II.3. Let us introduce the Weyl fields: $\begin{cases} \psi_L = \frac{1}{2}(1-\gamma_5)\psi \\ \psi_R = \frac{1}{2}(1+\gamma_5)\psi \end{cases}$

where ψ is a Dirac Spinor

Derive the equation of motion for these fields. Show that they are decoupled in the case of massless spinors.

We use the γ-matrices of Weyl that are defined by $\gamma^\mu = \begin{pmatrix} \cdot & \sigma^\mu \\ \bar{\sigma}^\mu & \cdot \end{pmatrix}$

where we introduce the notation

$\sigma^\mu = (\mathbb{1}, \vec{\sigma})$

$\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$
is a "vector" of Pauli matrices

- Pauli matrice verify $(\sigma^i)^2 = \mathbb{1}_2$
- γ-matrices are 4×4 matrices and verify the algebra: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

From the 4 γ-matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$, we define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
with the property: $\{\gamma^5, \gamma^{\overset{(0,1,2,3)}{\mu}}\} = 0 \Leftrightarrow \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$

[Note that γ-matrices in Dirac and Weyl representation are not exactly the same.

Dirac spinor equation of motion is the Dirac equation: $(i\not{\partial} - m)\psi = 0$

$\Leftrightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$

We are looking for equation of motion of the Weyl spinor ψ_L and ψ_R defined as:

$$\begin{cases} \psi_L = \frac{1}{2}(1-\gamma_5)\psi & (1) \\ \psi_R = \frac{1}{2}(1+\gamma_5)\psi & (2) \end{cases}$$

We see that $\underline{(1)+(2)} \Rightarrow \psi_L + \psi_R = \psi$

$\underline{(1)-(2)} \Rightarrow \psi_L - \psi_R = -\gamma_5 \psi$

it is thus a good idea to find the equation of motion for $\gamma_5 \psi$.

From Dirac equation: $\gamma_5 \left[(i\not{\partial} - m)\psi = 0 \right]$

$$\Leftrightarrow (i\gamma_5 \not{\partial} - m\gamma_5)\psi = 0$$

$$\Leftrightarrow (i\gamma_5 \gamma_\mu \partial^\mu - m\gamma_5)\psi = 0$$

$$\Leftrightarrow (-i\gamma_\mu \gamma_5 \partial^\mu - m\gamma_5)\psi = 0$$

$$\Leftrightarrow -(i\not{\partial} + m)\gamma_5 \psi = 0$$

Thus, if we apply the Dirac equation to $\psi_L + \psi_R$ and $\psi_L - \psi_R$ we write:

$$\begin{cases} [i\not{\partial} - m](\psi_L + \psi_R) = 0 & \boxed{*} \\ [i\not{\partial} + m](\psi_L - \psi_R) = 0 & \boxed{**} \text{ from above} \end{cases}$$

and $\boxed{*} + \boxed{**} \rightarrow i\not{\partial}\psi_L + i\not{\partial}\psi_R - m\psi_L - m\psi_R$

$$+ i\not{\partial}\psi_L - i\not{\partial}\psi_R + m\psi_L - m\psi_R = 0$$

$$\hookrightarrow \boxed{i\not{\partial}\psi_L = m\psi_R}$$

$\boxed{*} - \boxed{**} \rightarrow i\not{\partial}\psi_L + i\not{\partial}\psi_R - m\psi_L - m\psi_R$

$$- i\not{\partial}\psi_L + i\not{\partial}\psi_R - m\psi_L + m\psi_R = 0$$

$$\hookrightarrow \boxed{i\not{\partial}\psi_R = m\psi_L}$$

Eq. for ψ_L and ψ_R are clearly decoupled when $m = 0$.

