III.2. Prove that the sum over polarization gives

$$\sum_s u_p^s \bar{u}_p^s = \gamma.p + m \quad ; \quad \sum_s v_p^s \bar{v}_p^s = \gamma.p - m$$

Reminder: $\cdot$ $u_p^s = \begin{pmatrix} \sqrt{p.\sigma}\ \xi^s \\ \sqrt{p.\bar{\sigma}}\ \xi^s \end{pmatrix}$; $v_p^s = \begin{pmatrix} \sqrt{p.\sigma}\ \eta^s \\ -\sqrt{p.\bar{\sigma}}\ \eta^s \end{pmatrix}$

- $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ spin-up ; spin-down
- same for η
- $\sum_s \xi^s \xi^{s\dagger} = \sum_s \eta^s \eta^{s\dagger} = \mathbb{1}_2$
- $\bar{u}_p^s = u_p^\dagger \gamma^0$ with $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$

And $(p.\sigma)(p.\bar{\sigma}) = p^2 = m^2$; $(\sigma^i)^2 = \mathbb{1}_2$

Indeed: $p.\sigma = p^\mu \sigma_\mu = p^0 \mathbb{1}_2 + p^i \sigma_i$

$p.\bar{\sigma} = p^\mu \bar{\sigma}_\mu = p^0 \mathbb{1}_2 - p^i \sigma_i$

Thus: $(p.\sigma)(p.\bar{\sigma}) = (p^0 \mathbb{1}_2 + p_i \sigma^i)(p^0 \mathbb{1}_2 - p_i \sigma^i)$

$= (p^0)^2 \mathbb{1}_2 - (p_i \sigma^i)^2$ [! Square of the sum !]

$= (p_0^2 - p_1^2 - p_2^2 - p_3^2)\mathbb{1}_2$

$- [p_1 p_2 \sigma^1 \sigma^2 + p_1 p_3 \sigma^1 \sigma^3 + p_2 p_3 \sigma^2 \sigma^3$

$+ p_2 p_1 \sigma^2 \sigma^1 + p_3 p_1 \sigma^3 \sigma^1 + p_3 p_2 \sigma^3 \sigma^2]$

$= \mathbb{1}_2 (p_\mu p^\mu)$

$- [p_1 p_2 \{\sigma^1, \sigma^2\} + p_1 p_3 \{\sigma^1, \sigma^3\} + p_2 p_3 \{\sigma^2, \sigma^3\}]$

$= \mathbb{1}_2 (p_\mu p^\mu) = \mathbb{1}_2\ m^2$ (on-shell)

because $\{\sigma^i, \sigma^j\} = \delta^{ij} \mathbb{1}_2$

$$\boxed{*} \quad \sum_s u_p^s \bar{u}_p^s = \sum_s u_p^s u_p^{s\dagger} \gamma_0$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}^\dagger \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}$$

and $(A.B)^T = B^T A^T$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\sigma}^\dagger & \xi^{s\dagger}\sqrt{p\cdot\bar\sigma}^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\bar\sigma}^\dagger & \xi^{s\dagger}\sqrt{p\cdot\sigma}^\dagger \end{pmatrix}$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma}^\dagger & \sqrt{p\cdot\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}^\dagger \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma}^\dagger & \sqrt{p\cdot\bar\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}^\dagger \end{pmatrix}$$

Use $\sum_s \xi^s \xi^{s\dagger} = 1\!\!1_2$ and $(\sigma^\mu)^\dagger = \sigma^\mu$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix} = p\cdot\gamma + m\,1\!\!1_4$$

Similarity:

$$\sum_s v_p^s \bar{v}_p^s = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix} \begin{pmatrix} -\eta^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \eta^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \sum_s \begin{pmatrix} -\sqrt{p\cdot\sigma}\,\eta^s\eta^{s\dagger}\sqrt{p\bar{\sigma}} & \sqrt{p\cdot\sigma}\,\eta^s\eta^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\bar{\sigma}}\,\eta^s\eta^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & -\sqrt{p\cdot\bar{\sigma}}\,\eta^s\eta^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\sigma & -m \end{pmatrix}$$

$$= p\cdot\gamma - m\,\mathbb{1}_4$$

III. 3. Check that $\gamma_0^\dagger = \gamma_0$ and $\gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^\dagger$

$$\gamma^\mu = \begin{pmatrix} \cdot & \sigma^\mu \\ \bar{\sigma}^\mu & \cdot \end{pmatrix} \qquad \begin{aligned} \sigma^\mu &= (\mathbb{1}, \vec{\sigma}) \\ \bar{\sigma}^\mu &= (\mathbb{1}, -\vec{\sigma}) \end{aligned}$$

it is our convention here.

