QED cross sections

$e^-\mu^- \to e^-\mu^-$

- spin average and sum
- $|\overline{\mathcal{M}}|^2$ and crossing
- cross sections
- Mandelstam invariants

$e^+e^- \to e^+e^-$

$e^-\gamma \to e^-\gamma$

cutting rules

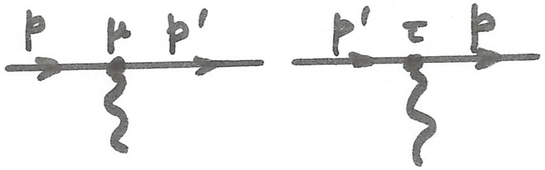
Spin sums

$$\left.\begin{array}{l} \sum_s u^s_\alpha(p)\,\bar{u}^s_\beta(p) = \gamma^\mu_{\alpha\beta}\, p_\mu + m\,\delta_{\alpha\beta} \\[1em] \sum_\lambda \varepsilon^\lambda_\mu(p)\,\varepsilon^{\lambda *}_\nu(p) = g_{\mu\nu} \end{array}\right\} \ \tfrac{1}{i} \times \text{top of propagator}$$

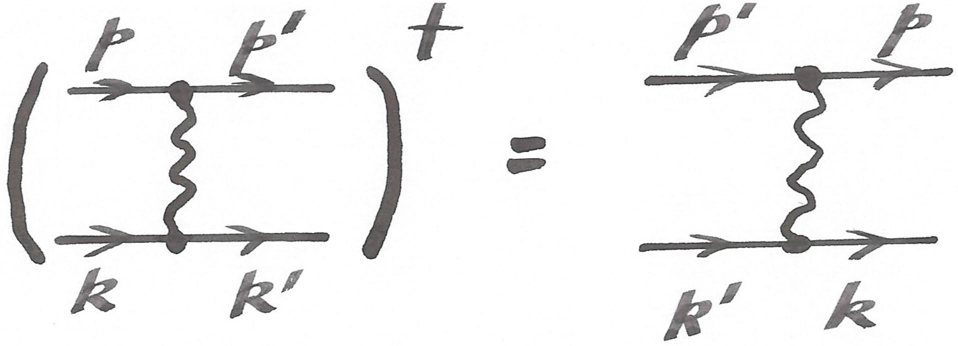
unpolarised beams $\left(\frac{1}{2}\right)^2 \sum_{s_i}$ for pp

no spin analysis $\sum_{s_f}$

$$|\bar{\mathcal{M}}|^2 \sim \frac{1}{4} \sum_{s_1 s_2} \sum_{s_3 s_4} \left[\bar{u}(p', s_1)\,\gamma^\mu\, u(p, s_2)\right]\left[u^\dagger(p, s_2)\,\underbrace{\gamma^{\tau\dagger}}_{\gamma_0\gamma^\tau\gamma_0}\,\underbrace{\gamma_0^\dagger}_{\gamma_0}\, u(p', s_1)\right]\ldots$$



$$\left[\bar{u}(p, s_2)\,\gamma^\tau\, u(p', s_1)\right]$$



→ calculate $|\bar{\mathcal{M}}|^2$ directly

$$|\bar{\mathcal{M}}|^2 = \frac{1}{4}\sum_{s_1 s_2 s_3 s_4}\left(\frac{e^2}{q^2+i\varepsilon}\right)^2\left[\bar{u}(p',s_1)\gamma^\mu u(p,s_2)\bar{u}(p,s_2)\gamma^\tau u(p',s_1)\right]$$
$$\times\left[\bar{u}(k',s_3)\gamma_\mu u(k,s_4)\bar{u}(k,s_4)\gamma_\tau u(k',s_3)\right]$$

