Interacting fields

possible interactions

S matrix

evaluation in perturbation theory

relation with observables σ, Γ

Wick's theorem

Feynman rules

free $\mathcal{L}$ $\longleftrightarrow$ propagators

interaction $\mathcal{L}$ $\longleftrightarrow$ vertices

Possible interactions

causality $\rightarrow$ locality $\mathcal{L}(\phi(x))$

Lorentz $\rightarrow$ $\mathcal{L}$ is Lorentz invariant

renormalisability

$$[\mathcal{L}] = \text{GeV}^4 \qquad [\phi] = \text{GeV} \qquad [\psi] = \text{GeV}^{3/2}$$

correct dimension $\rightarrow$ renormalisable
too many GeV $\rightarrow$ non renorm.
too few GeV $\rightarrow$ super renorm.

all renormalisable terms (up to γ_5)

$\lambda\phi^4$ or $\lambda(\phi^*\phi)^2$ .... Higgs $\qquad$ $e^2\phi^*\phi A_\mu A^\mu$ ... QED

$g\bar{\psi}\psi\phi$ .... Yukawa $\qquad$ $g(A_\mu A^\mu)^2$ ... QCD

$e\bar{\psi}\gamma_\mu\psi A^\mu$ .... QED $\qquad$ $A_\mu A^\nu \partial^\mu A_\nu$ ... QCD

Gauge invariance

$$\begin{cases} \psi(x) \to e^{i\Lambda(x)}\psi(x) \\ A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \Lambda(x) \end{cases}$$ local phase

$$\boxed{D_\mu \equiv \partial_\mu + ieA_\mu}$$ gauge covariant derivative

$\Rightarrow$ interacting theory:

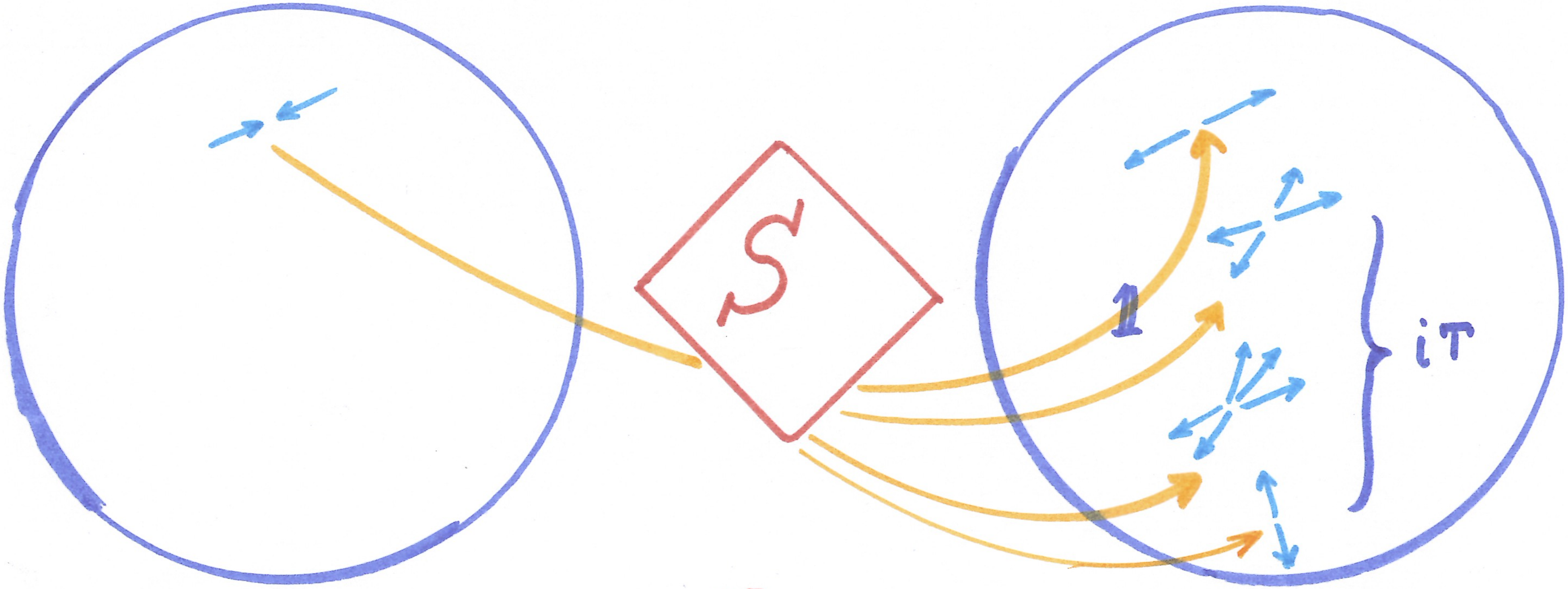
$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\gamma\cdot D - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - m^2|\phi|^2}$$

