Model Independent Analysis Of The Proton Magnetic Radius

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Outline

• Motivation
• How do we extract the proton magnetic radius
• Our model-independent approach to extract the radius
• Parameterization and minimizing techniques
• Results
• Conclusion
Motivation

• Proton is one of the fundamental constituent of matter.
• It is an extended object and should have a size which is not known to us precisely.
• The electric radius values extracted from e-p scattering e.g. $r_E^p = 0.871\pm0.009$ fm, e.g. Hill and Paz(2010), from Muonic Hydrogen $r_E^p = 0.84184\pm0.0006$ fm, Pohl et al.(2010).
• PDG 2014 reports the magnetic radius of proton as $r_M^p = 0.777 \pm 0.014$ fm according to Bernauer et al. (2010).
Other values mentioned in PDG are:
$r_M^p = 0.854\pm0.005$ fm Belushkin et al. (2007)
$r_M^p = 0.876\pm0.020$ fm D. Borisyuk (2010)
Form Factors

• In general there are two form factors known as Dirac and Pauli form factors, defined as

\[
\langle N(p') | J_{\mu}^{em} | N(p) \rangle = \bar{u}(p') \left[ \gamma_{\mu} F_{1}^{N}(q^2) + \frac{i\sigma_{\mu\nu}}{2m_N} F_{2}^{N}(q^2) q^\nu \right] u(p)
\]

• The experimental data given by Sachs electric and magnetic form factors which are related to the Dirac and Pauli basis by

\[
G_E^N(t) = F_1^N(t) + \frac{t}{4(m_N^2)} F_2^N(t); \quad G_M^N(t) = F_1^N(t) + F_2^N(t)
\]

where \( t = q^2 = -Q^2 \).
• \( G_M^p(0) = \mu_p \approx 2.793 \); \( G_M^n(0) = \mu_n \approx -1.913 \)

• The isoscalar and isovector form factors are defined as

\[
G_{M,E}^{I=0} = G_M^p + G_M^n, \quad G_{M,E}^{I=1} = G_M^p - G_M^n
\]

such that at \( t = 0, \quad G_{M}^{I=0}(0) = \mu_p + \mu_n \approx 0.88 ; \quad G_{M}^{I=1}(0) = \mu_p - \mu_n \approx 4.7 \]
Measurement of Magnetic Radius

• In the Breit frame \((q^2 = -\vec{q}^2)\) the magnetic form factor is the Fourier transform of proton’s magnetic moment density \(\mu(\vec{r})\):

\[
G_M^p(q^2) = \int d^3x \, \mu(\vec{r}) \, e^{i\vec{q} \cdot \vec{r}} = \mu[1 - \frac{1}{6} \vec{q}^2 \langle r^2 \rangle_M + \cdots]
\]

• The magnetic radius of proton is defined as

\[
\langle r^2 \rangle^p_M = \frac{6}{G_M^p(0)} \frac{dG_M^p(q^2)}{dq^2} \bigg|_{q^2=0}
\]

• Usually a form factor parameterization is chosen and then a fit is performed to experimentally measured form factors. The radius is then obtained from above equation.
Analyticity of Form Factors

• Form Factors are analytic in the cut plane.

Conformal mapping of the cut plane to the unit circle

• If any complex function \( f(z) \) is analytic, we can express it as a series expansion

\[
 f(z) = \sum_{k=0}^{\infty} a_k z^k.
\]
• Choosing $z(t, t_{cut}, t_0) = \frac{\sqrt{t_{cut} - t} - \sqrt{t_{cut} - t_0}}{\sqrt{t_{cut} - t} + \sqrt{t_{cut} - t_0}}$, where $t_0$ is a free parameter (determines the location of $z = 0$) and $t_{cut} = 4m_\pi^2$, maps the cut plane to the unit circle.

• For $t_0 = 0$, $t_{cut} = 4m_\pi^2 = 0.0784$ GeV$^2$ and $t = -0.5$ GeV$^2$, $z_{max} = 0.70$. 
• We can expand the magnetic form factor as

\[ G_M(q^2) = \sum_{k=0}^{\infty} a_k z(q^2)^k \]

which always converges.