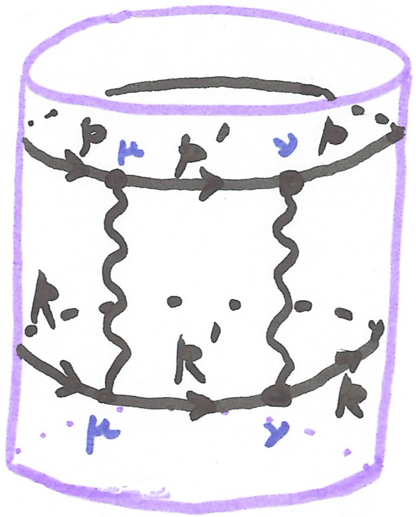
$$\sum_{\alpha..\eta}\sum_{s_1 s_2}\underbrace{\bar{u}_\alpha(p',s_1)\,\gamma^\mu_{\alpha\beta}\,\underbrace{u_\beta(p,s_2)\;\bar{u}_\delta(p,s_2)}_{(\gamma\cdot p+m)_{\beta\delta}}\,\gamma^\tau_{\delta\eta}\,u_\eta(p',s_1)}_{(\gamma\cdot p'+m)_{\eta\alpha}}$$

$$= \sum_{\alpha..\eta}(\gamma\cdot p'+m)_{\eta\alpha}\,\gamma^\mu_{\alpha\beta}\,(\gamma\cdot p+m)_{\beta\delta}\,\gamma^\tau_{\delta\eta}$$

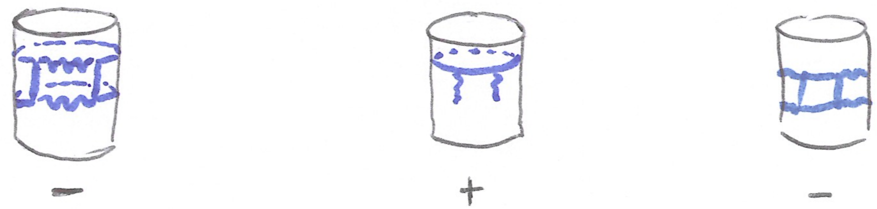
$$= \mathrm{Tr}\left[(\gamma\cdot p'+m)\gamma^\mu(\gamma\cdot p+m)\gamma^\tau\right] \equiv L^{\mu\tau}$$

Okubo diagram

- write the diagram on a cylinder
- Trace the fermion lines



- numerator of propagator if external lines
- − per fermion loop at the surface



Traces of γ matrices

$$\mathrm{Tr}(\gamma^\mu) = 0$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}\,\mathrm{Tr}(\mathbb{1}) = 4g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma_5^2) = -\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\tau\gamma_5) = 0$$

$$\mathrm{Tr}(\text{odd number}) = 0$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\tau\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\tau\sigma} + g^{\mu\sigma}g^{\nu\tau} - g^{\mu\tau}g^{\nu\sigma}\right)$$

$$L^{\mu\tau} = \mathrm{Tr}\left[(\gamma.p'+m)\gamma^\mu(\gamma.p+m)\gamma^\tau\right] = \mathrm{Tr}(\gamma.p'\gamma^\mu\gamma.p\,\gamma^\tau) + m^2\,\mathrm{Tr}(\gamma^\mu\gamma^\tau)$$

$$= \left[p'^\mu p^\tau + p'^\tau p^\mu - p.p'\,g^{\mu\tau} + m^2 g^{\mu\tau}\right]\times 4$$

$$|\bar{\mathcal{M}}|^2 = \left(\frac{e^2}{2q^2}\right)^2 L^{\mu\tau}(m=m_e, p, p') \, L_{\mu\tau}(m_\mu, k, k')$$

$$= \left(\frac{2e^2}{q^2}\right)^2 \left(p'^\mu p^\tau + p'^\tau p^\mu + (m_e^2 - p.p')g^{\mu\tau}\right)\left(k'_\mu k_\tau + k'_\tau k_\mu + (m_\mu^2 - k.k')g_{\mu\tau}\right)$$

$$= \left(\frac{2e^2}{q^2}\right)^2 2\big(p'.k' \, p.k + p'.k \, p.k' + (m_e^2 - p.p') \, k.k' + (m_\mu^2 - k.k') \, p.p' + 2(m_e^2 - p.p')(m_\mu^2 - k.k')\big)$$

$$= \frac{8e^4}{(q^2)^2}\left(p'.k' \, p.k + p'.k \, p.k' - m_e^2 \, k.k' - m_\mu^2 \, p.p' + 2 m_e^2 m_\mu^2\right)$$

center of mass + neglect masses $\qquad \sqrt{s} = E_{c.m.}$

$$p = \left(\frac{\sqrt{s}}{2}, 0, 0, \frac{\sqrt{s}}{2}\right) \qquad k = \left(\frac{\sqrt{s}}{2}, 0, 0, -\frac{\sqrt{s}}{2}\right)$$

