^2 (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$

$$\int \underbrace{\delta^{(3)}(\vec{p}-\vec{q})}_{\text{not invariant}} \underbrace{d^3p}_{\text{not invariant}} = \underbrace{1}_{\text{invariant}}$$

$$\int d^4p \; \delta(p^2-m^2)\, \theta(p^0) = \int \frac{d^3p}{2p_0} \Rightarrow 2p^0 \delta^{(3)} \text{ invariant}$$

invariant

$$\Rightarrow |N|^2 = 2p^0$$

$$|\vec{p}\rangle = \sqrt{2E_p}\, a_{\vec{p}}^{\dagger} |0\rangle$$

Complex scalar field

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi^* \phi$$

coefficient to obtain $[a, a^+] = (2\pi)^3 \delta^{(3)}$

$$\pi = \dot{\phi}^* \qquad \pi^* = \dot{\phi}$$

$$\phi = \int \frac{d^3p}{(2\pi)^3 \sqrt{2p^0}} \left(a_{\vec{p}} \, e^{i\vec{p}\cdot\vec{x}} + b^+_{\vec{p}} \, e^{-i\vec{p}\cdot\vec{x}} \right)$$

complex

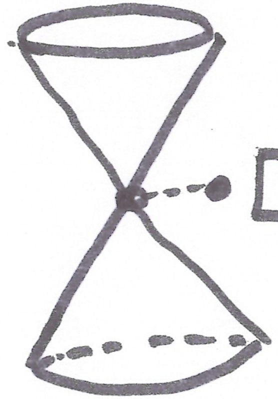
$$H = \int d^3x \left(\pi \pi^* + \vec{\nabla}\phi \cdot \vec{\nabla}\phi^* + m^2 \phi^* \phi \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \left(a^+_{\vec{p}} a_{\vec{p}} + b^+_{\vec{p}} b_{\vec{p}} \right) + C$$

$$j^\mu = i\left[(\partial^\mu \phi^*) \phi - \phi^* \partial_\mu \phi \right] \Rightarrow Q = \int d^3x \, j_0 = \int \frac{d^3p}{(2\pi)^3 2E_0} \left(a^+_{\vec{p}} a_{\vec{p}} - b^+_{\vec{p}} b_{\vec{p}} \right)$$

a = particle $\qquad$ b = antiparticle

Causality and antiparticles



$$[\phi(x), \phi^*(y)] \stackrel{?}{=} 0 \quad \text{for } (x-y)^2 < 0$$

$$[\phi(x), \phi^*(y)] = \int \frac{d^3p}{(2\pi)^3 2p_0} \left(e^{-ip.(x-y)} - e^{ip.(x-y)}\right)$$

$$\equiv \quad D(x-y) - D(y-x)$$

$$(x-y)^2 < 0 \quad \Rightarrow \begin{cases} \text{choose} \quad x^0 = y^0 \\ \text{rotate} \quad \vec{x}-\vec{y} \text{ into } \vec{y}-\vec{x} \end{cases} \Rightarrow [\phi(x), \phi^*(y)] = 0 \text{ if } (x-y)^2 < 0$$

$$[\phi(x), \phi^*(y)] = \langle 0 | [\phi, \phi^*] | 0 \rangle = \langle 0 | \phi \phi^* | 0 \rangle - \langle 0 | \phi^* \phi | 0 \rangle$$

particle   antiparticle

Quantisation of the Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma.\partial - m)\psi \qquad \bar{\psi} = \psi^\dagger \gamma_0$$

Euler Lagrange $\quad (i\gamma.\partial - m)\psi = 0$

$$\psi = e^{-ip.x} u^{(s)}(p) \qquad u^{(s)} = \begin{pmatrix} \sqrt{p.\sigma}\,\xi \\ \sqrt{p.\bar{\sigma}}\,\xi \end{pmatrix} \qquad \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$p_0 = E_p$$

$$\psi = e^{ip.x} v^{(s)}(p) \qquad v^{(s)} = \begin{pmatrix} \sqrt{p.\sigma}\,\xi \\ -\sqrt{p.\bar{\sigma}}\,\xi \end{pmatrix}$$

$$p_0 = E_p$$

$j^\mu = \bar{\psi}\gamma^\mu\psi \quad$ conserved current

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left(a_{\vec{p}}^{(s)} u^{(s)} e^{-i\vec{p}.\vec{x}} + b_{\vec{p}}^{(s)\dagger} v^{(s)} e^{+i\vec{p}.\vec{x}} \right)$$

$$\pi = i\psi^\dagger$$

$$H = \int d^3x \; \bar{\psi}\,(-i\,\vec{\nabla}\cdot\vec{\gamma} - m)\,\psi$$

- $[\psi_a(x), \psi_b^\dagger(y)] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$

  $\rightarrow$ wrong algebra $[a, a^\dagger] = [b^\dagger, b] = (2\pi)^3 \delta^{(3)}$

  $\langle 0| b b^\dagger |0\rangle < 0$

- $b \rightleftharpoons b^\dagger \quad \rightarrow \quad H \sim \sum E_p\,(a^\dagger a - b^\dagger b)$

  not positive definite

$$\left.\begin{array}{l}\text{causality}\\ H > 0\\ \text{Hilbert space}\end{array}\right\} \Rightarrow \text{use anticommutators}$$

