IV Cross Section and Feynman rules

IV.1. Calculate the invariant amplitude of Møller scattering, $e^-e^- \to e^-e^-$ via Wick's theorem and directly from Feynman rules

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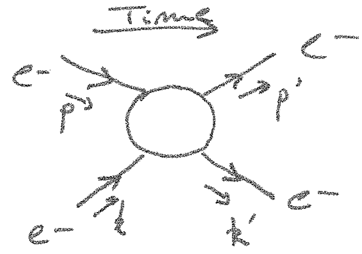
The Møller scattering: 

The amplitude of the process is computed through

$$\langle p', k' | \frac{T}{2!} \int d^4x : \bar\Psi(x)(-ie)\gamma^\mu \Psi(x) A_\mu(x) : \int d^4y : \bar\Psi(y)(-ie)\gamma^\nu \Psi(y) A_\nu(y) : | p, k \rangle$$

where the spinor field has the form:

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(\underbrace{a_p u_p e^{-ip.x}}_{\text{Destroyes positive energy to the right}} + \underbrace{b_p^+ v_p e^{ip.x}}_{\text{Produces negative energy to the left}} \Big)$$

and $$\bar\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(\underbrace{a_p^+ \bar u_p e^{ipx}}_{\text{Produces positive energy to the left}} + \underbrace{b_p \bar v_p e^{-ip.x}}_{\text{Destroy negative energy to the right}} \Big)$$

In Møller scattering, there are particles only and photon exchange. There is no antiparticle $\Rightarrow$ no neg. energy $\Rightarrow$ no "v's".

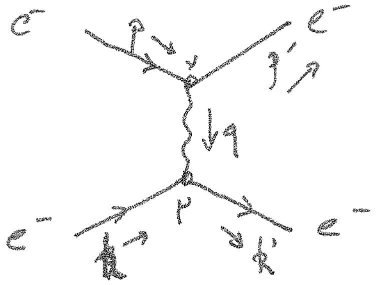
We will have something like this

$$\langle p', k' | \frac{T}{2!} \int d^4x : (a^+\bar u)(-ie)\gamma^\mu (a u) A_\mu : \int d^4y \,(a^+\bar u)\gamma^\nu(-ie)(a u) A_\nu : | p, k \rangle$$

where we have to compute all possible contractions.

* Possible contractions are :

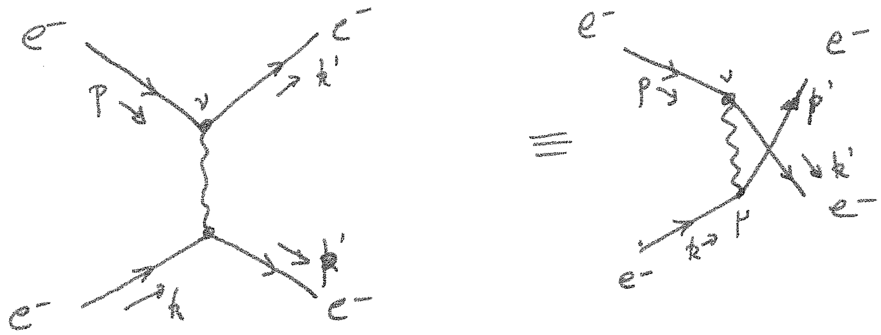
① : $\langle p', k' | \frac{T}{2!} \int d^4x : \bar{\psi} \gamma^\mu \psi A_\mu (-ie) : \int d^4y : \bar{\psi} \gamma^\nu \psi A_\nu (-ie) : | p, k \rangle$



" t - channel "

+ the "same" but inverting $x \longleftrightarrow y$

② $\langle p', k' | \frac{T}{2!} \int d^4x : \bar{\psi} \gamma^\mu \psi A_\mu (-ie) : \int d^4y : \bar{\psi} \gamma^\nu \psi A_\nu (-ie) | p, k \rangle$



" u - channel "

+ the "same" with $x \longleftrightarrow y$

Note that contraction lines cross one time $\Rightarrow$ (-1) for the amplitude.