$$p' = \left(\frac{\sqrt{s}}{2}, *, *, \frac{\sqrt{s}}{2}\cos\theta\right) \qquad k' = \left(\frac{\sqrt{s}}{2}, *, *, -\frac{\sqrt{s}}{2}\cos\theta\right)$$

$$p'.k' = p.k = \frac{s}{2} \qquad \text{as } s = (p+k)^2 = (k'+p')^2$$

$$p'.k = p.k' = \frac{s}{2}(1+\cos\theta) \qquad q^2 = (p-p')^2 = \frac{s}{2}(1-\cos\theta)$$

$$\Rightarrow |\bar{\mathcal{M}}|^2 = (8e^4)\,\frac{\frac{s^2}{4} + \frac{s^2}{4}(1+\cos\theta)^2}{\frac{s^2}{4}(1-\cos\theta)^2} = 8e^4 \, \frac{1 + (1+\cos\theta)^2}{(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\cos\theta\, d\varphi} = \frac{|\bar{\mathcal{M}}|^2}{64\pi^2 s} \Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{8e^4}{32\pi s} \frac{2+2\cos\theta+\cos^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{8(4\pi\alpha)^2}{32\pi s} \frac{1+4\cos^4\frac{\theta}{2}}{4\sin^4\frac{\theta}{2}}$$

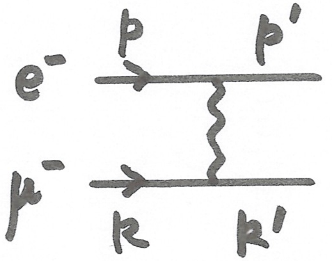
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s} \frac{1+4\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}$$

Rutherford

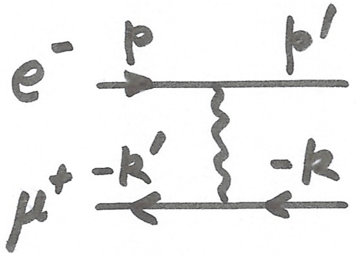
$q^2 \to 0$

$\Rightarrow$ perturbation theory breaks down

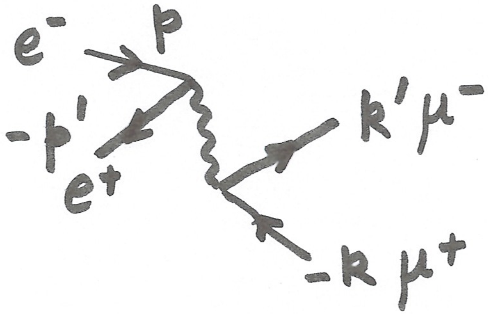
Associated processes



$$\mathcal{M}(p, p', k, k') = \mathcal{M}(s, t, u)$$



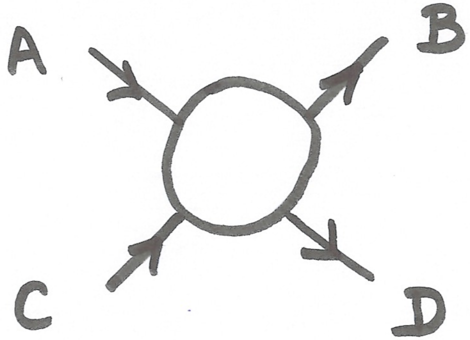
$$\mathcal{M}(p, p', -k', -k) = \mathcal{M}(u, t, s)$$



$$\mathcal{M}(p, -k, -p', k') = \mathcal{M}(t, s, u)$$

(keep $p + k - p' - k' = 0$)