• Advantage of \( z \) expansion is that its coefficients can be bound using the knowledge of imaginary part of \( G_M \).
Coefficients

• The analytic structure in the $t$-plane implies a dispersion relation

$$G^p(t) = \frac{1}{\pi} \int_{t_{cut}}^{\infty} dt' \frac{\text{Im}G^p(t'+i0)}{t'-t}$$

• Parameterizing by $z(t) = e^{i\theta}$ and solving $z = \frac{\sqrt{t_{cut}-t} - \sqrt{t_{cut}-t_0}}{\sqrt{t_{cut}-t} + \sqrt{t_{cut}-t_0}}$ for $t$, we get

$$t = t_0 + \frac{2(t_{cut}-t_0)}{1 - \cos \theta} \equiv t(\theta)$$
• Using the orthogonality over the unit circle, we get

\[ a_k = -\frac{2}{\pi} \int_0^\pi \text{Im}G[t(\theta) + i0] \sin(k\theta) \, d\theta \]

• Using \( \theta = 2 \sin^{-1} \frac{t_{\text{cut}} - t_0}{\sqrt{t - t_{\text{cut}}}} \) we derive the final expression for the coefficients

\[
a_k = -\frac{2}{\pi} \int_{t_{\text{cut}}}^{\infty} \frac{dt}{t - t_0} \sqrt{\frac{t_{\text{cut}} - t_0}{t - t_{\text{cut}}}} \text{Im}G[t(\theta) + i0] \sin((k\theta(t)) \]
Bounds on coefficients

• To extract $r_E^p$ Hill and Paz (2010) used bounds of $|a_k| \leq 5.10$.

• Since $a_0 = G_M^{I=1}(0) = \mu_p - \mu_n \approx 4.7$, bound of 5 is too stringent.

• Using vector dominance ansatz we find:
  $$I = 1 \ (\rho \text{ exchange}) \ |a_k| \leq 5.1$$

• Using $\pi\pi$ continuum we find:
  $$a_0 \approx 7.9, a_1 \approx -5.5, a_2 \approx -6.1, a_4 \approx 1.1.$$  
  Using $|\sin(k\theta)| \leq 1$ gives $|a_k| \lesssim 7.2$ for $k \geq 1$.

• Above $t = 4m_N^2$ use $e^+e^- \rightarrow N\bar{N}$: negligible contribution to $a_k$
• We conclude:
  \[ |a_k| \leq 5 \] is not conservative enough, use \[ |a_k| \leq 10 \] and \[ |a_k| \leq 15 \]

• We used higher bounds like 20, but did not observe significant changes in the extracted values.
Minimizing technique

• We fit our data by ‘z expansion’ method as described

\[ G_M^p = a_0 + a_1 z(q^2) + a_2 z^2(q^2) + \ldots \]

with \( z(q^2) = z(q^2, t_{cut}, t_0 = 0) \).

• We minimize a \( \chi^2 \) function

\[ \chi^2 = \sum (\text{data} - \text{theory})^2 / (\delta G_M)^2 \]

for ‘data’ we used proton, neutron and \( \pi\pi \) datasets, \( \delta G_M \) is the corresponding error bar and z-expansion provided our ‘theory’.
Datasets used for $r^p_M$ Extraction

• For the Proton Data we use

$$t_{cut} = 4m^2_{\pi} \Rightarrow z(q^2) \equiv z(q^2, 4m^2_{\pi}, 0)$$

• Including the Neutron data allows us to separate the $I = 1$ and $I = 0$ isospin component.

$$I = 0, \; t_{cut} = 9m^2_{\pi} \Rightarrow \text{decrease } z_{max}$$

• Inclusion of $\pi\pi$ data raise the effective threshold for the isovector. In this case

$$I = 1, \; t_{cut} = 16m^2_{\pi} \Rightarrow \text{decrease } z_{max}$$
Results for Proton Data

Arrington et.al (2007)

\[ Q^2 \leq 0.5 \text{ GeV}^2 \]

<table>
<thead>
<tr>
<th>Bound on ( a_k )</th>
<th>( r^p_M )</th>
<th>+( \sigma )</th>
<th>-( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.89</td>
<td>0.03</td>
<td>0.05</td>
</tr>
<tr>
<td>10</td>
<td>0.91</td>
<td>0.03</td>
<td>0.06</td>
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<tr>
<td>15</td>
<td>0.92</td>
<td>0.04</td>
<td>0.07</td>
</tr>
<tr>
<td>20</td>
<td>0.93</td>
<td>0.04</td>
<td>0.07</td>
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\[ Q^2 \leq 1.0 \text{ GeV}^2 \]