Note: (See also Slide 56)

$$\psi(\vec{x})\,|p\rangle = \underbrace{\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}}\sum_{s'} a_{p'}^{s'}\, u_{p'}^{s'}\, e^{-ip'\cdot x}\, \underbrace{\underbrace{\sqrt{2E_p}}\, a_p^{s\dagger}\,|0\rangle}_{=:|p\rangle}$$

Remember that this is Lorentz invariant because it can be seen as a 4dim. $\int$.

This term is for normalization purpose

and $\left[a_q^r, a_p^{s\dagger}\right] = + g^{rs}(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})$

$$= a_q^r a_p^{s\dagger} - \underbrace{a_p^{s\dagger} a_q^r}$$

↳ gives 0 when apply to $|0\rangle$

↳ $\psi(x)|p\rangle = u_p^s\, e^{-ipx}\,|0\rangle$

and similarily to the left:

$$\langle p'|\,\bar{\psi}(x) = \langle 0|\,\bar{u}_{p'}^s\, e^{ip'x}$$

1^{st} diagram:

$$\langle p', k' | \frac{T}{2!} \int d^4x : \bar{\psi}(-ie)\gamma^\mu \psi A_\mu : \int d^4y : \bar{\psi}(-ie)\gamma^\nu \psi A_\nu : | p, k \rangle$$

$$\int \frac{d^4q}{(2\pi)^4} \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} e^{-iq(x-y)}$$

propagator

crossing of contraction line

2 vertices

$$= (-1)\frac{(-e^2)}{2!} \int \frac{d^4q}{(2\pi)^4} \int d^4x \; \bar{u}_{k'} \gamma^\mu u_k e^{-i(k-k'+q)x} \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \int d^4y \; \bar{u}_{p'} \gamma^\nu u_p e^{-i(p-p'-q)y}$$

$$= \frac{e^2}{2} \int \frac{d^4q}{(2\pi)^4} \; \bar{u}_{k'} \gamma^\mu u_k (2\pi)^4 \underbrace{\delta^{(4)}(k-k'+q)}_{q=k'-k} \left[\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\right] \bar{u}_{p'} \gamma^\nu u_p (2\pi)^4 \underbrace{\delta^{(4)}(p-p'-q)}_{q=p-p'}$$

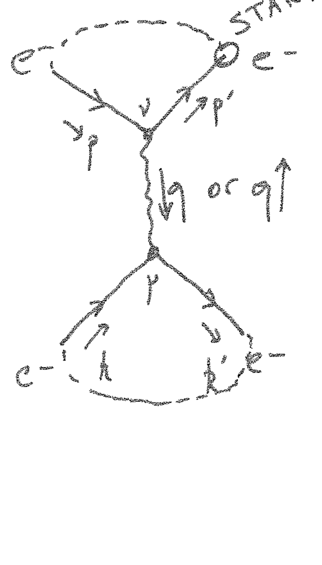
$$= \frac{e^2}{2} (2\pi)^4 \delta^{(4)}(k-k'-(p-p')) \; \bar{u}_{k'} \gamma^\mu u_k \left[\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\right]_{\substack{q=k'-k \\ =p-p'}} \bar{u}_{p'} \gamma^\nu u_p$$

$$i\mathcal{M}_{1} = \left[\frac{e^2}{2} (\bar{u}_{k'} \gamma^\mu u_k) \left.\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\right|_{\substack{q=k'-k \\ =p-p'}} (\bar{u}_{p'} \gamma^\nu u_p)\right] \cdot 2$$

"because we don't know if x happens before y or the opposite"

Using Feynman rules:

1) Draw the diagram
2) Close the current loop
3) follow the line AGAINST current
4) Ask 4-momentum conservation at vertices



$$\bar{u}_{p'}(-ie\gamma^\nu) u_p \left[\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\right] \bar{u}_{k'}(-ie\gamma^\mu) u_k$$

with $q = p - p'$ (if $\{q\downarrow$)
$\quad = k' - k$

or $q = p' - p$ if $q\uparrow\}$
$\quad = k - k'$

2^{nd} diagram:

$$\langle p', k' | \frac{T}{2!} \int d^4x : \bar{\Psi} (-ie) \gamma^\mu \Psi A_\mu : \int d^4y : \bar{\Psi} (-ie) \gamma^\nu \Psi A_\nu : | p, k \rangle$$

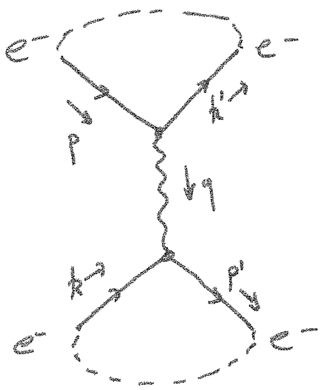
$$\int \frac{d^4q}{(2\pi)^4} \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} e^{-iq(x-y)}$$

$$= (-1) \frac{(-e^2)}{2!} \int \frac{d^4q}{(2\pi)^4} \int d^4x \; \bar{u}_{p'} \gamma^\mu u_k \, e^{+i(p'-k-q)x} \left[\frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \right] \int d^4y \; \bar{u}_{k'} \gamma^\nu u_p \, e^{-i(p-k'-q)}$$

Remark that it is the same as before but with $p' \longleftrightarrow k'$

$$i\mathcal{M}_{(2)} = e^2 \left(\bar{u}_{p'} \gamma^\mu u_k \right) \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \Bigg|_{\substack{q = p'-k \\ = p-k'}} \left(\bar{u}_{k'} \gamma^\nu u_p \right)$$

Using Feynman rules:



$$(-ie)^2 \; \bar{u}_{p'} \gamma^\mu u_k \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \bar{u}_{k'} \gamma^\nu u_p$$

with $q = p - k' = p' - k$ $\quad (q \downarrow \xi)$

Invariant Amplitude

$$|\mathcal{M}|_{Møller} = \frac{1}{4}\sum_{spin} \left| \mathcal{M}_{1} + \mathcal{M}_{2} \right|^2$$

$$= \frac{1}{4}\sum_{spin} \left\{ |\mathcal{M}_{1}|^2 + |\mathcal{M}_{2}|^2 + \text{Interference terms} \right\}$$

1st diagram:

$$\frac{1}{4}\sum_{spin}|\mathcal{M}_{1}|^2 = \frac{e^4}{4t^2}\left[\bar{u}_{k'}\gamma^\mu u_k \left(-ig_{\mu\nu}\right) \bar{u}_{p'}\gamma^\nu u_p\right]$$

$$\cdot\left[\bar{u}_{k'}\gamma^\sigma u_k \left(\cancel{-}ig_{\sigma\rho}\right)\bar{u}_{p'}\gamma^\rho u_p\right]^\dagger$$

with the sum over spin

$$= \frac{e^4}{4t^2}\left\{ \left(\bar{u}_{k'}\gamma^\mu u_k\right)\left(\bar{u}_k\gamma^\sigma u_{k'}\right)\left(-ig_{\mu\nu}\right)\left(ig_{\sigma\rho}\right) \left(\bar{u}_{p'}\gamma^\nu u_p\right)\left(\bar{u}_p\gamma^\rho u_{p'}\right)\right\}$$

$$= \frac{e^4}{4t^2}\left\{\left(\gamma^\mu u_k\bar{u}_k\gamma^\sigma u_{k'}\bar{u}_{k'}\right).\left(\gamma_\mu u_p\bar{u}_p\gamma_\sigma u_{p'}\bar{u}_{p'}\right)\right\}$$

$$= \frac{e^4}{4t^2}\, Tr\left(\gamma^\mu(\gamma.k+m)\gamma^\nu(\gamma.k'+m)\right).Tr\left(\gamma_\mu(\gamma.p+m)\gamma_\nu(\gamma.p'+m)\right)$$

