Talk at Matey Mateev Symposium,
Sofia, 17 April 2015

Invariant Differential Operators for Non-Compact Lie Algebras Parabolically Related to Conformal Lie Algebras

V.K. Dobrev

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory. Thus, it is important for the applications in physics to study systematically such operators.

In a recent paper we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study. One first choice would be non-compact groups that have discrete series of representations. By the Harish-Chandra criterion these are groups where holds:

$$\text{rank}\, G = \text{rank}\ K,$$

where K is the maximal compact subgroup of the non-compact group G. Another formulation is to say that the Lie algebra $\mathcal{G}$ of G has a compact Cartan subalgebra.

Example: the groups $SO(p, q)$ have discrete series, except when both p, q are odd numbers.

This class is rather big, thus, we decided to consider a subclass, namely, the class of Hermitian symmetric spaces. The practical criterion is that in these cases, the maximal compact subalgebra $\mathcal{K}$ is of the form:

$$\mathcal{K} \ = \ so(2) \oplus \mathcal{K}'$$

The Lie algebras from this class are:

$$so(n,2), \quad sp(n,R), \quad su(m,n),$$

$$so^*(2n), \quad E_{6(-14)}\,, \quad E_{7(-25)}$$

These groups/algebras have highest/lowest weight representations, and relatedly *holomorphic* discrete series representations.

The most widely used of these algebras are the conformal algebras so(n,2) in n-dimensional Minkowski space-time. In that case, there is a maximal Bruhat decomposition that has direct physical meaning:

$$so(n,2) \;=\; \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}} \;,$$

$$\mathcal{M} \;=\; so(n-1,1) \;, \quad \dim \mathcal{A} = 1,$$

$$\dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n$$

where $so(n-1,1)$ is the Lorentz algebra of n-dimensional Minkowski space-time, the subalgebra $\mathcal{A} = so(1,1)$ represents the dilatations, the conjugated subalgebras $\mathcal{N}$, $\tilde{\mathcal{N}}$ are the algebras of translations, and special conformal transformations, both being isomorphic to n-dimensional Minkowski space-time.

The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ ($\cong \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$) is a maximal parabolic subalgebra.

There are other special features of the conformal algebra which are important. In particular, the complexification of the maximal compact subalgebra is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\mathcal{K}^{\mathbb{C}} \; = \; so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) \; \cong$$

$$\cong \; so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}}$$

In particular, the coincidence of the complexification of the semi-simple sub-algebras:

$$\mathcal{K}'^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \qquad (*)$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $\mathcal{K}'$.

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(n, 2)$.

This subclass consists of:

$$so(n,2), \quad sp(n,\mathbb{R}), \quad su(n,n),$$

$$so^*(4n), \quad E_{7(-25)}$$

In view of applications to physics, we proposed to call these algebras 'conformal Lie algebras', (or groups).

We have started the study of all algebras in the above class in the framework of the present approach, and we have considered also the algebra $E_{6(-14)}$.

Lately, we discovered an efficient way to extend our considerations beyond this class introducing the notion of 'parabolically related non-compact semisimple Lie algebras' [D].

• *Definition:* Let $\mathcal{G}, \mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them parabolically related if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ $(\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}})$.$\diamondsuit$

Certainly, there are many such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G}, \mathcal{G}'$ may be parabolically related with different parabolic subalgebras.

We summarize the algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling $(*)$ in the following table [D]:

Table of conformal Lie algebras (CLA) $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling $(*)$ and the corresponding parabolically related algebras $\mathcal{G}'$

$\mathcal{G}$	$\mathcal{K}'$	$\mathcal{M}$ $\dim V$	$\mathcal{G}'$	$\mathcal{M}'$
$so(n,2)$ $n \geq 3$	$so(n)$	$so(n-1,1)$ n	$so(p,q)$, $p+q=$ $=n+2$	$so(p-1,q-1)$
$su(n,n)$ $n \geq 3$	$su(n) \oplus su(n)$	$sl(n,\mathbb{C})_{\mathbb{R}}$ n^2	$sl(2n,\mathbb{R})$ $su^*(2n)$, $n=2k$	$sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$ $su^*(2k) \oplus su^*(2k)$
$sp(2r,\mathbb{R})$ rank $=2r \geq 4$	$su(2r)$	$sl(2r,\mathbb{R})$ $r(2r+1)$	$sp(r,r)$	$su^*(2r)$
$so^*(4n)$ $n \geq 3$	$su(2n)$	$su^*(2n)$ $n(2n-1)$	$so(2n,2n)$	$sl(2n,\mathbb{R})$
$E_{7(-25)}$	e_6	$E_{6(-26)}$ 27	$E_{7(7)}$	$E_{6(6)}$
below not CLA !				
$E_{6(-14)}$	$so(10)$	$su(5,1)$ 21	$E_{6(6)}$ $E_{6(2)}$	$sl(6,\mathbb{R})$ $su(3,3)$

Conformal algebras $so(n,2)$ and parabolically related

Let $\mathcal{G} = so(n,2)$, $n > 2$. We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\begin{aligned} \chi \ &= \ \{\, n_1\,, \ldots, n_{\tilde{h}}\,;\, c\,\}\ , \\ &n_j \in \mathbb{Z}/2\ , \quad c = d - \tfrac{n}{2}\ , \quad \tilde{h} \equiv [\tfrac{n}{2}], \\ &|n_1| < n_2 < \cdots < n_{\tilde{h}}\ , \quad n \text{ even}\ , \\ &0 < n_1 < n_2 < \cdots < n_{\tilde{h}}\ , \quad n \text{ odd}\ , \end{aligned}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first $\tilde{h}$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M} \cong so(n-1,1)$.

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simple intuitive way:

$$\begin{aligned}
\chi_1^{\pm} &= \{\epsilon n_1\,, \dots, n_{\tilde{h}}\,;\, \pm n_{\tilde{h}+1}\}\,, \\
& \qquad n_{\tilde{h}} < n_{\tilde{h}+1}\,, \\
\chi_2^{\pm} &= \{\epsilon n_1\,, \dots, n_{\tilde{h}-1}\,, n_{\tilde{h}+1}\,;\, \pm n_{\tilde{h}}\} \\
\chi_3^{\pm} &= \{\epsilon n_1, \dots, n_{\tilde{h}-2}, n_{\tilde{h}}, n_{\tilde{h}+1}\,;\, \pm n_{\tilde{h}-1}\} \\
\dots \\
\chi_{\tilde{h}}^{\pm} &= \{\epsilon n_1\,, n_3\,, \dots, n_{\tilde{h}}\,, n_{\tilde{h}+1}\,;\, \pm n_2\} \\
\chi_{\tilde{h}+1}^{\pm} &= \{\epsilon n_2\,, \dots, n_{\tilde{h}}\,, n_{\tilde{h}+1}\,;\, \pm n_1\}
\end{aligned}$$