$\underline{II.5}$. Compute the commutators $[\phi(x), \phi(x')]_{t=t'}$ and $[\pi(x), \phi(x')]_{t=t'}$

- $$\phi(x) = \frac{1}{(2\pi)^3} \int_{k_0>0} \frac{d^3k}{\sqrt{2k_0}} \left(a_{\vec{k}} \, e^{-i\cancel{kx}} + a_{\vec{k}}^\dagger \, e^{i\vec{k}.\vec{x}} \right) \qquad k.x$$

- $$\pi(x) \cancel{=} = \frac{-ik_0}{(2\pi)^3} \int_{k_0>0} \frac{d^3k}{\sqrt{2k_0}} \left(a_{\vec{k}} \, e^{-ikx} - a_{\vec{k}}^\dagger e^{ikx} \right)$$

and we have the identities: $[a, a] = 0 = [a^\dagger, a^\dagger]$

$$[a, a^\dagger] = 1.$$

and more specifically: $[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$

$$\begin{cases} a_k \equiv \text{Annihilation Operator} \\ a_k^\dagger \equiv \text{Creation Operator.} \end{cases}$$

Other way of writing $\phi(x)$:

$$\phi(x) = \frac{1}{(2\pi)^3} \left\{ \int \frac{d^3k}{\sqrt{2k_0}} \, a_{\vec{k}} \, e^{-i\vec{k}\vec{x}} + \int \frac{d^3k}{\sqrt{2k_0}} \, a_k^\dagger \, e^{+i\vec{k}\vec{x}} \right\}$$

$k \to -k$

$dk \to -dk$

then integral $-\infty \to \infty$ becomes $+\infty \to -\infty$

Add "$-$" to restor the things

$$\hookrightarrow \phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2k_0}} \left(a_{\vec{k}} \, e^{-i\vec{k}.\vec{x}} + a_{\vec{k}}^\dagger \, e^{-i\vec{k}.\vec{x}} \right)$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2k_0}} \, e^{-i\vec{k}.\vec{x}} \left(a_{\vec{k}} + a_{-\vec{k}}^\dagger \right)$$

$$\pi(x) = \frac{-ik_0}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2k_0}} \, e^{-ik.x} \left(a_{\vec{k}} \pm a_{-\vec{k}}^\dagger \right)$$

This one

$$[\phi(x), \phi(y)]_{t=t'} = \frac{1}{(2\pi)^6} \int \frac{d^3k}{\sqrt{2k_0}} \int \frac{d^3p}{\sqrt{2p_0}} \; e^{-ikx} \, e^{-ip\cdot y}$$

$$\cdot \Big\{ \underbrace{\big[(a_k + a^\dagger_{-k}), (a_p + a^\dagger_{-p}) \big]} \Big\}$$

$$a_k a_p + a_k a^\dagger_{-p} + a^\dagger_{-k} a_p + a^\dagger_{-k} a^\dagger_{-p}$$

$$\underbrace{- a_p a_k}_{=0} - a^\dagger_{-p} a_k - a_p a^\dagger_{-k} \underbrace{- a^\dagger_{-p} a^\dagger_{-k}}_{=0}$$

$$= \frac{1}{(2\pi)^6} \int \frac{d^3k}{\sqrt{2k_0}} \int \frac{d^3p}{\sqrt{2p_0}} \; e^{-ikx} e^{-ipy} \Big\{ + \underbrace{[a_k, a^\dagger_{-p}]}_{(2\pi)^3 \delta^{(3)}(k+p)} - \underbrace{[a_p, a^\dagger_{-k}]}_{(2\pi)^3 \delta^{(3)}(p+k)} \Big\}$$

$$\cancel{= \frac{-1}{(2\pi)^3} \int \frac{d^3k}{(2k_0)} \Big\{ e^{-ikx} e^{-i(-ky)} + \text{same} \Big\}}$$

$$\cancel{= \frac{-1}{(2\pi)^3} \int \frac{d^3k}{2k_0} e^{-ik(x-y)}}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2k_0}} \int \frac{d^3p}{\sqrt{2p_0}} \; e^{-ikx} e^{-ipy} \, (2\pi)^3 \underbrace{\left(\delta^{(3)}(k+p) - \delta^{(3)}(p+k) \right)}_{=0}$$