Mandelstam invariants



$$s = (p_A + p_C)^2 = (p_B + p_D)^2$$

$$t = (p_A - p_B)^2 = (p_C - p_D)^2$$

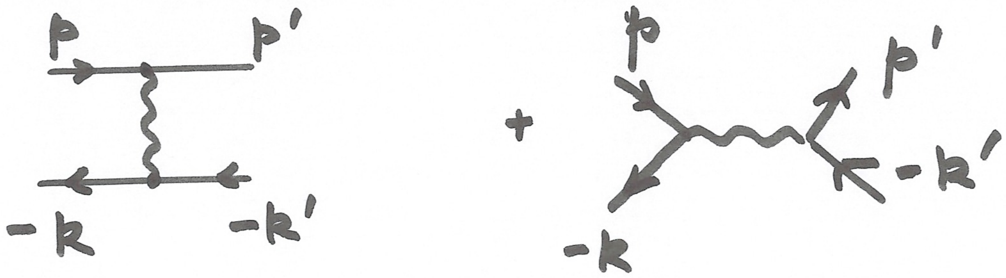
$$u = (p_A - p_D)^2 = (p_C - p_B)^2$$

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

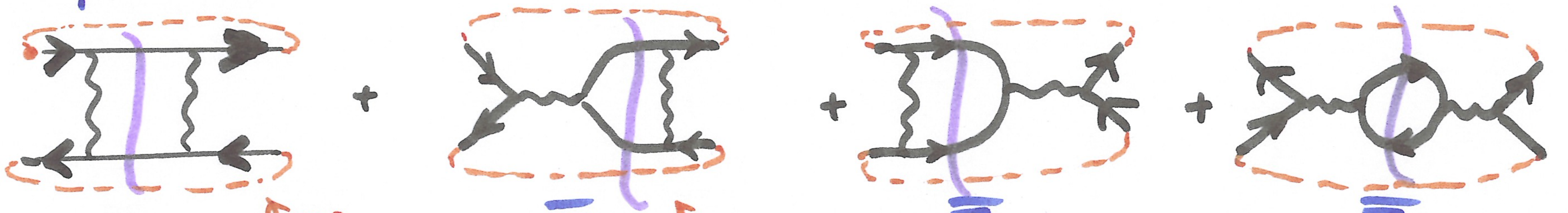
$$|\overline{\mathcal{M}}|^2 = \frac{8e^4}{t^2}\left(\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right)$$

$e^+e^- \to e^+e^-$ (Bhabha)

$i\mathcal{M} =$



square:



$\overline{T}_1$ ↑ no loops

$\overline{T}_2$ ↑ one loop

$\overline{T}_3$

$\overline{T}_4$

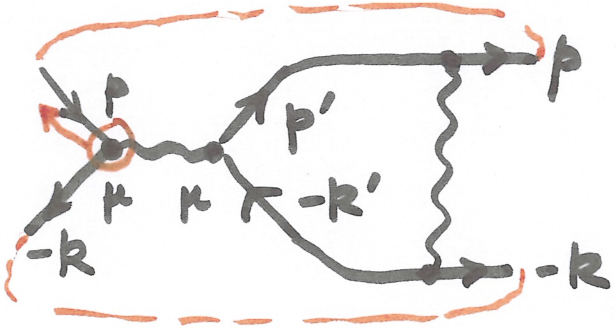
↑ known

$2e^4 \dfrac{s^2+u^2}{t^2}$

↑ crossing

$2e^4 \dfrac{t^2+u^2}{s^2}$

T_2 :



$$T_2 = (-1)\, e^4 \; \frac{\mathrm{Tr}\big(\gamma^\mu \underbrace{p\cdot\gamma\; \overbrace{\gamma^\nu\, p'\cdot\gamma\; \gamma_\mu}^{2\,k'\cdot\gamma\,\gamma_\mu\, p'\cdot\gamma} (-k'\cdot\gamma)}_{4\,k'\cdot p}\, \gamma_\nu (-k\cdot\gamma)\big)}{st}$$

using $\gamma^\mu \gamma_\nu \gamma_\mu = -2\gamma_\nu$

$\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\mu = 4 g_{\nu\rho}$

$\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu = -2\gamma_\sigma \gamma_\rho \gamma_\nu$

$$= 8e^4\, k'\cdot p\; \mathrm{Tr}(p'\cdot\gamma\; k\cdot\gamma)\, \frac{1}{st}$$

$$= 32 e^4 \frac{k'\cdot p\;\; k\cdot p'}{st} = 8e^4 \frac{u^2}{st}$$

$$T_3 = T_2 (p \rightleftarrows -k',\; p' \rightleftarrows -k) = T_2$$

$$\overline{T_3} + \overline{T_2} = 4e^4 \frac{u^2}{st}$$

$$|\overline{\mathcal{M}}|^2_{Bh} = 2e^4\left(\left(\frac{s}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{u}{s} + \frac{u}{t}\right)^2\right)$$