$$\epsilon = \begin{cases} \pm\,, & n \ even \\ 1, & n \ odd \end{cases}$$

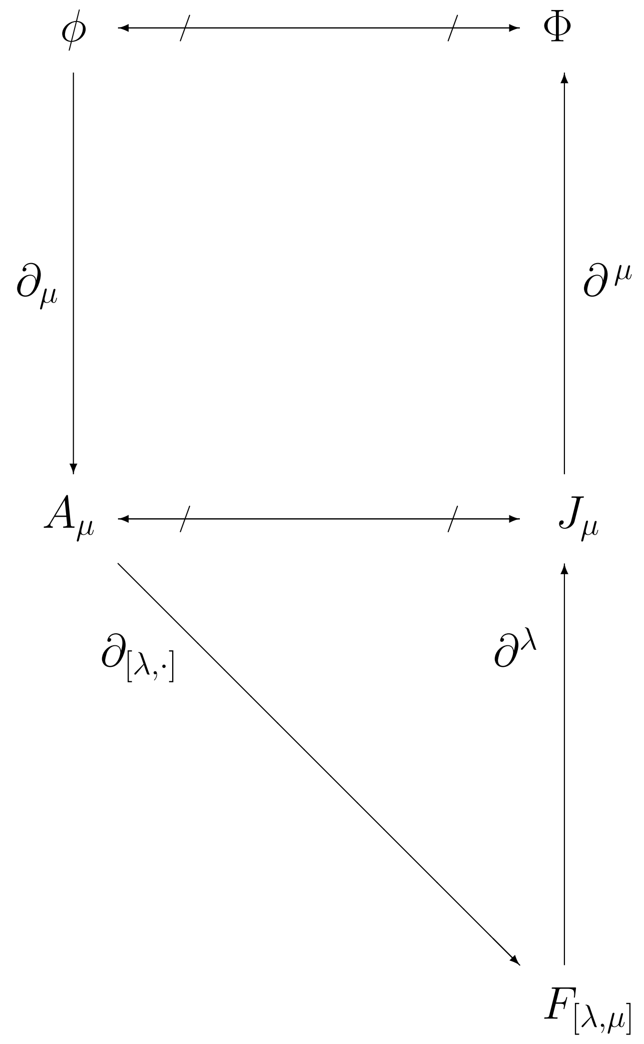
Further, we denote by $\tilde{\mathcal{C}}_i^\pm$ the representation space with signature $\chi_i^\pm$.

The number of ERs in the corresponding multiplets is equal to:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|\,/\,|W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| \;=\; 2(1+\tilde{h})$$

where $\mathcal{H}^{\mathbb{C}}$, $\mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$, $\mathcal{M}^{\mathbb{C}}$, resp.

At this moment we show the simplest example for the most common conformal group in 4-dimensional Minkowski space-time $so(4,2)$.

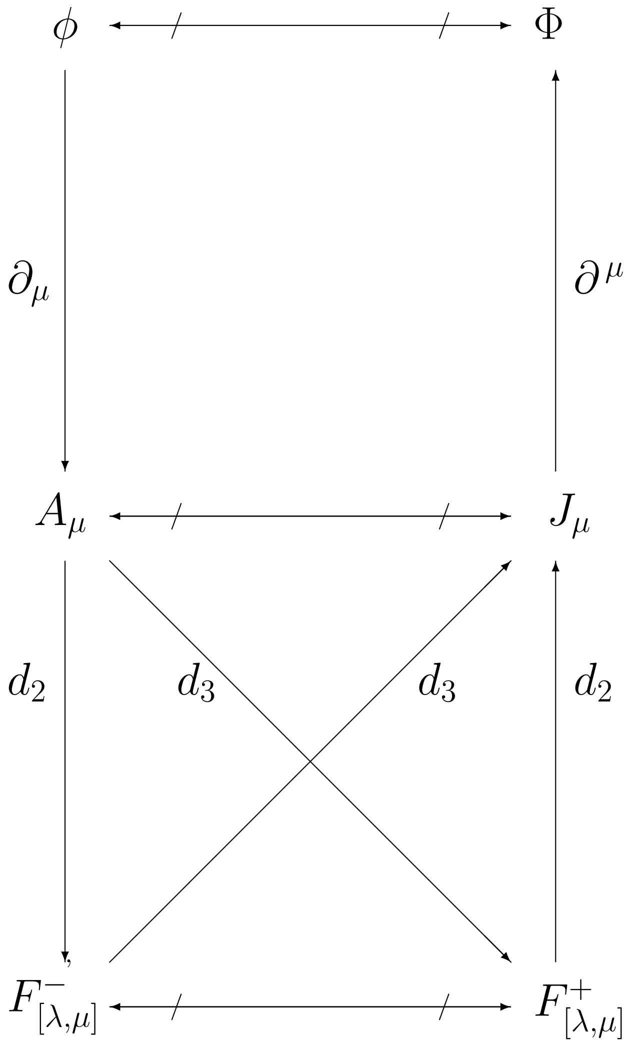


Simplest example of diagram with conformal invariant operators
(arrows are differential operators, dashed arrows are integral operators)

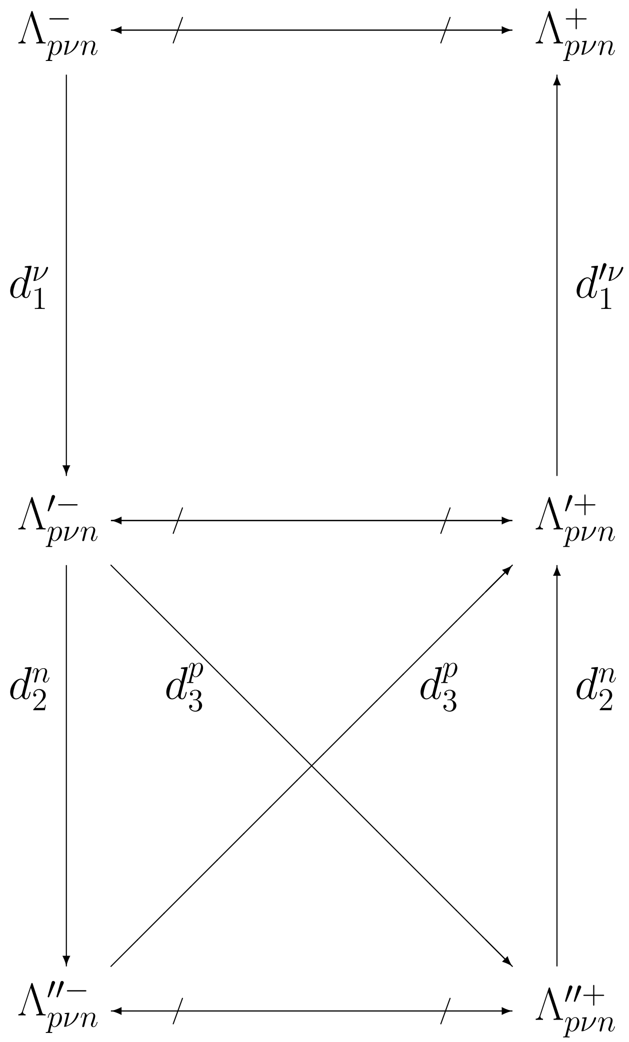
$$\partial_\mu = \frac{\partial}{\partial x_\mu}\,,\ \ A_\mu \text{ electromagnetic potential,}\quad \partial_\mu\,\phi = A_\mu$$

$$F \text{ electromagnetic field,}\quad \partial_{[\lambda}A_{\mu]} = \partial_\lambda A_\mu - \partial_\mu A_\lambda \ = \ F_{\lambda\mu}$$

$$J_\mu \text{ electromagnetic current,}\ \ \partial^\lambda F_{\lambda\mu} = J_\mu,\ \ \partial^\mu J_\mu = \Phi$$



More precise showing of the simplest example
$F = F^+ \oplus F^-$ electromagnetic field
d_2, d_3 linear invariant operators