non abelian

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \to \left[e^{i\sum_{j=1}^{N}\Lambda_j(x)\tau_j} \right] \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$ $\Rightarrow$ N gauge fields

matrices

S matrix



$t = -\infty$
free states
$|i\rangle$

H_I
→ evolution

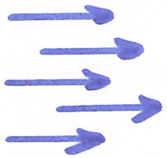
$t = +\infty$
free states
$|f\rangle$

$$|\psi\rangle = \sum_f S_{fi}\,|f\rangle$$

S unitary

$S = \mathbb{1} + iT$ ← transition matrix

Scattering cross section



flux

$cm^{-2} s^{-1}$

$\mathcal{J}$

density

cm^{-3}

ρ

cross section

$$\underbrace{dN}_{\text{number of collisions per } cm^3 \text{ and per } s} = \mathcal{J} \rho \sigma$$

$$\frac{dN}{d^3p_1 \dots d^3p_n} = \mathcal{J} \rho \frac{d\sigma}{d^3p_1 \dots d^3p_n}$$

differential cross section

σ from T

$$ {}_f\langle p_1 \cdots p_n \,|\, k_A k_B \rangle_i = \langle p_1 \cdots p_n \,|\, iT \,|\, k_A k_B \rangle $$

$$ = (2\pi)^4 \, \delta^{(4)}(k_{in} - p_{out}) \times i\mathcal{M} $$

4-mom conservation

invariant amplitude

$$ 2\pi\,\delta(k)\Big|_{k=0} = \int e^{ix.k}\, dx \Big|_{k=0} = \int dx = L $$

$$ \Rightarrow P = \int \cancel{L^3}\,\cancel{T}\,(2\pi)^4\,\delta^{(4)}(k_{in} - p_{out})\,|\mathcal{M}|^2 \overbrace{\frac{d^3p_1}{(2\pi)^3\,2E_1} \cdots \frac{d^3p_n}{(2\pi)^3\,2E_n}}^{\text{LIPS}} $$

$$ = \cancel{L^3}\,\cancel{T}\; d\sigma\; \mathcal{F}\,\rho $$

$\mathcal{F} = 2E_A v_A$

$\rho = 2E_B$

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \prod_j \frac{d^3 p_j}{(2\pi)^3 2E_j} |\mathcal{M}|^2$$

$$\times (2\pi)^4 \delta^{(4)}(k_{in} - p_{out})$$

2→2 c.m. frame:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E_A E_B |v_A - v_B|} \frac{|\vec{p}|}{(2\pi)^3 4E_{c.m.}} |\mathcal{M}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2} \quad \text{(identical masses)}$$

$$d\Gamma = \frac{1}{2M_A} \prod_j \frac{d^3 p_j}{(2\pi)^3 2E_j} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(k_{in} - p_{out})$$

width

Evaluation of $\mathcal{M}$:

$$H = H_0 + H_{int}$$

Schrödinger:

$$\begin{cases} i\partial_t |\psi\rangle = H|\psi\rangle \\ |\psi\rangle = e^{-iHt}|\psi\rangle \\ \hat{O} = \hat{O}(t=0) \end{cases}$$

Heisenberg:

$$\begin{cases} |\psi(t)\rangle = |\psi(t=0)\rangle \\ i\partial_t \hat{O} = [\hat{O}, H] \\ \hat{O}(t) = e^{iHt}\hat{O}(t=0)e^{-iHt} \end{cases}$$

Interaction:

$$|\psi(t)\rangle = e^{-iH_{int}t}|\psi(0)\rangle$$

the Hilbert space evolves because of the interaction

$$\hat{O}(t) = e^{iH_0 t}\hat{O}(0)e^{-iH_0 t}$$

the fields are the same as in free theory

$$|\psi(t=-\infty)\rangle = |i\rangle$$

$$|\psi(t=+\infty)\rangle = S|i\rangle = \sum_f S_{fi}|f\rangle$$

$$i\frac{d}{dt}|\psi(t)\rangle = H_I(t)\,|\psi(t)\rangle$$

perturbation:

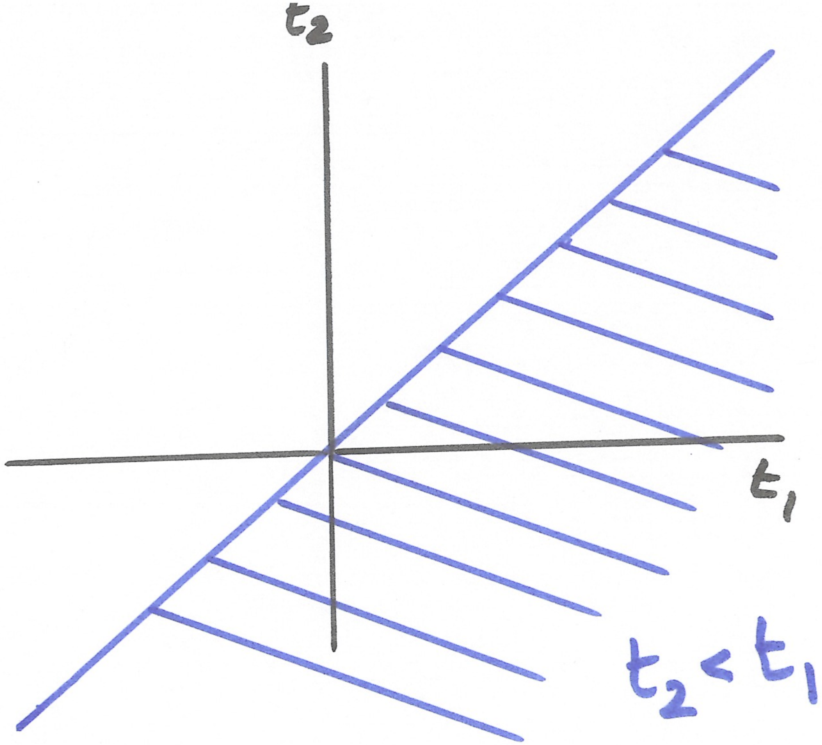
$$|\psi(t)\rangle = |i\rangle + \int_{-\infty}^{t} dt_1 \, \frac{H_I(t_1)}{i} |\psi(t_1)\rangle$$

$$= |i\rangle + \int_{-\infty}^{t} dt_1 \, \frac{H_I(t_1)}{i} |i\rangle + \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \, \frac{H_I(t_1)}{i} \frac{H_I(t_2)}{i} |\psi(t_2)\rangle$$

$$S = \sum_{n=0}^{\infty} \frac{1}{i^n} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n \, H_I(t_1) H_I(t_2) \dots H_I(t_n)$$

$$\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_2} H_I(t_1)\, H_I(t_2)\; dt_2$$

$$= \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{t_1} H_I(t_2)\, H_I(t_1)\; dt_1$$



$$= \frac{1}{2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\; T\; H_I(t_1)\, H_I(t_2)$$

time ordering

$t \longleftarrow$

$$S = \sum_{0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n \; T\left(:\mathcal{H}_I(t_1): \ldots :\mathcal{H}_I(t_k): \right)$$

$$\equiv T \exp\left(-i \int d^4x \; :\mathcal{H}_I(x): \right)$$

Wick's theorem

Every T product can be written in terms of normal-ordered products

Split the fields:

$$\psi = \psi^+ + \psi^-$$
$$A = A^+ + A^-$$
$$\phi = \phi^+ + \phi^-$$

annihilation (ϕ^+) creation (ϕ^-)

$$:\psi\psi': = \psi^+\psi'^+ + \psi^-\psi'^+ + \psi^-\psi'^- - \psi'^-\psi^+$$

$$:\phi\phi': = \phi^+\phi'^+ + \phi^-\phi'^+ + \phi^-\phi'^- + \phi'^-\phi^+$$

$$\psi\psi' - :\psi\psi': = \{\psi^+, \psi'^-\} = \langle 0|\psi\psi'|0\rangle$$

$$\phi\phi' - :\phi\phi': = [\phi^+, \phi'^-] = \langle 0|\phi\phi'|0\rangle$$

$$T(AB) - :AB: = \langle 0|TAB|0\rangle \qquad \text{propagator}$$

$$\equiv \underbrace{A \quad B}$$

theorem : sum over all possible contractions and normal-order the rest

$$T(ABC\dots XYZ) = \,:ABC\dots XYZ:$$
$$+ :\overbracket{AB}C\dots XYZ: + :A\overbracket{BC}\dots XYZ:\dots$$
$$+ :\underbracket{AB}\,\underbracket{C\dots X}YZ: + :\underbracket{ABC}\dots X\underbracket{YZ}:\dots$$
$$+ \;\dots$$