Spin-statistics theorem: $\left\{\begin{array}{ll}\text{spin } n & \text{commutators}\\ \text{spin } n+\frac{1}{2} & \text{anticomm.}\end{array}\right.$

$$\{\psi_a(x), \psi_b^\dagger(y)\}_{ET} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

$$\{\psi_a, \psi_b\}_{ET} = \{\psi_a^\dagger, \psi_b^\dagger\}_{ET} = 0$$

$$\Rightarrow \quad \{a_{\vec{p}}^{(r)}, a_{\vec{q}}^{(s)\dagger}\} = \{b_{\vec{p}}^{(r)}, b_{\vec{q}}^{(s)\dagger}\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,\delta_{rs}$$

$$\{a, a\} = \{b, b\} = \{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\} = 0$$

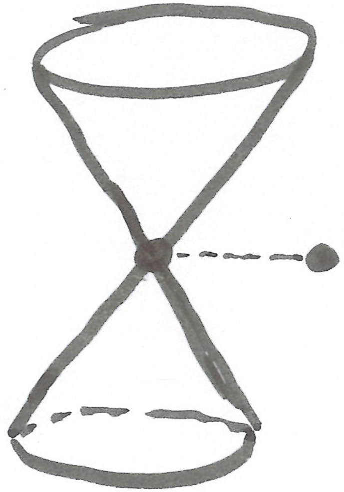
$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a_{\vec{p}}^{(s)\dagger} a_{\vec{p}}^{(s)} + b_{\vec{p}}^{(s)\dagger} b_{\vec{p}}^{(s)} \right)$$

$(a^\dagger)^2 = 0 \Rightarrow$ 1 or 0 particle / state

$a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger |0\rangle = - a_{\vec{q}}^\dagger a_{\vec{p}}^\dagger |0\rangle \Rightarrow$ antisymmetric state

$\{a^\dagger, a\} = \{a, a^\dagger\} \Rightarrow$ symmetry hole / particle

Observables for fermions



$[\psi(x), \psi^\dagger(x)] \neq 0 \quad \Rightarrow \quad \psi(x)$ not observable

$\{\psi(x), \psi^\dagger(x)\} = 0$

bilinear operators ok:

$$[ab, cd] = a\{bc\}d - ac\{bd\} + \{ac\}db - c\{ad\}b$$

cf. H, j^μ

Quantisation of gauge fields

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

no $\partial_0 A_0$!

$\Rightarrow \pi^0 = 0$

2 possibilities:

- get rid of A^0 e.g. axial gauge $A^0 = 0$
- change $\mathcal{L}$ and restrict the Hilbert space

$$\boxed{\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\lambda}{2} (\partial \cdot A)^2}$$

Euler-Lagrange

$$\Box A^\mu - (1-\lambda)\, \partial^\mu (\partial \cdot A) = 0$$

choose $\lambda = 1$ (Feynman) $\Rightarrow \boxed{\Box A^\mu = 0}$

$$A^\mu = \varepsilon^\mu(p)\, e^{-ip.x}$$

$\Rightarrow$ 4 polarisations $\quad \varepsilon^{(0)} = \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad \varepsilon_L = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \quad \varepsilon_T = \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ 1 \\ \cdot \end{pmatrix}$

if $\vec{p}$ along p_3

$$\pi_0 = -\partial . A = -(\dot{A}_0 + \vec{\nabla}.\vec{A})$$

$$\pi_i = \partial_i A_0 - \partial_0 A_i = -(\dot{A}_i - \vec{\nabla}.\vec{A})$$

$$[A_\mu, \pi_\nu] = i\, \delta^{(3)}(x-y)\, \delta_{\mu\nu}$$

$$\Rightarrow \begin{cases} [\dot{A}_\rho(x), A_\nu(y)] = i g_{\mu\nu}\, \delta^{(3)}(\vec{x}-\vec{y}) \\ [\dot{A}, \dot{A}] = [A, A] = 0 \end{cases}$$

almost 4 scalars

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2k_0}} \left[\sum_\lambda a_\lambda(k)\, \varepsilon_\mu^{(\lambda)} e^{-ik.x} + a_\lambda^\dagger \varepsilon_\mu^{(\lambda)*} e^{ik.x} \right]$$

$$\Rightarrow \quad [a_{\vec q}^{\lambda}, a_{\vec p}^{\beta\dagger}] = -g^{\lambda\beta} (2\pi)^3 \delta^{(3)}(\vec p - \vec q)$$

problem with a_0 : $a_0^\dagger |0\rangle$ has negative norm

restrict the Hilbert space

$$\partial_\mu A^{\mu +} |\psi\rangle = 0$$

$$\sim \sum \varepsilon^\lambda \cdot k \; a_\lambda \Rightarrow \lambda = 0, 3 \; :$$

$$|\psi\rangle = |\psi_T\rangle \, |\phi\rangle \Rightarrow (a_0 - a_3)|\phi\rangle = 0$$

transverse 1, 2

longitudinal + timelike 0, 3

States of indefinite norm:

$a_0^\dagger |0\rangle$ has negative norm from [ ]

but $\langle 0 | a_0 a_0^\dagger | 0 \rangle = \langle 0 | a_3 a_0^\dagger | 0 \rangle = 0$

$\boxed{\text{They do not enter observables}}$

e.g. $H = \int d^3x \, \pi_\mu \dot{A}^\mu - \mathcal{L}$

$$= \frac{1}{2} \int d^3x \sum_1^3 \dot{A}_i^2 + (\vec{\nabla}.\vec{A})^2 - A_0^2 - (\vec{\nabla} A^0)^2$$

$$= \int \frac{d^3k}{(2\pi)^3} E_k \left[(a_1^\dagger a_1 + a_2^\dagger a_2) + \underbrace{a_3^\dagger a_3 - a_0^\dagger a_0}_{\substack{\text{always } 0 \\ \text{on restricted } \mathcal{H}}} \right]$$

N.B.: off-shell photons $\Rightarrow k^0 \neq k^3 \Rightarrow$ longitudinal polarisation matters