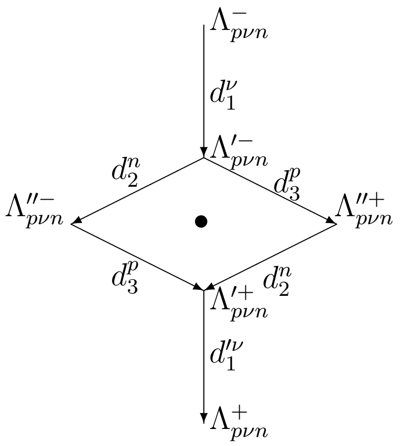


The general classification of conformal invariant operators

p, ν, n are three natural numbers

the shown simplest case is when $p = \nu = n = 1$

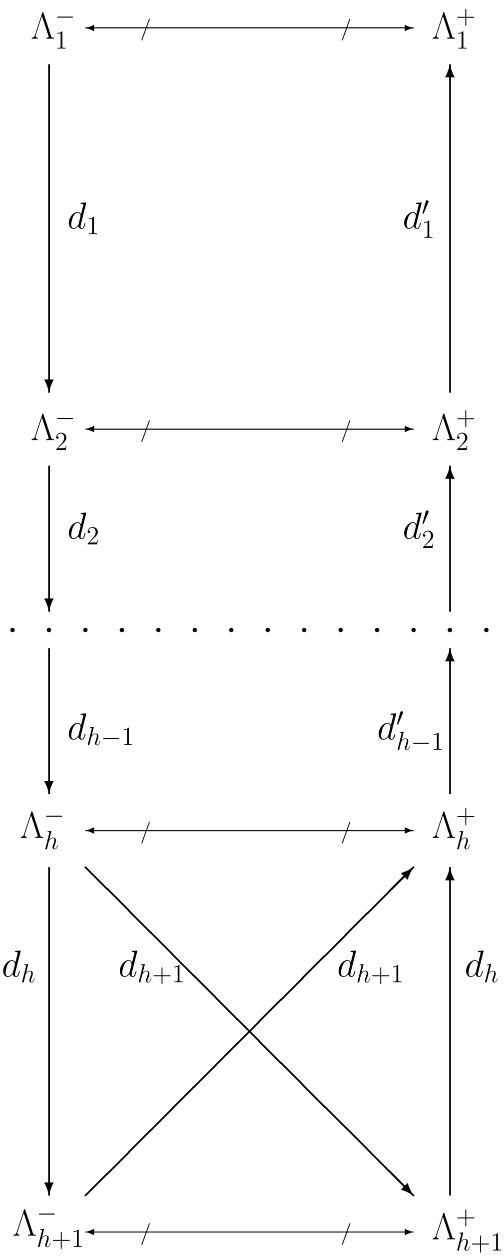
d_1^ν is a linear differential operator of order ν, similarly $d_1'^\nu$, d_2^n, d_3^p



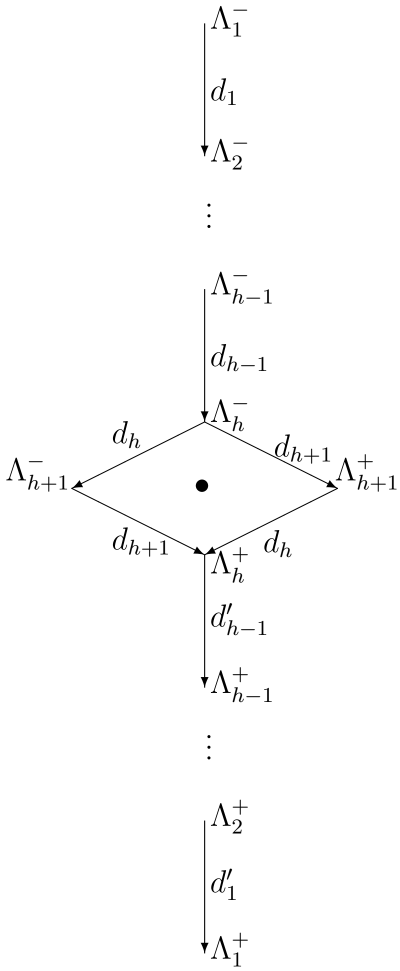
Alternative showing of the conformal invariant operators -

- showing only the differential operators

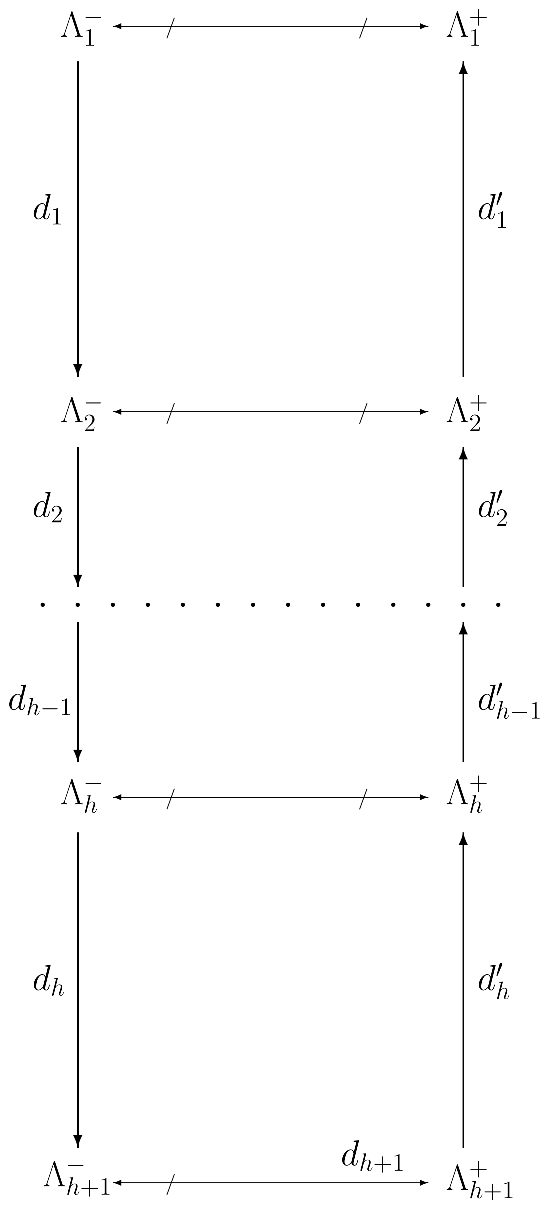
The integral operators are assumed as symmetry w.r.t. the bullet in the centre.



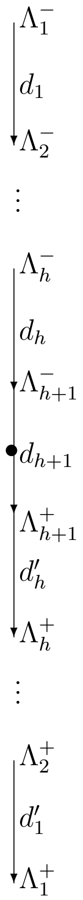
The general classification of conformal invariant operators in $2h$-dimensional space-time. By parabolic relation the diagram above is valid for all algebras $so(p,q)$, $p+q=2h+2$.



Alternative showing of the conformal invariant operators in 2h-dimensional space-time



The general classification of conformal invariant operators in $2h+1$ dimensional space-time. By parabolic relation the diagram above is valid for all algebras $so(p,q)$, $p+q=2h+3$.



Alternative showing of the conformal invariant operators in $2h+1$ dimensional space-time

The ERs in the multiplet are related by intertwining integral and differential operators. The integral operators were introduced by Knapp and Stein. They correspond to elements of the restricted Weyl group of $\mathcal{G}$. These operators intertwine the pairs $\tilde{\mathcal{C}}_i^{\pm}$

$$G_i^{\pm} \; : \; \tilde{\mathcal{C}}_i^{\mp} \longrightarrow \tilde{\mathcal{C}}_i^{\pm} \;, \qquad i \;=\; 1, \ldots, 1 + \tilde{h}$$

The intertwining differential operators correspond to non-compact positive roots of the root system of $so(n+2, \mathbb{C})$, cf. [D]. [In the current context, compact roots of $so(n+2,\mathbb{C})$ are those that are roots also of the subalgebra $so(n,\mathbb{C})$, the rest of the roots are non-compact.]