contractions at different times

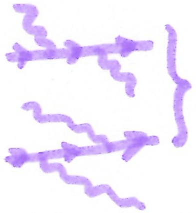
Example $\mathcal{H}_I(x) = -e :\bar{\Psi}(x)\,\gamma.A\,\Psi(x):$

1st order : $e^+e^- \rightarrow \gamma$

forbidden by 4-momentum conservation

$$2^{nd} \text{ order}: \quad -\frac{e^2}{2!}\int d^4x_1\, d^4x_2\, s = S'$$

$$s = :(\bar\psi A\cdot\gamma\psi)_1(\bar\psi A\cdot\gamma\psi)_2:$$



$$+ :(\bar\psi A\cdot\gamma\psi)_1(\bar\psi A\cdot\gamma\psi)_2: + :(\bar\psi A\cdot\gamma\psi)_1(\bar\psi A\cdot\gamma\psi)_2:$$

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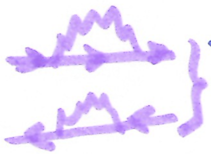

$$+ :(\bar\psi A\cdot\gamma\psi)_1(\bar\psi A\cdot\gamma\psi)_2: + :(\bar\psi A\cdot\gamma\psi)_1(\bar\psi A\cdot\gamma\psi)_2:$$

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Propagators

scalar $\quad \langle 0 | T\,\phi(x)\,\phi(y) | 0 \rangle = D_F(x-y)$

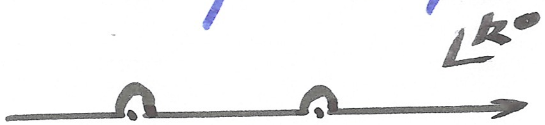
solution of $(\Box + m^2) D = -i\,\delta^{(4)}(x-y)$

general solution: $\quad D(x) = \int \frac{d^4k}{(2\pi)^4}\, D(k)\, e^{-ik\cdot x}$

$\Rightarrow (-k^2 + m^2)\, D(k) = -i \qquad D(k) = \frac{i}{k^2 - m^2}$

Fourier back: poles at $k_0 = \pm\sqrt{\vec{k}^2 + m^2}$

$\rightarrow$ prescription:



k^0

$$\int \frac{dk_0}{(2\pi)^4}\, \frac{1}{(k_0-\omega)(k_0+\omega)}\, e^{-ik_0 t}$$

retarded

$= 0$ if $t < 0$

Feynman

$k_0 > 0$ if $t > 0$ particles

$k_0 < 0$ if $t < 0$ antiparticles

$$D_F(k) = \frac{i}{k^2 - m^2 + i\varepsilon}$$

All fields obey $(\Box + m^2) \ldots = 0$
$\Rightarrow$ always the same denominator

- fermions: $\langle 0 | T \psi^\alpha(x) \bar{\psi}^\beta(y) | 0 \rangle = S_F^{\alpha\beta}(x-y)$

$$S(k) = \frac{i(\gamma \cdot k + m)}{k^2 - m^2 + i\varepsilon}$$

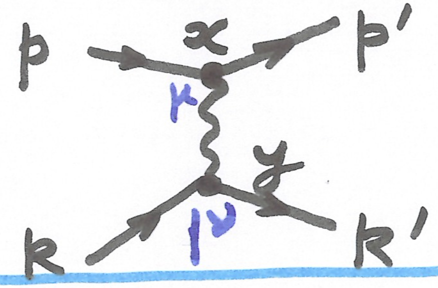
- photons: $\langle 0 | T A^\mu(x) A^\nu(y) | 0 \rangle = D_F^{\mu\nu}(x-y)$

$$D^{\mu\nu}(k) = \frac{-i g^{\mu\nu}}{k^2 + i\varepsilon}$$ (Feynman gauge)