$\hookrightarrow$ $[\phi(x), \phi(y)]_{t=t'} = 0$ ok.

$$\phi(\vec{x}) = \int \frac{d^3p}{\sqrt{2p_0}} \frac{1}{(2\pi)^3} \left(a_{\vec{p}}\, e^{ip.x} + a^+_{\vec{p}}\, e^{-ip.x} \right)$$

$$\pi(\vec{x}) = \frac{-ip_0}{(2\pi)^3} \int \frac{d^3p}{\sqrt{2p_0}} \left(a_{\vec{p}}\, e^{i\vec{p}\vec{x}} - a^+_{\vec{p}}\, e^{-i\vec{p}\vec{x}} \right)$$

$$[\phi(x), \phi(y)] = \frac{1}{(2\pi)^6} \int \frac{d^3p}{\sqrt{2p_0}} \int \frac{d^3k}{\sqrt{2k_0}} \left[\left(a_p e^{ip.x} + a^+_p e^{-ipx}\right), \left(a_k e^{iky} + a^+_k e^{-iky}\right) \right]$$

$$= 0 \quad \text{ok}$$

$$[\phi(x), \pi(y)]_{t=t'} = \frac{ik_0}{(2\pi)^6} \int \frac{d^3p}{\sqrt{2p_0}} \int \frac{d^3k}{\sqrt{2k_0}}$$

$$\underbrace{\left(\left[\left(a_p e^{ipx} + a^+_{-p} e^{ipx}\right), \left(a_k e^{iky} - a^+_{-k} e^{iky}\right) \right] \right)}$$

$$e^{ipx} e^{ipy} \left(-a_p a^+_{-k} + a^+_{-p} a_k - a_k a^+_{-p} + a^+_{-k} a_p \right)$$

$$= e^{ipx} e^{ipy} \left([a^+_{-k}, a_p] + [a^+_{-p}, a_k] \right)$$

$$= - e^{ipx} e^{ipy} \left([a_p, a^+_{-k}] + [a_k, a^+_{-p}] \right)$$

$$= - e^{ipx} e^{ipy} \left((2\pi)^3 \delta^{(3)}(\underbrace{p+k}_{\to\, p=-k}) + (2\pi)^3 \delta^{(3)}(k+p) \right)$$

$$[\phi(x), \pi(y)]_{t=t'} = \frac{-i\cancel{k_0}}{(2\pi)^3} \int \frac{d^3k}{\cancel{2}\cancel{k_0}} e^{-ik(x-y)} \cancel{2}$$

$$= \frac{-i}{(2\pi)^3} \int d^3k\, e^{-ik(x-y)} = -i\delta^{(3)}(x-y)$$

II. 6. Interpret the following Lagrangian densities and find the corresponding Euler-Lagrange equations:

(i). $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$

(ii) $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4$

(iii) $\mathcal{L} = (\partial_\mu \phi - ie A_\mu \phi)(\partial^\mu \phi^* + ie A^\mu \phi^*) - m^2 \phi\phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

(iv) $\mathcal{L} = \bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \frac{1}{4!}\lambda\phi^4 - ig\bar{\Psi}\gamma_5\Psi\phi$

(i) $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$

↳ Lagrangian density for a massive vector field A^μ

Euler-Lagrange equation: $\boxed{\partial^\alpha \frac{\partial \mathcal{L}}{\partial \partial^\alpha A^\beta} - \frac{\partial \mathcal{L}}{\partial A^\beta} = 0}$

- $$\frac{\partial \mathcal{L}}{\partial A^\beta} = \frac{\partial}{\partial A^\beta}\left(\frac{m^2}{2} A_\mu A^\mu\right)$$
$$= \frac{m^2}{2} \frac{\partial}{\partial A^\beta}\left(g_{\mu\nu} A^\nu A^\mu\right) = \frac{m^2}{2} g_{\mu\nu}\left(\frac{\partial A^\nu}{\partial A^\beta} A^\mu + A^\nu \frac{\partial A^\mu}{\partial A^\beta}\right)$$
$$= \frac{m^2}{2} g_{\mu\nu}\left(\delta^\nu_\beta A^\mu + A^\nu \delta^\mu_\beta\right) = \frac{m^2}{2}\left(g_{\mu\beta} A^\mu + g_{\beta\nu} A^\nu\right)$$
$$= m^2 A_\beta$$