Matters are arranged so that in every multiplet only the ER with signature χ_1^- contains a finite-dimensional nonunitary subrepresentation in a subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional unitary irrep of $so(n+2)$ with signature $\{n_1, \ldots, n_{\tilde{h}}, n_{\tilde{h}+1}\}$. The subspace $\mathcal{E}$ is annihilated by the operator G_1^+, and is the image of the operator G_1^-.

Although the diagrams are valid for arbitrary $so(p,q)$ $(p+q \geq 5)$ the contents is very different. We comment only on the ER with signature χ_1^+. In all cases it contains an UIR of $so(p,q)$ realized on an invariant subspace $\mathcal{D}$ of the ER χ_1^+. That subspace is annihilated by the operator G_1^-, and is the image of the operator G_1^+. (Other ERs contain more UIRs.)

If $pq \in 2\mathbb{N}$ the mentioned UIR is a discrete series representation. (Other ERs contain more discrete series UIRs.)

And if $q = 2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^+ \bigoplus \mathcal{D}^-$, in which are realized a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Above we restricted to $n > 2$.
The case $n = 2$ is reduced to $n = 1$ since $so(2,2) \cong so(1,2) \oplus so(1,2)$.
The case $so(1,2)$ is special and must be treated separately. But in fact, it is contained in what we presented already. In that case the multiplets contain only two ERs which may be depicted by the top pair $\chi_1^\pm$ in both pictures that we presented. And they have the properties that we described. That case was the first given already in 1947 independently by Bargmann and Gel'fand et al.

The Lie algebra $su(n,n)$ and parabolically related

Let $\mathcal{G} = su(n,n)$, $n \geq 2$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{C})_{\mathbb{R}}$. The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| = \binom{2n}{n}$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{ n_1 \,, \ldots, n_{n-1} \,, n_{n+1} \ldots, n_{2n-1} \,; c \}$$
$$n_j \in \mathbb{N} \,, \quad c = d - n$$

The Knapp–Stein restricted Weyl reflection is given by:

$$G_{KS} \ : \ \mathcal{C}_\chi \ \longrightarrow \ \mathcal{C}_{\chi'} \ ,$$

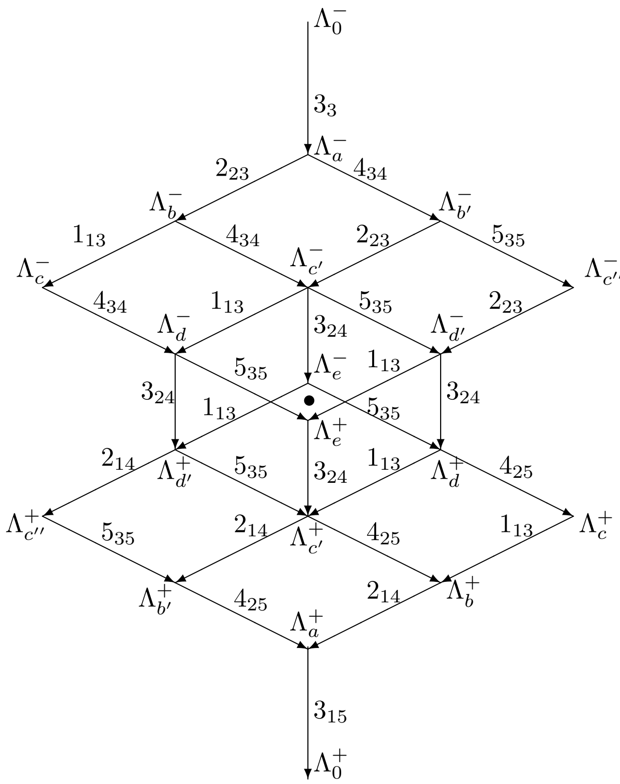
$$\chi' = \{(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^*; -c\}$$

$$(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^* \ \doteq \ (n_{n+1}, \ldots, n_{2n-1}, n_1, \ldots, n_{n-1})$$

Below we give the diagrams for $su(n,n)$ for $n = 2, 3, 4$. These are diagrams also for the parabolically related $sl(2n, \mathbb{R})$ and for $n = 2k$ these are diagrams also for the parabolically related $su^*(4k)$ [D].

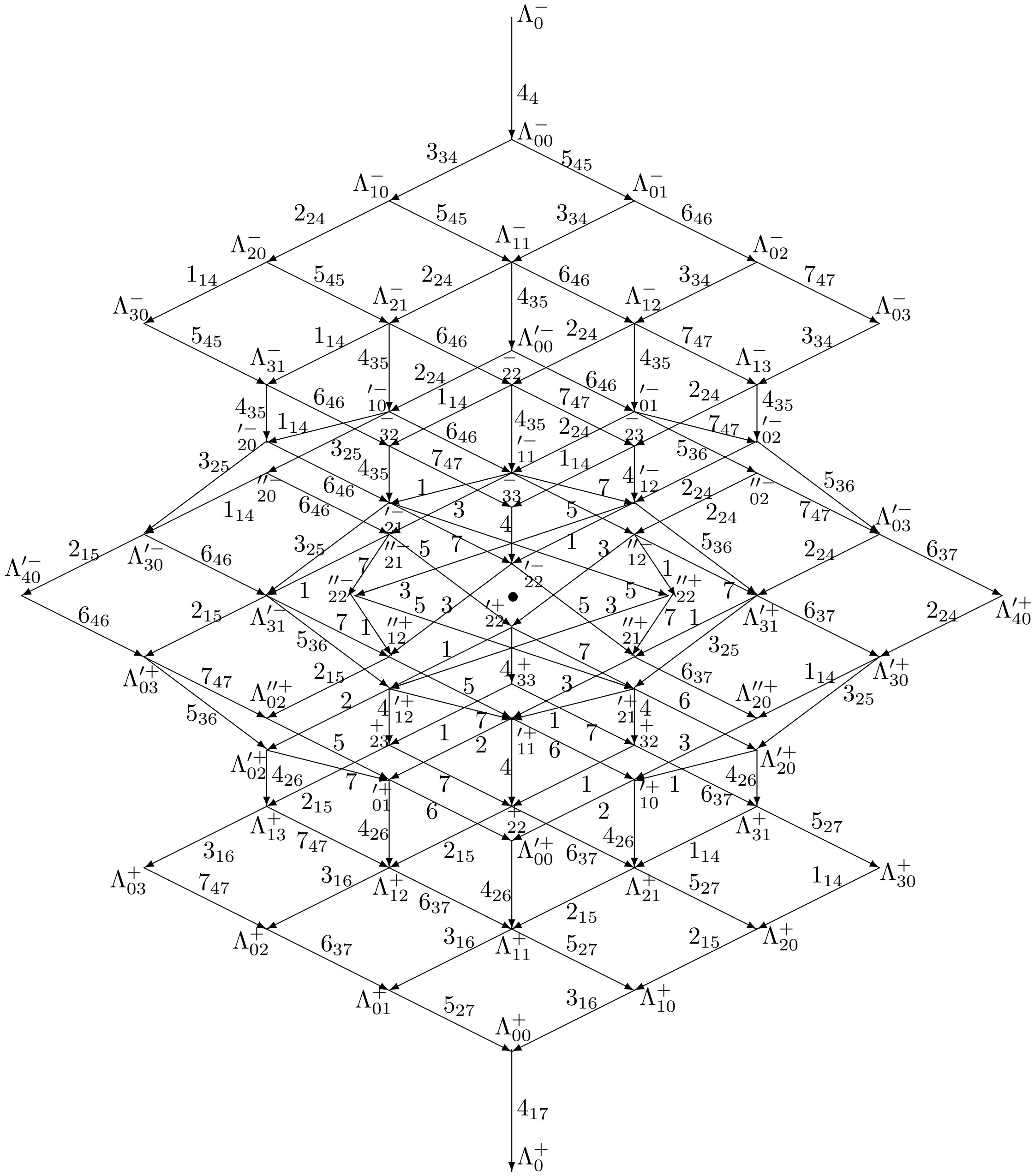
We only have to to take into account that the latter two algebras do not have discrete series representations.