Explicit example: $e\mu \to e\mu$

$|i\rangle = |e\mu\rangle \qquad p, k$

$|f\rangle = |e\mu\rangle \qquad p', k'$



$$S_{fi} = \langle p'k' | T \frac{(-i)^2}{2!} \int e:\bar{\psi}_e \gamma^\mu \psi_e A_\mu : d^4x \int e:\bar{\psi}_\mu \gamma^\nu \psi_\mu A_\nu : d^4y \, | pk \rangle$$

$$\psi_e |pk\rangle = \underbrace{\int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_{s'} a_{p'}^{s'} u^{s'}(p') e^{-ip'\cdot x}}_{\psi_e^+} \underbrace{\sqrt{2E_p}\, a_p^{s\dagger} |k\rangle}_{|pk\rangle}$$

$$= e^{-ip\cdot x} u^{s}(p) |k\rangle \equiv \psi(x) |pk\rangle$$

external lines

$$\langle p'k' | \bar{\psi}(x) = \langle k' | e^{ip'\cdot x} \bar{u}^s(p) \equiv \langle p'k' | \bar{\psi}(x)$$

$$S_{fi} = \frac{(-ie)^2}{2} \int d^4x \, d^4y \left[\bar{u}_e(p) \gamma^\mu u_e(p) \right] \langle 0 | T A_\mu^{(x)} A_\nu^{(y)} | 0 \rangle \left[\bar{u}_\mu(k) \gamma^\nu u_\mu(k) \right]$$

$$\times \, e^{-i(p-p')\cdot x} \quad e^{-i(k-k')\cdot y}$$

$$S_{fi} = \frac{(-ie)^2}{2} \int d^4q\, d^4x\, d^4y \;\; e^{-i(p-p')\cdot x}\; e^{-i(k-k')\cdot y}\; e^{-i q\cdot(x-y)}$$

$$[\bar{u}\,\gamma^{\mu} u]\, \frac{-i g_{\mu\nu}}{q^2+i\varepsilon}\, [\bar{u}\,\gamma^{\nu} u] \qquad A$$

$$= \int d^4q \;\underbrace{\int d^4x\; e^{-i(p-p'+q)}}_{(2\pi)^4\,\delta^{(4)}(p-p'+q)}\; \underbrace{\int d^4y\; e^{-i(k-k'-q)}}_{(2\pi)^4\,\delta^{(4)}(k-k'-q)} \quad A$$

$$= (2\pi)^4\, \delta^{(4)}(p+p'-k-k') \quad \underbrace{\frac{(-ie)^2}{2}\, [\bar{u}\,\gamma^{\nu} u]\, \frac{-i g_{\mu\nu}}{(p-p')^2+i\varepsilon}\, [\bar{u}\,\gamma^{\nu} u]}_{\equiv\, -i\,\frac{\mathcal{M}}{2}}$$

$\times 2$ for $x \rightleftarrows y$

Feynman rules for $i\mathcal{M}$:

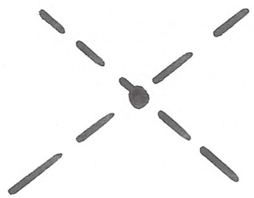
External lines:

scalars		$\phi\,\vert q\rangle$	1
fermions		$\psi\,\vert q,s\rangle$	$u^s(q)$
		$\langle q,s\vert\,\bar{\psi}$	$\bar{u}^s(q)$
antiparticles		$\langle q,s\vert\,\psi$	$v^s(q)$
		$\bar{\psi}\,\vert q,s\rangle$	$\bar{v}^s(q)$

$\bar{v}^s(q) \Longleftrightarrow \bar{u}^s(-q)$ $\quad v^s(q) \Longleftrightarrow u^s(-q)$

photons		$\varepsilon_\mu(q)$	$A_\mu\,\vert q\rangle$
		$\varepsilon^*_\mu(q)$	$\langle q\vert\,A_\mu$

Vertices



$-i\lambda$ $\qquad \mathcal{L}_I = \frac{1}{4!}\lambda\phi^4$

$-ig$ $\qquad \mathcal{L}_I = g\bar{\psi}\psi\phi$

$-ie\gamma^\mu$ $\qquad \mathcal{L}_I = -Q\bar{\psi}\gamma\cdot A\psi$

$Q_e = -e$

$-ie(p+p')^\mu$

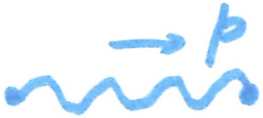
$\mathcal{L} \sim (D_\mu\phi)^*(D^\mu\phi)$