- $\gamma_0^\dagger = (\gamma_0^*)^T = \left(\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}^* \right)^T = \gamma_0$ <u>trivial!</u>

- $\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$

Already done for $\mu = 0$! Because $\gamma_0 \gamma_0 = \mathbb{1}_4$

<u>$\mu = i$</u> : $\gamma_i^\dagger = \begin{pmatrix} \cdot & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^\dagger$

$$= \begin{pmatrix} \cdot & -\sigma^{i\dagger} \\ \sigma^{i\dagger} & \cdot \end{pmatrix}$$

And Pauli matrices are Hermitian $\Rightarrow \sigma^{i\dagger} = \sigma^i$

Thus $\gamma_i^\dagger = -\gamma_i$

And $\gamma_0 \gamma_i \gamma_0 = \begin{pmatrix} \cdot & \mathbb{1} \\ \mathbb{1} & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \sigma_i \\ -\sigma_i & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \mathbb{1} \\ \mathbb{1} & \cdot \end{pmatrix}$

$$= \begin{pmatrix} \cdot & \mathbb{1} \\ \mathbb{1} & \cdot \end{pmatrix} \begin{pmatrix} \sigma_i & \cdot \\ \cdot & -\sigma_i \end{pmatrix}$$

$$= \begin{pmatrix} \cdot & -\sigma_i \\ \sigma_i & \cdot \end{pmatrix} = -\gamma_i$$

<u>Thus</u> : $\gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^\dagger$ is correct.

III.4. Compute $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta)$ and $\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \begin{array}{l} \sigma^\mu = (\mathbb{1}, \vec\sigma) \\ \bar\sigma^\mu = (\mathbb{1}, -\vec\sigma) \end{array}$$

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
- $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
- $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

⊛ $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = \mathrm{Tr}(-\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta + 2g^{\mu\nu}\gamma^\alpha\gamma^\beta)$

$= -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta) + 2g^{\mu\nu}\,\mathrm{Tr}(\gamma^\alpha\gamma^\beta)$

$= \mathrm{Tr}(\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta) - 2g^{\mu\alpha}\,\mathrm{Tr}(\gamma^\nu\gamma^\beta) + 2g^{\mu\nu}\,\mathrm{Tr}(\gamma^\alpha\gamma^\beta)$

$= -\mathrm{Tr}(\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu) + 2g^{\mu\beta}\,\mathrm{Tr}(\gamma^\nu\gamma^\alpha)$
$\quad - 2g^{\mu\alpha}\,\mathrm{Tr}(\gamma^\nu\gamma^\beta) + 2g^{\mu\nu}\,\mathrm{Tr}(\gamma^\alpha\gamma^\beta)$

cyclic permutation

$= -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) + 2g^{\mu\beta}4g^{\nu\alpha} - 4.2\,g^{\mu\alpha}g^{\nu\beta} + 2g^{\mu\nu}4g^{\alpha\beta}$

$\hookrightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$

⊛ $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = \gamma_\mu\gamma^\alpha(-\gamma^\mu\gamma^\beta + 2g^{\mu\beta})$

$= -\gamma_\mu\gamma^\alpha\gamma^\mu\gamma^\beta + 2g^{\mu\beta}\gamma_\mu\gamma^\alpha$

$= -\gamma_\mu(-\gamma^\mu\gamma^\alpha + 2g^{\mu\alpha})\gamma^\beta + 2\gamma^\beta\gamma^\alpha$

$= \gamma_\mu\gamma^\mu\gamma^\alpha\gamma^\beta - 2\gamma^\alpha\gamma^\beta + 2\gamma^\beta\gamma^\alpha$

$= 4\gamma^\alpha\gamma^\beta - 2\gamma^\alpha\gamma^\beta + 2(-\gamma^\alpha\gamma^\beta + 2g^{\alpha\beta})$

$= 4g^{\alpha\beta}$.

III.5. What are the Feynman rules for the Lagrangian densities of exercise II.6 ?

Easiest way to proceed:

- Take the interaction term(s)
- Remove the field(s)
- Multiply by "i"

→ you got the vertices

<u>Remark:</u>
- terms like $\sim m^2 \bar{\psi}\psi$ might be seen as an interaction term where the field interacts with the Higgs field. Or, it can be seen as a mass term → <u>we do that</u>
- terms with derivative such as $\partial_\mu \dots$ are kinematic term

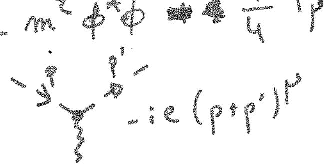
(Slide 5 of theoretical course)

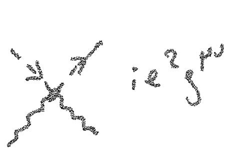
(i): $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$

↳ No interaction. The field A^μ is massive. That's it.