We use the following conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j...k}$ encoding the root $\beta_{j...k}$ and the number $m_{\beta_{j...k}}$ which encode the reducibility of the corresponding Verma module.



Pseudo-unitary symmetry $su(3,3)$

Pseudo-unitary symmetry $su(n,n)$ is similar to conformal symmetry in n^2 dimensional space, for $n = 2$ coincides with conformal 4-dimensional case. By parabolic relation the $su(3,3)$ diagram above is valid also for $sl(6,R)$.



Pseudo-unitary symmetry $su(4,4)$

(By parabolic relation the diagram above is valid also for $sl(8,R)$ and $su^*(8)$.)

The Lie algebras $sp(n,\mathbb{R})$ and $sp(\frac{n}{2},\frac{n}{2})$ (n–even)

Let $n \geq 2$. Let $\mathcal{G} = sp(n,\mathbb{R})$, the split real form of $sp(n,\mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{R})$. The number of ERs in the corresponding multiplets is:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| \;=\; 2^n$$

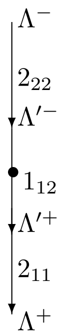
The signature of the ERs of $\mathcal{G}$ is:

$$\chi \;=\; \{\, n_1\,, \ldots, n_{n-1}\,;\, c\,\}\,, \qquad n_j \in \mathbb{N}\,,$$

The Knapp-Stein Weyl reflection acts as follows:

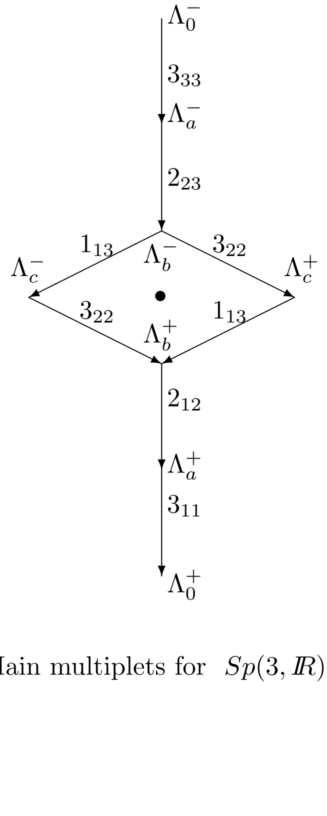
$$G_{KS} \ : \ \mathcal{C}_\chi \ \longrightarrow \ \mathcal{C}_{\chi'} \ ,$$
$$\chi' \ = \ \{\,(n_1,\ldots,n_{n-1})^*\,;\,-c\,\} \ ,$$
$$(n_1,\ldots,n_{n-1})^* \ \doteq \ (n_{n-1},\ldots,n_1)$$

Below we give pictorially the multiplets for $sp(n,\mathbb{R})$ for $n=2,3,4,5,6$. For $n = 2r$ these are also multiplets for $sp(r,r)$, $r=1,2,3$ [D]. We only have to to take into account that the latter algebra has discrete series representations but not highest/lowest weight representations.

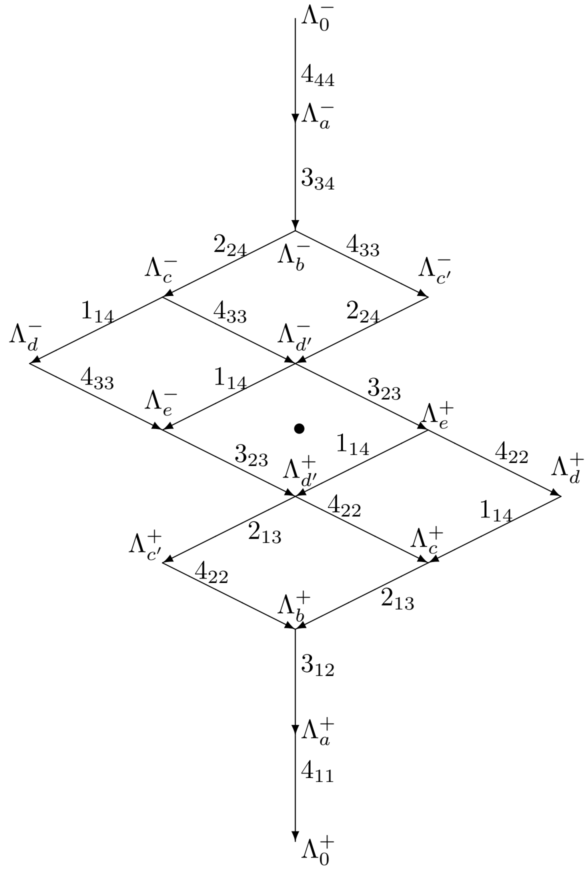


Simplest symplectic symmetry coinciding with 3-dimensional conformal case.

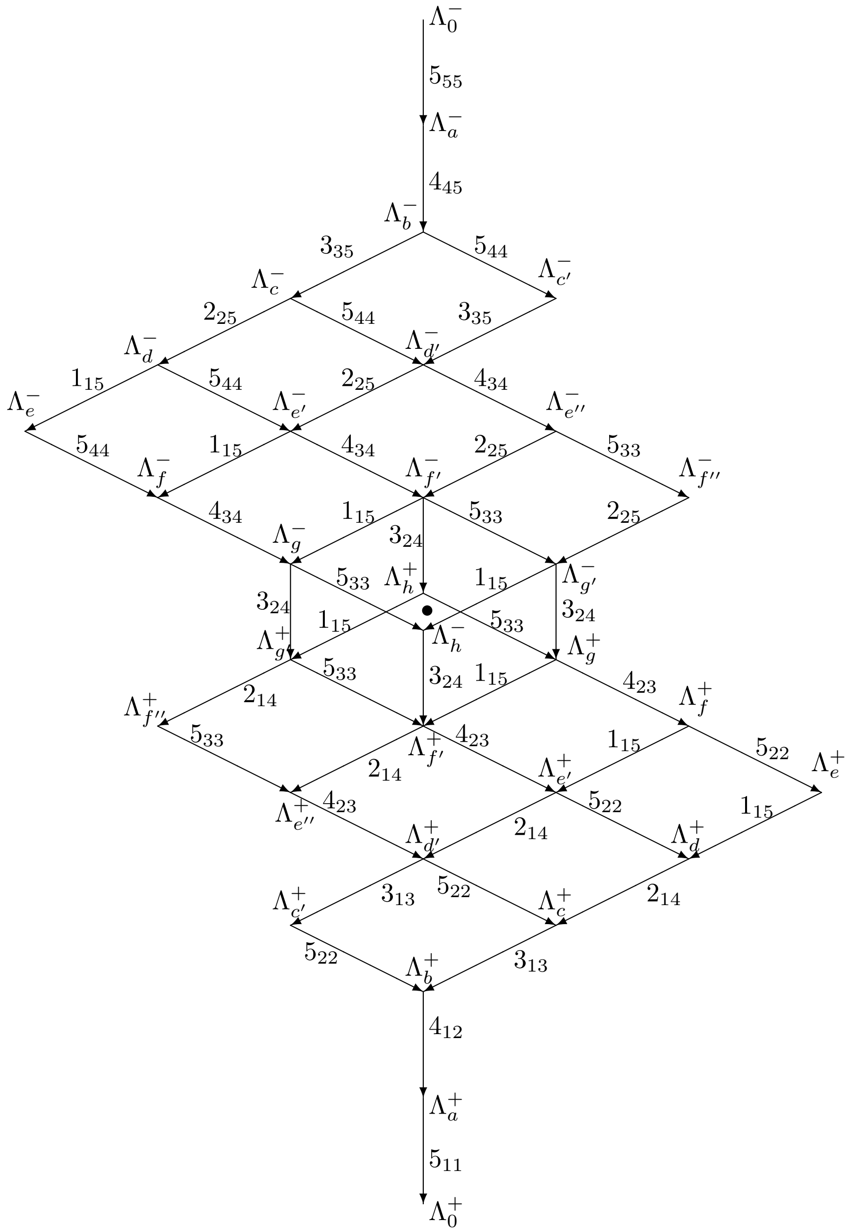
By parabolic relation the diagram above is valid
for $sp(2,R) \cong so(3,2)$ and $sp(1,1) \cong so(4,1)$.