$2ie^2g^{\mu\nu}$

Propagators :

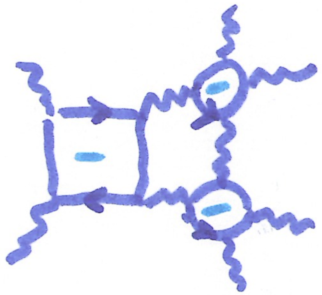
scalars  $\dfrac{i}{p^2 - m^2 + i\varepsilon}$

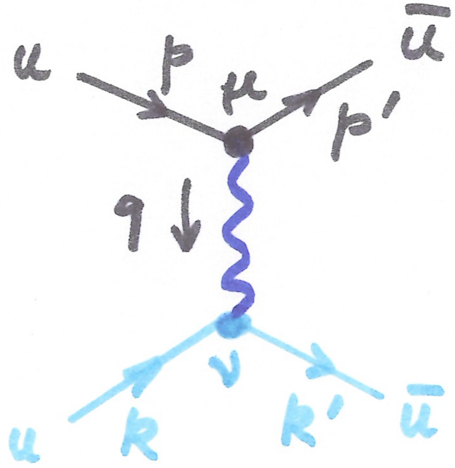
fermions  $\dfrac{i\,(p \cdot \gamma + m)}{p^2 - m^2 + i\varepsilon}$

photons  $\dfrac{-i g_{\mu\nu}}{p^2 + i\varepsilon}$

Loops

-1 per fermion loop



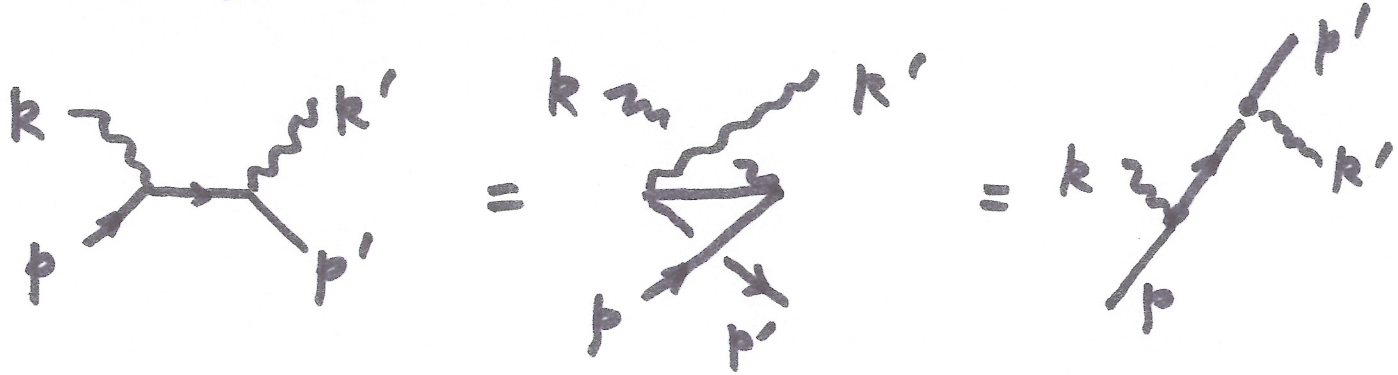


$$= \bar{u}(p')(-ie\gamma^{\mu})\,u(p)\,\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\,\bar{u}(k')(-ie\gamma^{\nu})\,u(k)$$

$$= i\mathcal{M}$$

Remarks

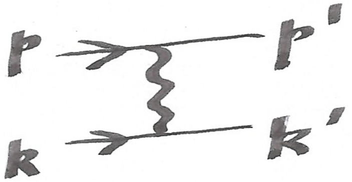
No time axis:



crossing symmetry: e^- with momentum $-p$

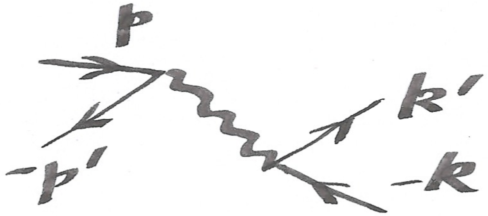
$= e^+$ for $\mathcal{M}$, not σ !

$\mathcal{M}(p,p';k,k') =$



elastic $e\mu$

$\mathcal{M}(p,-p';-k,k') =$



annihilation

$e^+e^- \to \mu^+\mu^-$