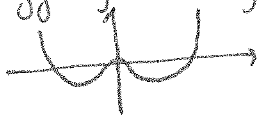
- $$\frac{\partial \mathcal{L}}{\partial \partial^\alpha A^\beta} = -\frac{1}{4} \frac{\partial}{\partial \partial^\alpha A^\beta}\left(F_{\mu\nu} F^{\mu\nu}\right) = -\frac{1}{4} \frac{\partial\left(g_{\mu\sigma} g_{\nu\rho} F^{\sigma\rho} F^{\mu\nu}\right)}{\partial \partial^\alpha A^\beta}$$
$$= -\frac{1}{4}\left(g_{\mu\sigma} g_{\nu\rho} \frac{\partial F^{\sigma\rho}}{\partial \partial^\alpha A^\beta} F^{\mu\nu} + g_{\mu\sigma} g_{\nu\rho} F^{\sigma\rho} \frac{\partial F^{\mu\nu}}{\partial \partial^\alpha A^\beta}\right)$$
$$= -\frac{1}{4}\left(g_{\mu\sigma} g_{\nu\rho} F^{\mu\nu}\left(\delta^\sigma_\alpha \delta^\rho_\beta - \delta^\sigma_\beta \delta^\rho_\alpha\right) + g_{\mu\sigma} g_{\nu\rho} F^{\sigma\rho}\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)\right)$$
$$= -\frac{1}{4}\left(F_{\alpha\beta} - F_{\beta\alpha} + F_{\alpha\beta} - F_{\beta\alpha}\right)$$
$$= -F_{\alpha\beta} = -\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)$$

E-L: $-\left(\Box A_\beta - \partial_\beta \partial^\alpha A_\alpha\right) - m^2 A_\beta = 0 \quad \Rightarrow \quad \boxed{\left(\Box + m^2\right) A_\beta = \partial_\beta \partial^\alpha A_\alpha}$

(ii) : $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \frac{1}{4}\lambda \phi^4$ $\qquad [\phi] = GeV$

$[\mathcal{L}] = GeV^4$

↳ Lagrangian density of a scalar field having mass m and that self-interact with a coupling constant λ

This can be the Higgs field if $m^2 < 0$ to have the Mexican hat 

- $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi + \lambda \phi^3$
- $\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} = \frac{2}{2} \partial^\alpha \phi$

indeed: $\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} = \frac{1}{2} \frac{\partial}{\partial \partial_\alpha \phi} \left((\partial_\mu \phi)(\partial^\mu \phi) \right)$

$$= \frac{1}{2}\left\{ \delta_{\mu\alpha} (\partial^\mu \phi) + (\partial_\mu \phi) \frac{\partial g^{\mu\nu} \partial_\nu \phi}{\partial \partial_\alpha \phi} \right\}$$

$$= \frac{1}{2}\left\{ \partial^\alpha \phi + (\partial^\nu \phi)\, \delta_{\nu\alpha} \right\} = \partial^\alpha \phi .$$

<u>Euler - Lagrange:</u> $\quad \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

$$\Leftrightarrow \partial_\alpha \left[\partial^\alpha \phi \right] - \lambda \phi^3 + m^2 \phi = 0$$

$$\Leftrightarrow \boxed{(\Box + m^2)\, \phi = \lambda \phi^3}$$

↳ Klein - Gordon equation with a damping term which is due to the self-interaction.