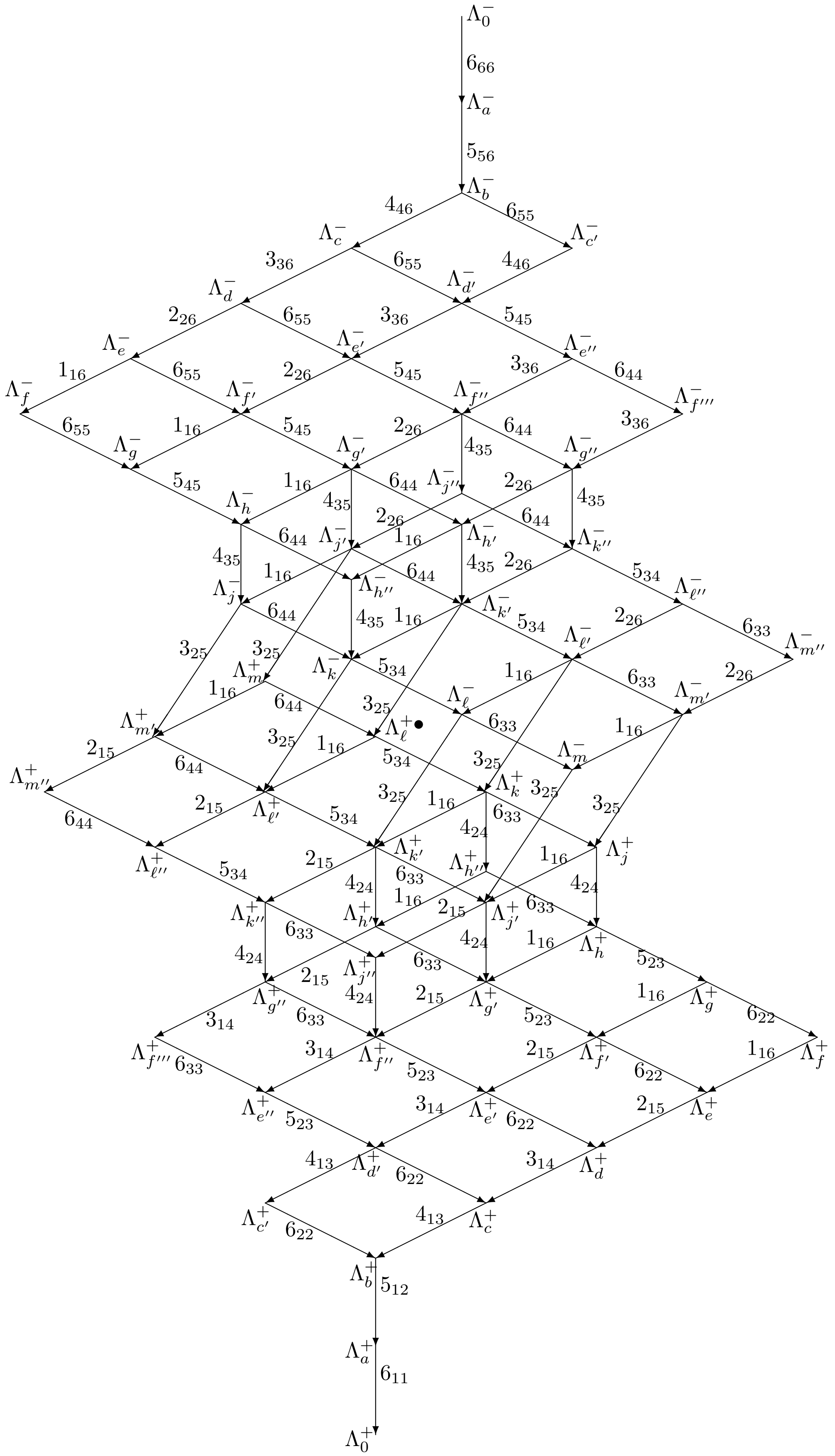
Main multiplets for $Sp(3, I\!R)$



Main multiplets for $Sp(4,I\!\!R)$ and $Sp(2,2)$



Main multiplets for $Sp(5, I\!R)$



Main multiplets for $Sp(6,\mathbb{R})$ and $Sp(3,3)$

The Lie algebra $so^*(12)$

The Lie algebra $\mathcal{G} = so^*(2n)$ is given by:

$$\begin{aligned} so^*(2n) &\doteq \{X \in so(2n, \mathbb{C}) : J_n C X = X J_n C\} \\ &= \{\ X = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \ \mid \ a, b \in gl(n, \mathbb{C}), \\ &\qquad {}^t a = -a, \quad b^\dagger = b\ \}\ . \end{aligned}$$

$\dim_R \mathcal{G} = n(2n-1)$, $\mathrm{rank}\, \mathcal{G} = n$.

The maximal compact subalgebra is $\mathcal{K} \cong u(n)$. Thus, $\mathcal{G} = so^*(2n)$ has discrete series representations and highest/lowest weight representations. The split rank is $r \equiv [n/2]$.

The maximal parabolic subalgebras have $\mathcal{M}$-factors as follows [D]:

$$\mathcal{M}_j^{\mathrm{max}} = so^*(2n-4j) \oplus su^*(2j) \ , \quad j = 1, \ldots, r \ .$$

For even $n = 2r$ we choose a *maximal* parabolic $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ such that $\mathcal{A} \cong so(1,1)$, $\mathcal{M} = \mathcal{M}_r^{\mathrm{max}} = su^*(n)$. We note also that

$$\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(n,\mathbb{C}) \cong \mathcal{A}^{\mathbb{C}} \oplus \mathcal{M}^{\mathbb{C}}$$

Thus, with this choice we utilize the property which distinguishes the class of 'conformal Lie algebras' to which class the algebras $so^*(4r)$ belong.

Further we restrict to our case of study $\mathcal{G} = so^*(12)$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1 , n_2 , n_3 , n_4 , n_5 ; c \} , \qquad n_j \in \mathbb{Z}_+ , \quad c = d - \tfrac{15}{2}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first five entries are labels of the finite-dimensional (nonunitary) irreps of $\mathcal{M} = su^*(6)$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$.