$$t = -\frac{s}{2}(1-\cos\theta)$$

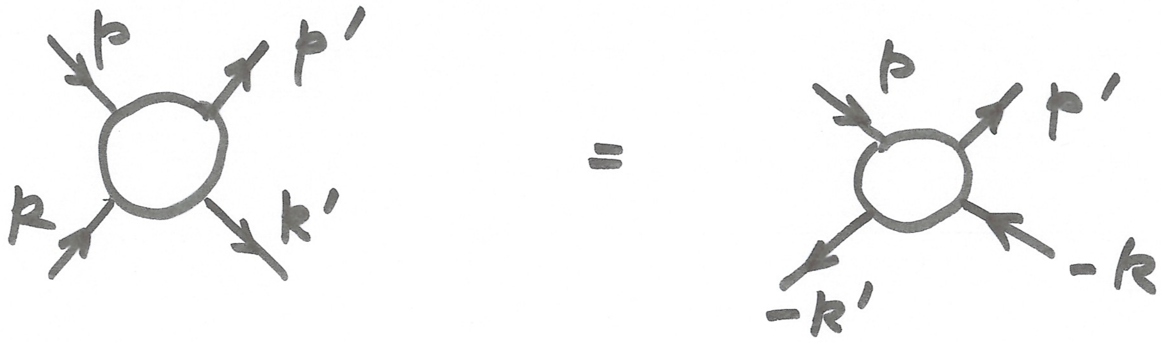
$$u = -\frac{s}{2}(1+\cos\theta)$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left(\frac{4}{(1-\cos\theta)^2} + \frac{(1-\cos\theta)^2}{4} + \left(\frac{1+\cos\theta}{2} + \frac{1+\cos\theta}{1-\cos\theta}\right)^2\right)$$

$$= \frac{\pi\alpha^2}{s}\left(\frac{1}{\sin^4\frac{\theta}{2}} + \sin^4\frac{\theta}{2} + \left(\cos^2\frac{\theta}{2} + \cot an^2\frac{\theta}{2}\right)^2\right)$$

Associated process

$e^-e^- \to e^-e^-$
(Møller)