(iii) $\mathcal{L} = (\partial_\mu \phi - ieA_\mu \phi)(\partial^\mu \phi^* + ieA^\mu \phi^*) - m^2 \phi\phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$\hookrightarrow$ Lagrangian density of electrodynamics with a covariant derivative for a massive scalar field and photon

$$(1a): \frac{\partial \mathcal{L}}{\partial \phi} = (-ieA_\mu)(\partial^\mu \phi^* + ieA^\mu \phi^*) - m^2 \phi^*$$

$$(2a): \frac{\partial \mathcal{L}}{\partial \phi^*} = (ieA^\mu)(\partial_\mu \phi - ieA_\mu \phi) - m^2 \phi$$

$$(3a): \frac{\partial \mathcal{L}}{\partial A_\beta} = (-ie\phi)(\partial^\beta \phi^* + ieA^\beta \phi^*) + (ie\phi^*)(\partial^\beta \phi - ieA^\beta \phi)$$

$$(1b): \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} = \partial^\alpha \phi^* + ieA^\alpha \phi^*$$

$$(2b): \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^*} = \partial^\alpha \phi - ieA^\alpha \phi$$

$$(3b): \frac{\partial \mathcal{L}}{\partial \partial_\alpha A_\beta} = -F_{\alpha\beta} \quad (\text{from earlier})$$
$$= \partial_\beta A_\alpha - \partial_\alpha A_\beta$$

We will have 3 different sets of Euler-Lagrange equations

Before, let me define
$$\begin{cases} D_\mu = \partial_\mu - ieA_\mu \\ D^*_\mu = \partial_\mu + ieA_\mu \end{cases}$$

and introduce the notation:
$$\begin{cases} \tilde{\Box} = D_\mu D^\mu \\ \tilde{\Box}^* = D^*_\mu D^{*\mu} \end{cases}$$

I: $\partial_\alpha \dfrac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} - \dfrac{\partial \mathcal{L}}{\partial \phi} = 0$

$\hookrightarrow \partial_\alpha (\partial^\alpha \phi^* + ie A^\alpha \phi^*) + ie A_\mu (\partial^\mu \phi^* + ie A^\mu \phi^*) + m^2 \phi^* = 0$

$\Leftrightarrow \left[(\partial_\alpha + ie A_\alpha)(\partial^\alpha + ie A^\alpha) + m^2 \right] \phi^* = 0$

$\Leftrightarrow \quad \left[D^*_\alpha D^{*\alpha} + m^2 \right] \phi^* = 0$

$\Leftrightarrow \quad \left[\tilde{\Box}^* + m^2 \right] \phi^* = 0$

$\hookrightarrow$ "Klein-Gordon"-like equation with covariante derivative.

II: $\partial_\alpha \dfrac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^*} - \dfrac{\partial \mathcal{L}}{\partial \phi^*} = 0$

$\hookrightarrow \partial_\alpha (\partial^\alpha \phi - ie A^\alpha \phi) + m^2 \phi - ie A^\mu (\partial_\mu \phi - ie A_\mu \phi) = 0$

$\Leftrightarrow \left[D_\alpha D^\alpha + m^2 \right] \phi = 0$

$\Leftrightarrow \left[\tilde{\Box} + m^2 \right] \phi = 0$

III: $\partial^\alpha \dfrac{\partial \mathcal{L}}{\partial \partial^\alpha A^\beta} - \dfrac{\partial \mathcal{L}}{\partial A^\beta} = 0$

$\hookrightarrow \partial^\alpha (\partial_\beta A_\alpha - \partial_\alpha A_\beta) + ie \phi (\partial^\beta \phi^* + ie A^\beta \phi^*)$
$\qquad - ie \phi^* (\partial^\beta \phi - ie A^\beta \phi) = 0$

$\Leftrightarrow \partial_\beta \partial^\alpha A_\alpha - \Box A_\beta = ie \left[\phi^* D^\beta \phi - \phi D^{*\beta} \phi^* \right]$

(iv) $\mathcal{L} = \underbrace{\bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi}_{\text{fermion Lagrangian}} + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \frac{1}{4!}\lambda \phi^4 - ig\underbrace{\bar{\Psi}\gamma_5\Psi}_{\text{pseudo-scalar}}\phi$

pseudo-scalar → has to be pseudo-scalar

↳ This is Lagrangian density for the interaction between fermion and massive pseudo-scalar field ϕ (such like pion) which is self-interacting