Trace of odd # of γ's $= 0$

$$= \frac{e^4}{4t^2}\left\{\left(Tr(\gamma^\mu\gamma.k\gamma^\nu\gamma.k') + m^2\,Tr(\gamma^\mu\gamma^\nu)\right) .\left(Tr(\gamma_\mu\gamma.p\gamma_\nu\gamma.p') + m^2\,Tr(\gamma_\mu\gamma_\nu)\right)\right\}$$

$$= \frac{e^4}{4t^2}\left\{\left(k_\alpha k'_\beta\, 4\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right) + m^2\, 4\, g^{\mu\nu}\right) .\left(p^\alpha p'^\beta\, 4\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha}\right) + m^2\, 4\, g_{\mu\nu}\right)\right\}$$

$$= \frac{4e^4}{t^2}\left\{\left(k^\mu k'^\nu - g^{\mu\nu}\,k.k' + k^\nu k'^\mu + m^2 g^{\mu\nu}\right) .\left(p_\mu p'_\nu - g_{\mu\nu}\,p.p' + p_\nu p'_\mu + m^2 g_{\mu\nu}\right)\right.$$

$$\frac{1}{4}\sum_{\text{Spin}} |\mathcal{M}_1|^2 = \frac{4e^4}{t^2}\Big\{ p.k\, p'.k' - p.p'\, k.k' + p.k'\, p'.k + m^2\, k.k'$$

$$- p.p'\, k.k' + 4\, p.p'\, k.k' - p.p'\, k.k' - 4\, m^2\, k.k'$$

$$+ p.k'\, p'.k - p.p'\, k.k' + p.k\, p'.k' + m^2\, k.k'$$

$$+ m^2 \left(p.p' - 4\, p.p' + p.p' + 4\, m^2 \right) \Big\}$$

$$= \frac{8e^4}{t^2}\Big\{ p.k\, p'.k' + p.k'\, p'.k - m^2 \left(k.k' + p.p' - 2m^2 \right) \Big\}$$

Mandelstam Variables :

$$\begin{cases} s = (p+k)^2 = (p'+k')^2 \longrightarrow & s = p^2 + k^2 + 2p.k = 2m^2 + 2p.k \\ t = (p-p')^2 = (k-k')^2 \longrightarrow & t = p^2 + p'^2 - 2p.p' = 2m^2 - 2p.p' \\ u = (p-k')^2 = (k-p')^2 \longrightarrow & u = p^2 + k'^2 - 2p.k' = 2m^2 - 2p.k' \end{cases}$$

If we neglect masses :

$$\frac{1}{4}\sum_{\text{Spin}} |\mathcal{M}_1|^2 = \frac{8e^4}{t^2}\Big\{ \frac{s}{2}\frac{s}{2} + \left(-\frac{u}{2}\right)\left(-\frac{u}{2}\right) \Big\}$$

$$= \frac{8e^4}{4t^2}\Big\{ s^2 + u^2 \Big\} = 2e^4 \left(\left(\frac{s}{t}\right)^2 + \left(\frac{u}{t}\right)^2 \right)$$

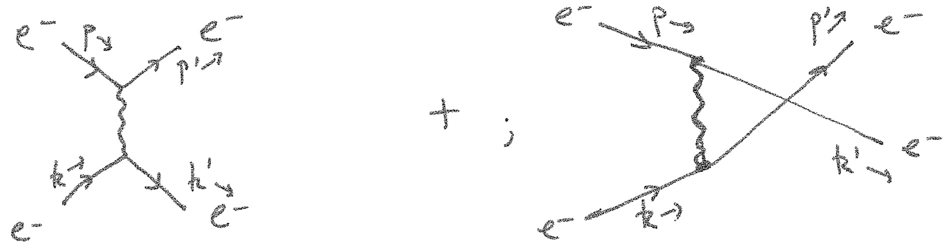
⊛ 2nd diagram there is the crossing $p' \longrightarrow k'$; $k' \longrightarrow p'$ which gives the crossing of Mandelstam Variables : $u \longleftrightarrow t$