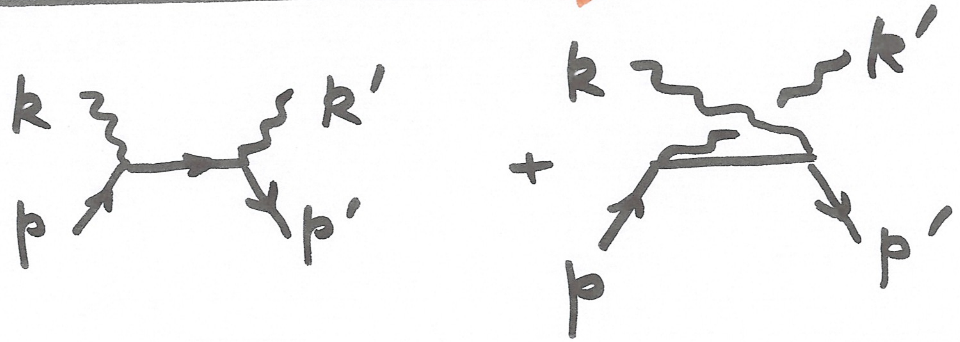


$s \rightleftarrows u$ crossing

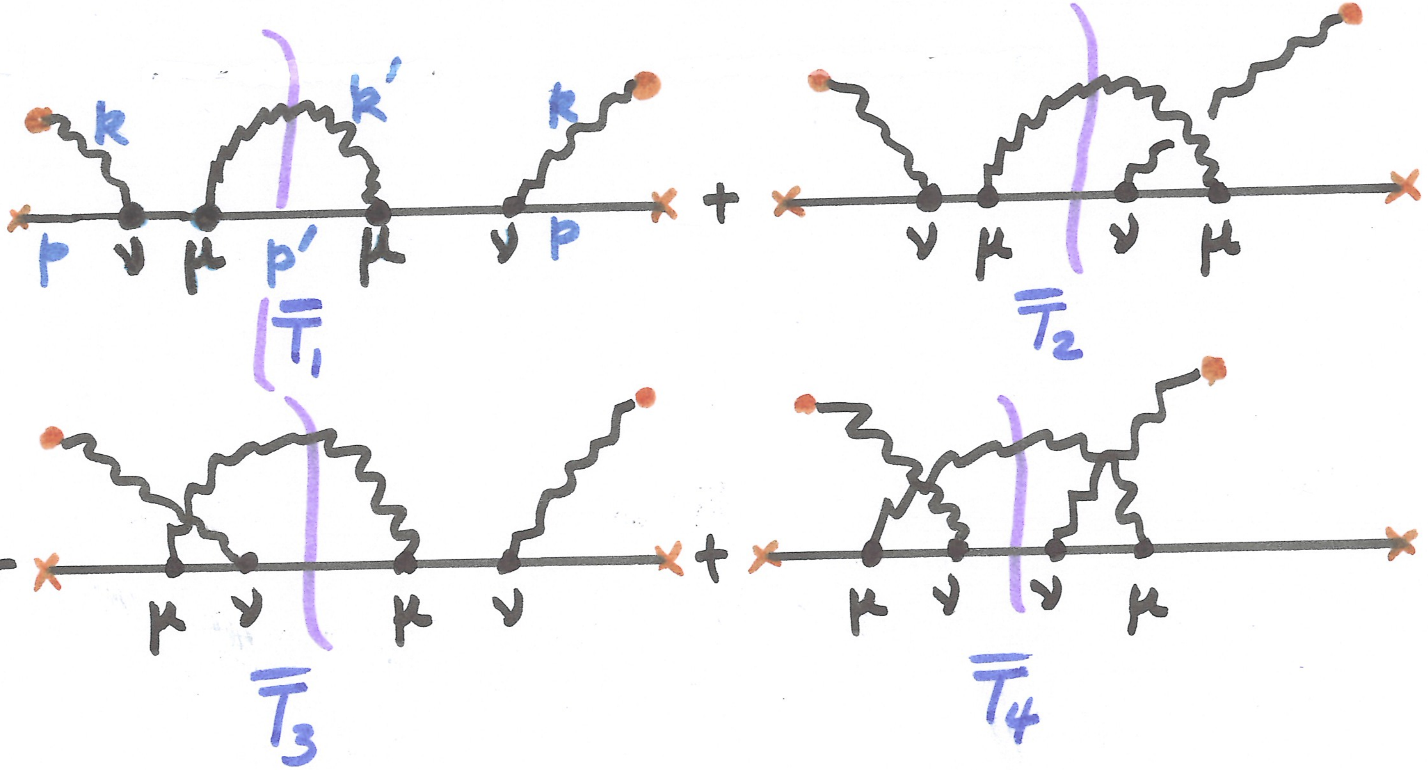
$$|\overline{\mathcal{M}}|^2 = 2e^4\left[\frac{u^2}{t^2} + \frac{t^2}{u^2} + \left(\frac{s}{u} + \frac{s}{t}\right)^2\right]$$

Compton scattering

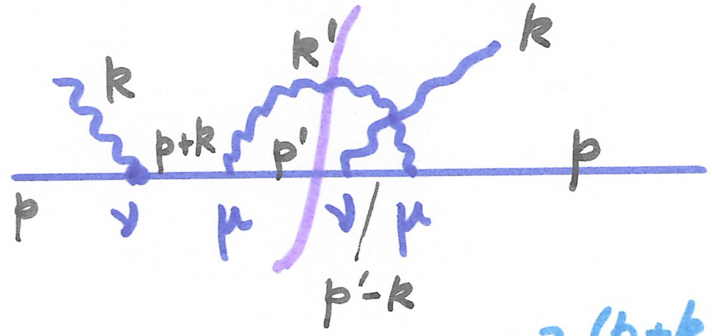
$i\mathcal{M} =$



$\overline{|\mathcal{M}|^2} =$



T_2



$$-2(p+k)\cdot\gamma\,\gamma_\mu\,p'\cdot\gamma$$

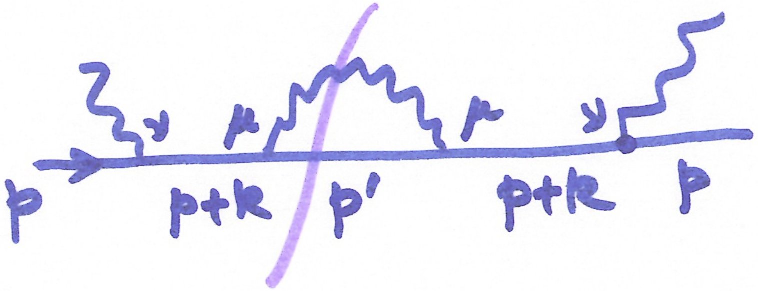
$$T_2 = \frac{\mathrm{Tr}\left(p\cdot\gamma\,\gamma^\mu\,(p'-k)\cdot\gamma\;\gamma^\nu\,p'\cdot\gamma\,\gamma_\mu\,(p+k)\cdot\gamma\,\gamma_\nu\right)}{(p+k)^2\,(p-k')^2}\,(e^4)$$