For instance, if Ψ are protons then ϕ can be pion

(1a): $\dfrac{\partial \mathcal{L}}{\partial \Psi} = -m\bar{\Psi} - ig\bar{\Psi}\gamma_5\phi$

(2a): $\dfrac{\partial \mathcal{L}}{\partial \bar{\Psi}} = -m\Psi - ig\gamma_5\Psi\phi$

(3a): $\dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi + \frac{1}{3!}\lambda\phi^3 - ig\bar{\Psi}\gamma_5\Psi$

(1b): $\dfrac{\partial \mathcal{L}}{\partial \partial_\alpha \Psi} = i\bar{\Psi}\gamma^\alpha$

(2b): $\dfrac{\partial \mathcal{L}}{\partial \partial_\alpha \bar{\Psi}} = i\gamma^\alpha\Psi$

(3b): $\dfrac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} = \partial_\alpha \phi$

<u>Euler-Lagrange</u>:

I : $i\bar{\Psi}\partial_\alpha\gamma^\alpha + \bar{\Psi}m + ig\bar{\Psi}\gamma_5\phi = 0$

$\bar{\Psi}[i\partial_\alpha\gamma^\alpha + m] = -ig\bar{\Psi}\gamma_5\phi$

II : $[i\partial_\alpha\gamma^\alpha + m]\Psi = -ig\gamma_5\Psi\phi$

III : $[\Box + m^2]\phi = \frac{1}{3!}\lambda\phi^3 - ig\bar{\Psi}\gamma_5\Psi$

II.8. For photons, compute the commutator $[a^\lambda, a^{\lambda\dagger}]$ from the canonical field commutations

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The fields have the form:

$$A(t,\vec{x}) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2k_0}} \sum_\lambda \left\{ a_k^\lambda \varepsilon_\mu^\lambda e^{-ik.x} + a_k^{\lambda\dagger} \varepsilon_\mu^{\lambda *} e^{ikx} \right\}$$

In the second term, make

$$\vec{k} \longrightarrow -\vec{k}$$
$$dk \longrightarrow -dk$$
$$\int_{-\infty}^{+\infty} \longrightarrow \int_{+\infty}^{-\infty} = -\int_{-\infty}^{+\infty}$$

thus: $$A_\mu(t,\vec{x}) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2k_0}} \sum_\lambda \left\{ a_k^\lambda \varepsilon_\mu^\lambda e^{-ikx} + a_{-k}^{\lambda\dagger} \varepsilon_\mu^{\lambda *} e^{-ikx} \right\}$$

And: $$\dot{A}_\mu(t,\vec{x}) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2k_0}} \sum_\lambda \left\{ (-ik_0)\, a_k^\lambda \varepsilon_\mu^\lambda e^{-ikx} + (ik_0)\, a_k^{\lambda\dagger} \varepsilon_\mu^{\lambda *} e^{ik.x} \right\}$$

conjugate momentum in Feynman gauge

$$= \frac{-i}{(2\pi)^3}\int \frac{d^3k\, k_0}{\sqrt{2k_0}} \sum_\lambda \left\{ a_k^\lambda \varepsilon_\mu^\lambda e^{-ik.x} - a_k^{\lambda\dagger} \varepsilon_\mu^{\lambda *} e^{ik.x} \right\}$$

make change in the 2nd term $\vec{k} \to -\vec{k}$

blablabla

$$= \frac{-i}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2k_0}}\, k_0 \sum_\lambda \left\{ a_k^\lambda \varepsilon_\mu^\lambda - a_{-k}^{\lambda\dagger} \varepsilon_\mu^{\lambda *} \right\} e^{-ikx}$$