So: $$\frac{1}{4}\sum_{\text{Spin}} |\mathcal{M}_2|^2 = 2e^4 \Big\{ \left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2 \Big\}$$

Before computing interference term, note that we could have done this in a much more faster way.

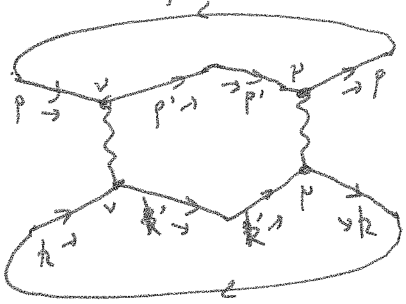
From the process $e^- e^- \longrightarrow e^- e^-$

1) Draw all possible Feynman diagram. Here: 2.



2) For amplitude, take the square (modulus) of the sum of the diagram. ⚠ to fermion loops!

$\left(\begin{matrix}1^{st}\\ \text{diag}\end{matrix}\right)^2$



- To compute the amplitude, close the current line
- Start somewhere and follow the line against the current.
- If independent lines ⇒ multiply
- Multiply by -1 for each fermion loop on the cylinder.
- Take the trace of each path. and count vertices

<u>For the 1^{st} diag:</u>

$$(-ie)^4 \left(\gamma^\nu u_p \bar{u}_p \gamma^\mu u_{p'} \bar{u}_{p'}\right)\left(\gamma_\nu u_k \bar{u}_k \gamma_\mu u_{k'} \bar{u}_{k'}\right)$$

$$= e^4 \,\mathrm{Tr}\left(\gamma^\nu \gamma.p\, \gamma^\mu \gamma.p'\right) \mathrm{Tr}\left(\gamma_\nu \gamma.k\, \gamma_\mu \gamma.k'\right)$$ (we neglect masses here.)

$$= e^4\, 4.4.\left(p_\alpha p'_\beta \left(g^{\nu\alpha} g^{\mu\beta} - g^{\nu\mu} g^{\alpha\beta} + g^{\nu\beta} g^{\mu\alpha}\right)\right)$$

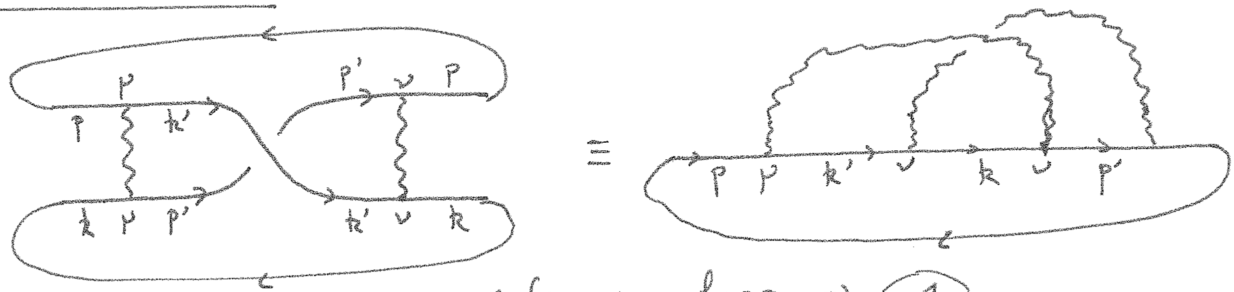
$$.\left(k^\alpha k'^\beta \left(g_{\nu\alpha} g_{\mu\beta} - g_{\nu\mu} g_{\alpha\beta} + g_{\nu\beta} g_{\mu\alpha}\right)\right)$$