$$4(p'-k)\cdot(p+k)$$

$$= -8e^4\,\frac{\mathrm{Tr}(p\cdot\gamma\;p'\cdot\gamma)\,(p\cdot p' + k\cdot p' - k\cdot p)}{su}$$

$$-\frac{t}{2} \quad -\frac{u}{2} \quad -\frac{s}{2} \quad = 0$$

similarly $T_3 = 0$



$$T_1 = e^4 \, \frac{\mathrm{Tr}\big(p\cdot\gamma\, \underbrace{\gamma^\nu (p+k)\cdot\gamma\, \overbrace{\gamma^\mu\, p'\cdot\gamma\, \gamma_\mu}^{-2\,p'\cdot\gamma}\, (p+k)\cdot\gamma\, \gamma_\nu}_{-2\,(p+k)\cdot\gamma\, p'\cdot\gamma\,(p+k)\cdot\gamma}\big)}{(p+k)^2}$$

$$= 4e^4 \, \frac{\mathrm{Tr}\big(p\cdot\gamma\, (\cancel{p}+k)\cdot\gamma\, p'\cdot\gamma\, (\cancel{p'}+k')\cdot\gamma\big)}{s^2}$$

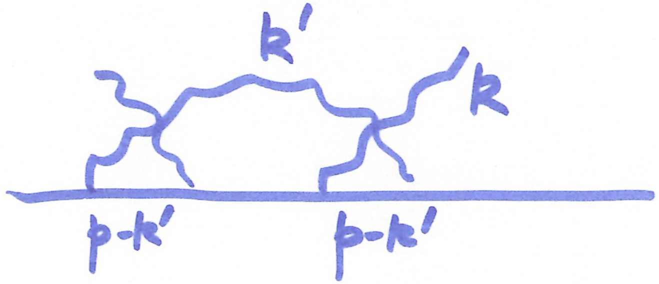
$$p^2 = p\cdot\gamma\, p\cdot\gamma$$

$$= 16e^4 \, \frac{p.k\, p'.k' + p.k'\, p'.k - p.p'\, k.k'}{s^2}$$

$$= 4\,e^4 \, \frac{s^2+u^2-t^2}{s^2} = +4e^4 \, \frac{-2su}{s^2}$$

$$= 8e^4 \frac{(-u)}{s}$$

T_4



same as T_1 if $p+k \to p-k'$

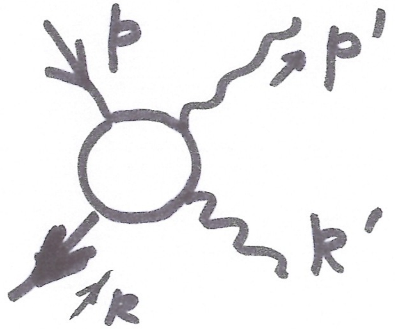
$k \rightleftharpoons -k'$

$s \rightleftharpoons u$

$$T_4 = -8e^4 \frac{s}{u}$$

$$\Rightarrow \quad |\bar{\mathcal{M}}|^2 = 2e^4 \left(\left|\frac{s}{u}\right| + \left|\frac{u}{s}\right| \right)$$

Associated process $e^+e^- \to \gamma\gamma$



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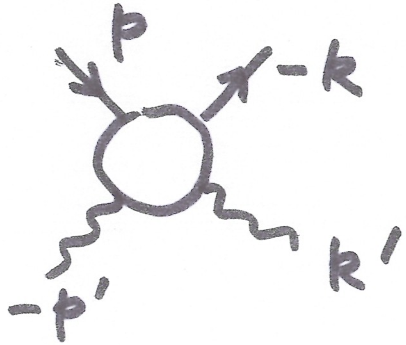
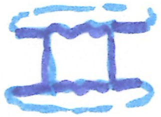


$$k \rightleftarrows -p'$$
$$s \rightleftarrows t$$

$$\Rightarrow |\overline{\mathcal{M}}|^2 = 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right)$$

(- sign for
fermion loop )