Go to Fourier space for A and $\dot{A}$

$$\boxed{*} \quad \int d^3x\, A_\mu(t,\vec{x})\, e^{ikx} = \frac{1}{\sqrt{2k_0}} \sum_\lambda \left\{ a_k^\lambda \varepsilon_\mu^\lambda + a_{-k}^{\lambda\dagger} \varepsilon_\mu^{\lambda *} \right\}$$

$$\boxed{**} \quad \int d^3x\, \dot{A}_\mu(t,\vec{x})\, e^{ikx} = \frac{-ik_0}{\sqrt{2k_0}} \sum_\lambda \left\{ a_k^\lambda \varepsilon_\mu^\lambda - a_{-k}^{\lambda\dagger} \varepsilon_\mu^{\lambda *} \right\}$$

Compute

$\boxed{*} + \frac{i}{k_0}\boxed{**} \Rightarrow \sqrt{\frac{2}{k_0}} \sum_\lambda a_k^\lambda \varepsilon_\mu^\lambda = \int d^3x \left\{ A(t,\vec{x}) + \frac{i}{k_0}\dot{A}(t,\vec{x}) \right\} e^{ikx}$

$\boxed{*} - \frac{i}{k_0}\boxed{**} \Rightarrow \sqrt{\frac{2}{k_0}} \sum_\lambda a_{-k}^{\lambda\dagger} \varepsilon_\mu^\lambda = \int d^3x \left\{ A(t,\vec{x}) - \frac{i}{k_0}\dot{A}(t,\vec{x}) \right\} e^{ikx}$

because we know the commutation relation between the field A_μ and its conjugate momentum $\dot{A}_\mu$:

$$[A_\mu(x), \dot{A}_\nu(y)] = -i g_{\mu\nu} \delta^{(3)}(\vec{x}-\vec{y})$$

and $[A, A] = [\dot{A}, \dot{A}] = 0$

$$\left[\sum_\lambda a_k^\lambda \varepsilon_\mu^\lambda \, , \, \sum_{\lambda'} a_p^{\lambda' \dagger} \varepsilon_\nu^{\lambda' *}\right]$$

$$= \frac{\sqrt{k_0 p_0}}{2} \int d^3x \int d^3y \left[\left(A_\mu + \frac{i}{k_0}\dot{A}_\mu\right) e^{ikx} , \left(A_\nu - \frac{i}{p_0}\dot{A}_\nu\right) e^{-ipy} \right]$$

$$= \frac{\sqrt{k_0 p_0}}{2} \int d^3x \int d^3y \; e^{i(kx-py)} \underbrace{\left[\left(A_\mu + \frac{i}{k_0}\dot{A}_\mu\right) , \left(A_\nu - \frac{i}{p_0}\dot{A}_\nu\right) \right]}_{C}$$

$$C = \underbrace{[A_\mu, A_\nu]}_{=0} + -\frac{i}{p_0} \underbrace{[A_\mu, \dot{A}_\nu]}_{=-i g_{\mu\nu} \delta^{(3)}(x-y)} + \frac{i}{k_0} \underbrace{[\dot{A}_\mu, A_\nu]}_{= i g_{\mu\nu} \delta^{(3)}(x-y)} + \frac{1}{k_0 p_0} \underbrace{[\dot{A}_\mu, \dot{A}_\nu]}_{=0}$$

$$= -\frac{i}{p_0}(-i) g_{\mu\nu} \delta^{(3)}(k-p) + \frac{i}{k_0} i g_{\mu\nu} \delta^{(3)}(p-k)$$

$$= \frac{\sqrt{k_0 p_0}}{2} \int d^3x \int d^3y \left\{ \frac{-1}{p_0} \delta^{(3)}(x-y) - \frac{1}{k_0} \delta^{(3)}(x-y) \right\} g_{\mu\nu} \, e^{i(kx-py)}$$

$$= \frac{\sqrt{k_0 p_0}}{2} \int d^3x \cancel{\int d^3y} \left(-\frac{1}{k_0} e^{ix(k-p)} - \frac{1}{k_0} e^{ix(k-p)} \right) g_{\mu\nu}$$