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, n_2, n_3, n_4, n_5)^* \doteq (n_5, n_4, n_3, n_2, n_1) .$$

The ERs in the multiplet are related also by intertwining integral operators introduced in [KnSt]. These operators are defined for any ER, the general action being:

$$G_{KS} \ : \ \mathcal{C}_\chi \ \longrightarrow \ \mathcal{C}_{\chi'} \ ,$$
$$\chi \ = \ \{ \, n_1, \ldots, n_5 \, ; \, c \, \} \ ,$$
$$\chi' \ = \ \{ \, (n_1, \ldots, n_5)^* \, ; \, -c \, \}.$$

Further, we give the correspondence between the signatures χ and the highest weight Λ. The connection is through the Dynkin labels:

$$m_i \ \equiv \ (\Lambda{+}\rho, \alpha_i^\vee) \ = \ (\Lambda{+}\rho, \alpha_i) \ , \quad i = 1, \ldots,$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. The explicit connection is:

$$\begin{aligned} n_i &= m_i \ , \\ c &= -\tfrac{1}{2}(m_1 + 2m_2 + 3m_3 + 4m_4 + \\ &\quad +2m_5 + 3m_6) \end{aligned}$$

Finally, we remind that according to [D] the above considerations are applicable also for the algebra $so(6,6)$ with parabolic $\mathcal{M}$-factor $sl(6,\mathbb{R})$.

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(12)$, i.e., they are labelled by the six positive Dynkin labels $m_i \in \mathbb{N}$.

The number of ERs/GVMs in the corresponding multiplets is [D]:

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| / |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| =$$
$$= |W(so(12, \mathbb{C}))| / |W(sl(6, \mathbb{C}))| = 32$$

where $\mathcal{H}$ is a Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$.

They are given explicitly in the Figure below. The pairs $\Lambda^{\pm}$ are symmetric w.r.t. to the bullet in the middle of the figure - this represents the Weyl symmetry realized by the Knapp-Stein operators

The statements made for the ER with signature χ_0^- in the previous cases remain valid also here. Also the conjugate ER χ_0^+ contains a unitary discrete series subrepresentation.

All the above is valid also for the algebra $so(6,6)$, cf. [D], however, the latter algebra does not have highest/lowest weight representations.