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<tbody>
<tr>
<td>5</td>
<td>0.89</td>
<td>0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>10</td>
<td>0.90</td>
<td>0.03</td>
<td>0.07</td>
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<tr>
<td>15</td>
<td>0.91</td>
<td>0.04</td>
<td>0.07</td>
</tr>
<tr>
<td>20</td>
<td>0.91</td>
<td>0.05</td>
<td>0.08</td>
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</tbody>
</table>
**Results for (Proton + Neutron) Data**

\[ Q^2 \leq 0.5 \text{ GeV}^2 \]

<table>
<thead>
<tr>
<th>Bound on ( a_k )</th>
<th>( r_M^p )</th>
<th>+( \sigma )</th>
<th>-( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
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<td>0.01</td>
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<tr>
<td>10</td>
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<td>0.05</td>
</tr>
<tr>
<td>15</td>
<td>0.87</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>20</td>
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\[ Q^2 \leq 1.0 \text{ GeV}^2 \]

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<td>0.05</td>
<td>0.06</td>
</tr>
</tbody>
</table>
Results for (Proton+ Neutron+ $\pi\pi\pi$) Data

$Q^2 \leq 0.5 \text{ GeV}^2$

<table>
<thead>
<tr>
<th>Bound on $a_k$</th>
<th>$r_p^M$</th>
<th>$+\sigma$</th>
<th>$-\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.013</td>
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<tr>
<td>10</td>
<td>0.871</td>
<td>0.011</td>
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<tr>
<td>15</td>
<td>0.873</td>
<td>0.012</td>
<td>0.016</td>
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<tr>
<td>20</td>
<td>0.876</td>
<td>0.012</td>
<td>0.018</td>
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$Q^2 \leq 1.0 \text{ GeV}^2$

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<tr>
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<th>$r_p^M$</th>
<th>$+\sigma$</th>
<th>$-\sigma$</th>
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</thead>
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<td>0.008</td>
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<tr>
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<td>0.875</td>
<td>0.013</td>
<td>0.016</td>
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</table>
Extraction of the Neutron Magnetic Radius

- We can extract the radius of the neutron similarly as proton, using

\[
\langle r^2 \rangle_M^n = \frac{6}{G_M^n(0)} \frac{d}{dq^2} G_M^n(q^2) |_{q^2=0}
\]

<table>
<thead>
<tr>
<th>Neutron Data</th>
<th>Neutron + Proton Data</th>
<th>Neutron + Proton + (\pi\pi) Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q^2) (GeV^2)</td>
<td>Bound</td>
<td>(r_M^n)</td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>0.74</td>
</tr>
<tr>
<td></td>
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<td>0.65</td>
</tr>
<tr>
<td>1.0</td>
<td>10</td>
<td>0.77</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.74</td>
</tr>
</tbody>
</table>

\(r_M^n = 0.7 \pm 0.2\)

\(r_M^n = 0.9 \pm 0.1\)

\(r_M^n = 0.89 \pm 0.03\)
Conclusion

• Calculations using the proton data gives
  \[ r_M^p = 0.91^{+0.03}_{-0.06} \pm 0.02 \text{ fm}. \]

• Calculations including the neutron data finds the radius as
  \[ r_M^p = 0.87^{+0.04}_{-0.05} \pm 0.01 \text{ fm}. \]

• Calculations including \( \pi\pi \) data along with proton and neutron data sets gives
  \[ r_M^p = 0.87^{+0.02}_{-0.02} \text{ fm}. \]

• Our results are more consistent with the values of
  \( 0.876 \pm 0.02 \text{ fm} \) or \( 0.854 \pm 0.005 \text{ fm} \), in contrast to the PDG 2014 value \( 0.777 \pm 0.014 \text{ fm} \).
• Even for higher bounds like $|a_k| \leq 20$, the values don’t change significantly.

• The extracted magnetic radius is independent of the numbers of parameters we fit or the cut on $Q^2$.

• It is interesting to note that within errors, magnetic radius of the neutron ($r_M^n = 0.89 \pm 0.03$ fm) is consistent with the magnetic radius of the proton ($r_M^p = 0.87 \pm 0.02$ fm).
Thank you!