$= [$what we found earlier.$]$

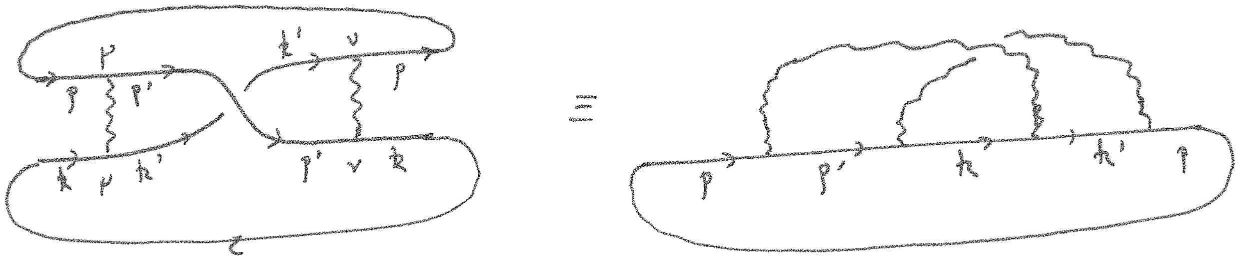
Divide by 4 (spin sum) and divide by momentum of propagators!

Interference Term

$\boxed{I_1}$:



$\boxed{I_2}$:



1 fermion loop ⇒ - 1 .

We see $\boxed{I_2} = \boxed{I_1(k' \leftrightarrow p')}$

$\boxed{I_1}$: $\gamma^\mu u_p \bar{u}_p \gamma^\nu u_{p'} \bar{u}_{p'} \gamma_\mu u_k \bar{u}_k \gamma_\nu u_{k'} \bar{u}_{k'}$

$$= \underbrace{\gamma^\mu \gamma.p\, \gamma^\nu \gamma.p'\, \gamma_\mu}_{-2\, \gamma.p'\, \gamma^\nu\, \gamma.p} \gamma.k\, \gamma_\nu\, \gamma.k'$$

$$= -2\, \gamma.p'\, \underbrace{\gamma^\nu \gamma.p\;\; \gamma.k\, \gamma_\nu}_{4\, p.k}\, \gamma.k'$$

$\gamma^\nu \gamma^\alpha \gamma^\beta \gamma_\nu = 4 g^{\alpha\beta}$

$$= -8\;\; \gamma.p'\, \gamma.k'\;\; k.p$$

$$= -8\;\; k.p\;\; \underbrace{Tr(\gamma^\alpha \gamma^\beta)}_{=4g^{\alpha\beta}}\, p'_\alpha k'_\beta$$

$$= -32\;\; k.p\;\, p'.k'$$

1 fermion → (-1) ; divide by 4 for spin sum

x e^4 since 4 vertices ; and 1 propagator "t"

1 propagator "u"

↳ $\boxed{I_1} = \dfrac{8e^4}{ut}\left(k.p\;\, p'.k'\right) = \dfrac{8e^4}{ut}\left(\dfrac{s}{2}\,\dfrac{s}{2}\right) = \dfrac{2e^4}{ut}\, s^2$

$\boxed{I_2} = \boxed{I_1(u \leftrightarrow t)} = \boxed{I_1}$.

Finally, the Møller scattering has an invariant amplitude of :

$$|\mathcal{M}|^2_{\text{Møller}} = 2e^4 \left\{ \left(\frac{s}{t}\right)^2 + \left(\frac{u}{t}\right)^2 + \left(\frac{s}{u}\right)^2 + \left(\frac{t}{u}\right)^2 + 2\,\frac{s^2}{tu} \right\}$$

$$= 2e^4 \left\{ \left(\frac{u}{t}\right)^2 + \left(\frac{t}{u}\right)^2 + \left(\frac{s}{t} + \frac{s}{u}\right)^2 \right\}$$

<u>ouf ti !</u>