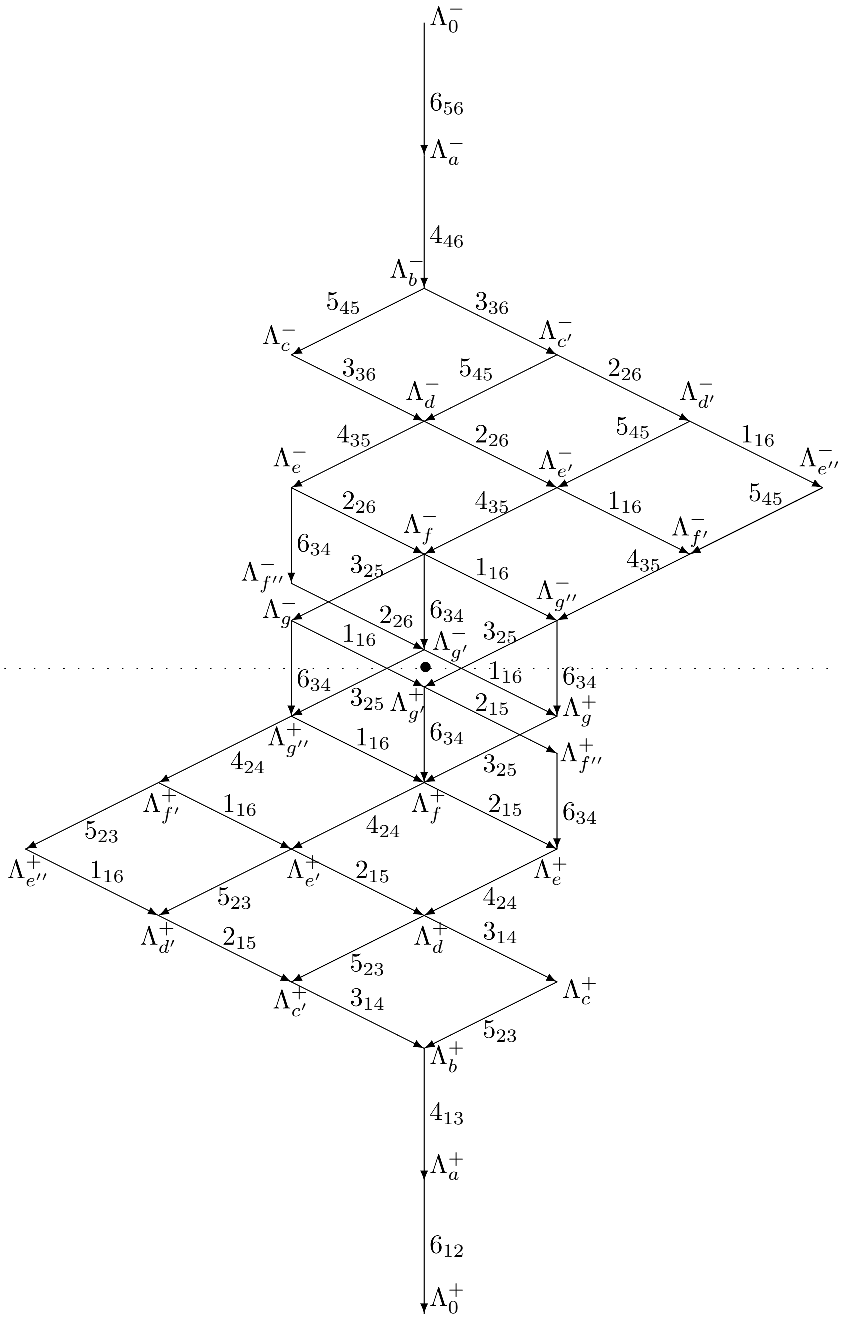


Fig. 1. $SO^*(12)$ main multiplets

The Lie algebras $E_{7(-25)}$ and $E_{7(7)}$

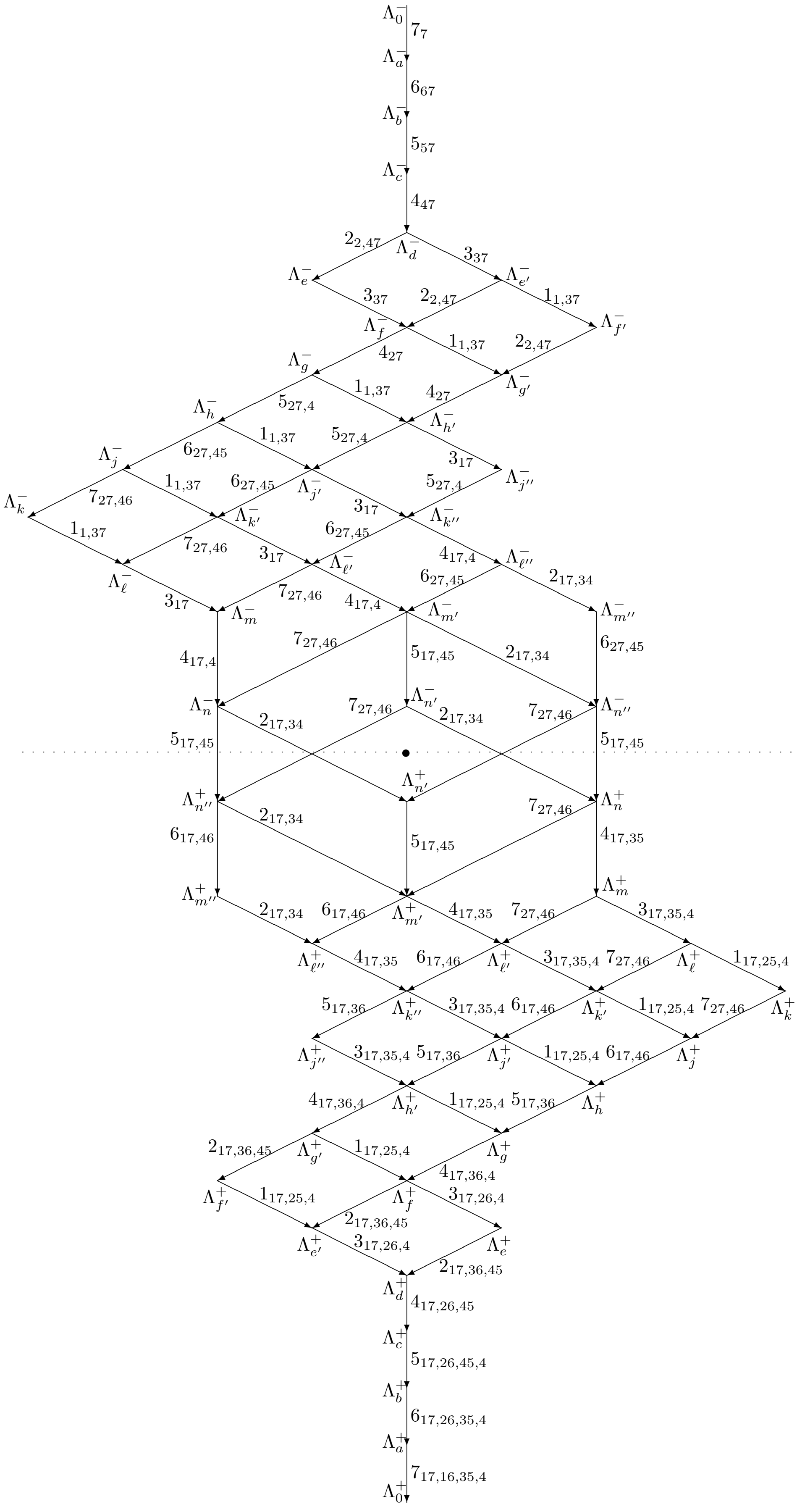
Let $\mathcal{G} = E_{7(-25)}$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$, while $\mathcal{M} \cong E_{6(-6)}$.

The signatures of the ERs of $\mathcal{G}$ are:

$$\chi \ = \ \{ n_1 \, , \ldots , n_6 \, ; \, c \} \ , \qquad n_j \in \mathbb{N} \ .$$

The same can be used for the parabolically related exceptional Lie algebra $E_{7(7)}$ [D

We only have to to take into account that the latter algebra has discrete series representations but not highest/lowest weight representations.



Main Type for $E_{7(-25)}$ and $E_{7(7)}$

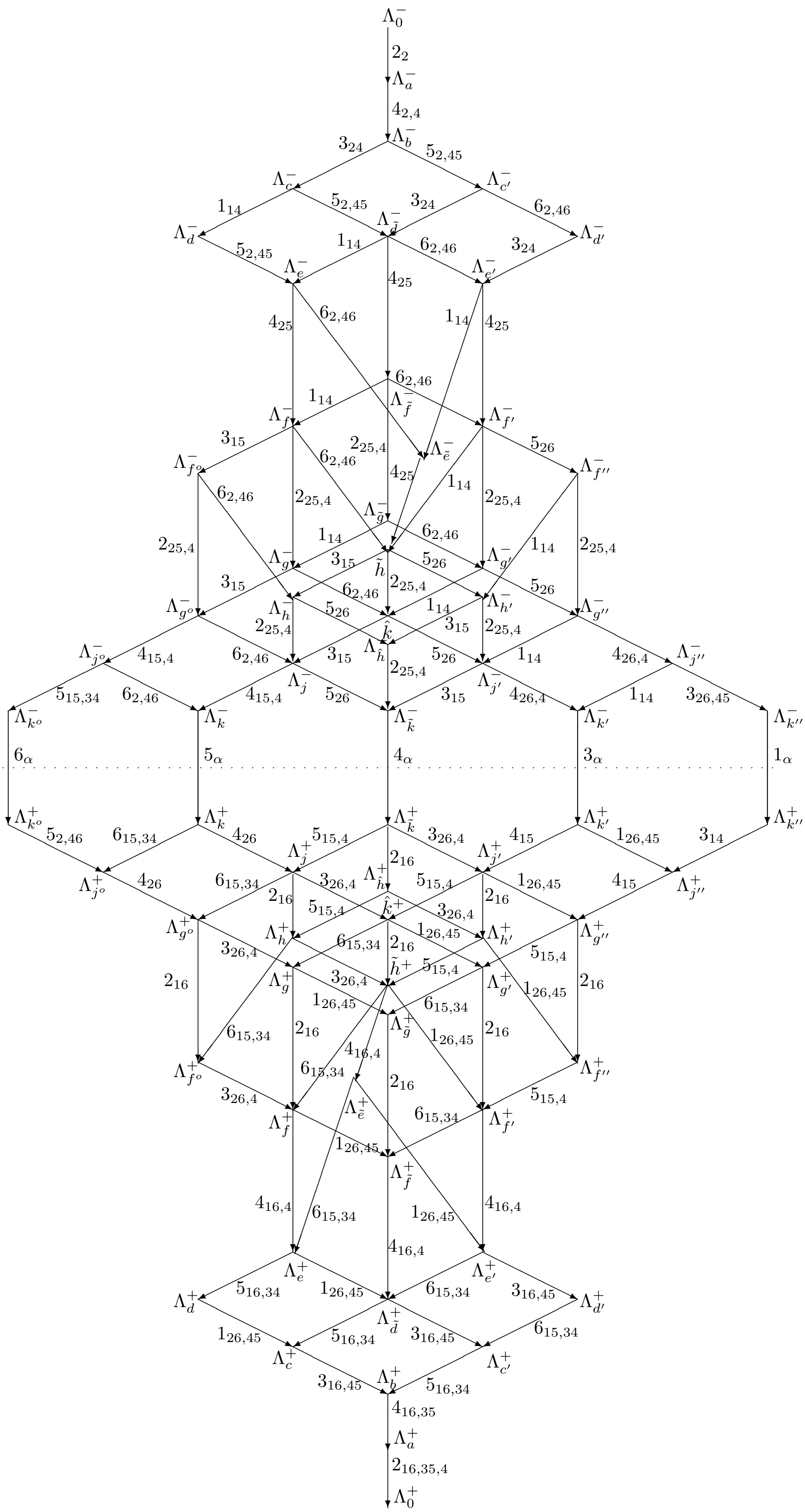
The Lie algebras $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$

Let $\mathcal{G} = E_{6(-14)}$. The maximal compact subalgebra is $\mathcal{K} \cong so(10) \oplus so(2)$, while $\mathcal{M} \cong su(5,1)$.

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{\, n_1\,, n_3\,, n_4\,, n_5\,, n_6\,;\, c\}\,, \qquad c = d - \tfrac{11}{2}\,.$$

The above can be used for the parabolically related exceptional Lie algebras $E_{6(6)}$ and $E_{6(2)}$ [D]. We only have to to take into account that only the algebra $E_{6(2)}$ has discrete series representations (but not highest/lowest weight representations).



Main Type for $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$