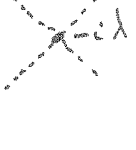
(ii) $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \underbrace{\frac{1}{4}\lambda\phi^4}_{\text{Self-Interaction}}$

(iii) $\mathcal{L} = \underbrace{D_\mu\phi D^\mu\phi^*}_{} - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$





(iv) $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + \underbrace{\frac{1}{4!}\lambda\phi^4}_{} - \underbrace{ig\bar{\psi}\gamma_5\psi\phi}_{}$



$$[\bar{u}_p \gamma^\mu v_k]$$

$$(\cdot \quad \cdot)\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$= (\cdot \quad \cdot)\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = 0$$

$$[\bar{u}_p \gamma^\mu v_k]^* = [\bar{u}_p \gamma^\mu v_k]^\dagger$$

$$= [v_k^\dagger \gamma^{\mu\dagger} \bar{u}_p^\dagger]$$

$$= v_k^\dagger \gamma^0 \gamma^\mu \gamma^0 (u_p^\dagger \gamma_0)^\dagger$$

$$= \cancel{\bar{v}_k \gamma^\mu \gamma^0 (u_p^\dagger)^\dagger \gamma_0^\dagger = \gamma_0}$$

$$= \cancel{\bar{v}_k}$$

$$= v_k^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma_0^\dagger (u_p^\dagger)^\dagger$$

$$= \bar{v}_k \gamma^\mu \underbrace{\gamma^0 \gamma^0}_{= \mathbb{1}} u_p$$

$$= \bar{v}_k \gamma^\mu u_p \qquad \text{OK?}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \quad \text{is Lorentz invariant.}$$

- $d^4p \equiv$ 4-volume $\equiv$ Lorentz invariant

$$d^4p \; \delta^{(4)}(p^\mu p_\mu - m^2)$$

$$= dp_0 \, d^3p \; \delta^{(4)}(p_0^2 - |\vec{p}|^2 - m^2)$$

$$= \frac{d(p_0^2)}{2p_0} \, d^3p \; \delta^{(4)}(p_0^2 - |\vec{p}|^2 - m^2) \longrightarrow \boxed{p_0^2 = |\vec{p}|^2 + m^2}$$

$$= \frac{1}{2p_0} \, d^3p \; \delta^{(3)}(|\vec{p}|^2 + m^2) \qquad p_0 > 0$$

$$= \frac{1}{2\sqrt{|\vec{p}|^2 + m^2}} \quad d^3p \; \delta^{(3)}(|\vec{p}|^2 + m^2)\Big|_{p_0>0} = \begin{cases} \frac{d^3p}{2p_0} & p_0 > 0 \\ 0 & p_0 < 0 \end{cases}$$

<u>So</u>: $$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4} (2\pi)\, \delta^{(4)}(p^\mu p_\mu - m^2)\Big|_{p_0 > 0}.$$

L.

$$|p\rangle = \sqrt{2E_p}\; a_p^\dagger |0\rangle$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} e^{+i\vec{p}\cdot\vec{x}} \left(a_p + a_{-p}^\dagger \right)$$

$$\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{+ip\cdot x}}{\sqrt{2p_0}} \Big(\underbrace{a_p|0\rangle}_{=0} + a_p^\dagger |0\rangle \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{e^{+ipx}}{\sqrt{2p_0}} \underbrace{a_p^\dagger|0\rangle}_{\frac{1}{\sqrt{2p_0}}|p\rangle}$$

$\phi(\vec{x})$ acting on vacuum $|0\rangle$ creates a "particle" at location $\vec{x}$ of momentum $\vec{p}$.

$$= \int \frac{d^3p}{(2\pi)^3} \frac{e^{+ipx}}{2p_0} |p\rangle$$

$\hookrightarrow$ <u>Lorentz invariant.</u>

$$\langle 0| \phi(\vec{x}) |\vec{p}\rangle = \langle 0| \int \frac{d^3q}{\sqrt{2q_0}} \frac{1}{(2\pi)^3} \left(a_q e^{-iqx} + a_q^\dagger e^{+iqx}\right) \sqrt{2p_0}\, a_p^\dagger |0\rangle$$

$$= \langle 0| \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{2p_0}{2q_0}}$$

$$\Big(e^{-iqx} a_q a_p^\dagger |0\rangle + e^{+iqx} \underbrace{a_q^\dagger a_p^\dagger |0\rangle}_{\hookrightarrow 0} \Big)$$

$$a_q a_p^\dagger = \underbrace{[a_q, a_p^\dagger]}_{(2\pi)^3 \delta^{(3)}(q-p)} - \underbrace{a_p^\dagger a_q}_{\text{give zero}}$$

$$= \langle 0| \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{2p_0}{2q_0}}\, e^{-iqx} (2\pi)^3 \delta^{(3)}(q-p) |0\rangle$$

$$= e^{-i\vec{p}\cdot\vec{x}} \underbrace{\langle 0|0\rangle}_{=1}$$

$$\boxed{\langle 0| \phi(\vec{x}) |\vec{p}\rangle = e^{-i\vec{p}\cdot\vec{x}}}\ .$$