$$= - \int d^3x \cancel{\int d^3y} \; e^{+ix(k-p)} g_{\mu\nu}$$

$$= - \int d^3x \; e^{ix(k-p)} g_{\mu\nu}$$

$$= - g_{\mu\nu} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})$$

$$\left[\sum_\lambda a_k^\lambda \varepsilon_\mu^\lambda \, , \, \sum_{\lambda'} a_p^{\lambda' \dagger} \varepsilon_\nu^{\lambda' *}\right] = - g_{\mu\nu} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})$$

$$= \sum_{\lambda\lambda'} \left(a_k^\lambda a_p^{\lambda'\dagger} \varepsilon_\mu^\lambda \varepsilon_\nu^{\lambda' *} - a_p^{\lambda'\dagger} a_k^\lambda \varepsilon_\mu^\lambda \varepsilon_\nu^{\lambda' *} \right)$$

and $\varepsilon_\mu^\lambda \varepsilon_\nu^{\lambda'} = g_{\mu\nu} g^{\lambda\lambda'}$

$$= g_{\mu\nu} g^{\lambda\lambda'} \left(a_k^\lambda a_p^{\lambda'\dagger} - a_p^{\lambda'\dagger} a_k^\lambda \right)$$

$$\boxed{\Rightarrow [a_k^\lambda, a_p^{\lambda'\dagger}] = - g^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})}$$

↳ Negative !

Quantization of gauge fields

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ $\Rightarrow$ There is no $\partial^0 A^0$! $\Rightarrow \pi^0 = 0$

<u>2 possibilities</u>: * get rid of A^0 $\to$ e.g. Axial gauge $A^0 = 0$.

∘ change $\mathcal{L}$ and restrict the Hilbertspace

$$\hookrightarrow \boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial \cdot A)^2}$$

$\hookrightarrow$ Then, Euler-Lagrange eq. read:

$$\Box A^\mu - (1-\lambda) \partial^\mu (\partial \cdot A) = 0 \qquad \Longrightarrow \pi^0 = -\lambda (\partial \cdot A)$$

<u>Feynman gauge</u>: $\underline{\lambda = 1}$ $\to$ E-L give: $\Box A^\mu = 0$.

Then: $A^\mu = \varepsilon^\mu(p) \, e^{-ip\cdot x}$

with 4 polarizations: $\varepsilon^{(0)} = \begin{pmatrix} 1 \\ \vdots \end{pmatrix}$

$\left\{ \begin{array}{l} \cdot \text{longitudinal}: \ \varepsilon_L = \begin{pmatrix} \vdots \\ 1 \end{pmatrix} \\ \cdot \text{transverse}: \ \varepsilon_T = \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix} ; \begin{pmatrix} \cdot \\ \cdot \\ 1 \\ \cdot \end{pmatrix} \end{array} \right.$

$\hookrightarrow$ if $\vec{p}$ is along p_3.

<u>Then</u>: $\left\{ \begin{array}{l} \pi_0 = -\partial \cdot A = -(\dot{A}_0 + \vec{\nabla} \cdot \vec{A}) \\ \pi_i = \partial_i A_0 - \partial_0 A_i \end{array} \right.$

and: $[A_\mu, \pi_\nu] = i \, \delta^{(3)}(\vec{x} - \vec{y}) \, \delta_{\mu\nu}$

become: $[\dot{A}_\mu(x), A_\nu(y)] = i g_{\mu\nu} \, \delta^{(3)}(\vec{x} - \vec{y})$

$[\dot{A}, \dot{A}] = [A, A] = 0$

Almost
4 scalars.

The fields take the form:

$$A_\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2k_0}} \sum_\lambda \left\{ a_\lambda(k) \, \varepsilon_\mu^{(\lambda)} e^{-ikx} + a_\lambda^\dagger(k) \, \varepsilon_\mu^{(\lambda)*} e^{ikx} \right\}$$

and $[a^\lambda_{\vec{q}}, a^{\beta\dagger}_{\vec{p}}] = -g^{\lambda\beta} (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{q})$